Limit at infinity

$$\lim_{x \to +\infty} f(x) = L$$

$f : D \to \mathbb{R}$ $\quad L \in \mathbb{R}$

$f: A \to B$ $\quad$ f is <u>constant</u> $\exists c \in B$ s.t. $\forall x \in A$ $f(x) = c$

<u>eventually constant</u>

D nonempty subset $\mathbb{R}$ $\quad$ $f: D \to \mathbb{R}$

$\exists c \in \mathbb{R}$ $\exists X \in D$ s.t. $\forall x > X$ we have $f(x) = c$

"constantish" approximations

$$\tanh 8 = 1$$

wrong

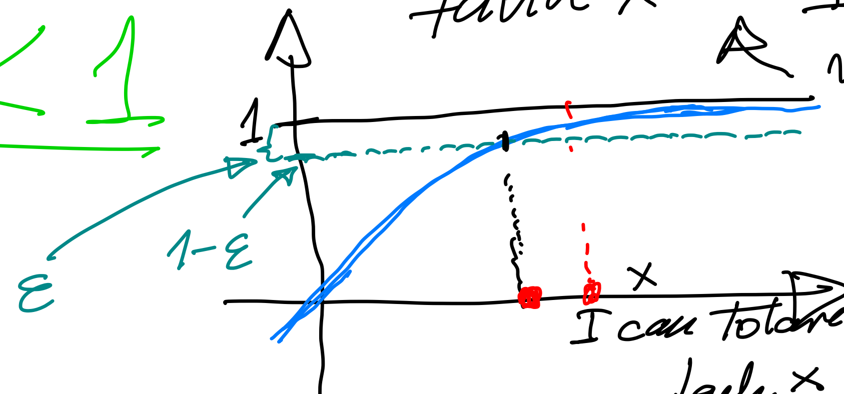
$$\tanh 8 < 1$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} < 1$$

If we are willing to tolerate $\varepsilon = 10^{-8}$ error for which x can we write

$$\tanh x = 1$$

never true



I can tolerate saying $\tanh x = 1$

Find such x

$$0 < 1 - \tanh x < \varepsilon$$ solve for x!

Solve $1-\tanh x < \varepsilon$

$$\underbrace{1-\tanh x}_{=} \qquad 1-\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = \frac{\cancel{e^{x}}+e^{-x}-\cancel{e^{x}}+e^{-x}}{e^{x}+e^{-x}} = \frac{2e^{-x}}{e^{x}+e^{-x}}$$

Solve $\boxed{\dfrac{2e^{-x}}{e^{x}+e^{-x}} < \varepsilon}$ hard!?

Make it easier by increasing $\dfrac{2e^{-x}}{e^{x}+e^{-x}}$

pizza →

party →

$$\boxed{\frac{2e^{-x}}{e^{x}+e^{-x}} \underbrace{<}_{\text{pizza-party-principle}} \frac{2e^{-x}}{e^{x}} \underset{\text{alg.}}{=} 2e^{-2x}}$$

huge

tiny

TRUE

Solving

$\varepsilon > 0$

**

$$2e^{-2x} < \varepsilon$$
$$e^{-2x} < \frac{\varepsilon}{2}$$
$$\ln e^{-2x} < \ln \frac{\varepsilon}{2}$$
$$-2x < \ln \frac{\varepsilon}{2}$$
$$x > -\frac{1}{2} \ln \frac{\varepsilon}{2}$$

much easier than what we had before

If $\varepsilon = 10^{-8}$ $x > -\frac{1}{2} \ln\left(\frac{1}{2} 10^{-8}\right) \approx 9.55\ldots$

$x > 10$ we are ok.

Definition of limit as x approaches $+\infty$

Let $D \subseteq \mathbb{R}$ and $L \in \mathbb{R}$. Assume $\exists X_0 \in D$ such that $[X_0, +\infty) \subseteq D$. Let $f : D \to \mathbb{R}$ be a function. We say that L is a limit of f as x approaches $+\infty$ if the following condition is satisfied

$$\forall \varepsilon > 0 \quad \exists X(\varepsilon) \geq X_0 \quad \text{s.t.}$$

$$x > X(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon$$

The notation for this is

$$\lim_{x \to +\infty} f(x) = L.$$

What we did above, we proved

$$\lim_{x \to +\infty} \tanh x = 1$$

How come? For every $\varepsilon > 0$ we calculated

$$X(\varepsilon) = -\frac{1}{2} \ln\left(\frac{\varepsilon}{2}\right)$$

Recall (*) $\boxed{\left|\tanh x - 1\right| = \frac{2e^{-x}}{e^{x}+e^{-x}} < 2e^{-2x}}$ proved by Pizza Party

and (**) $\boxed{\boxed{2e^{-2x} < \varepsilon} \iff x > -\frac{1}{2}\ln\left(\frac{\varepsilon}{2}\right)}$

Now prove

$$\boxed{x > -\frac{1}{2}\ln\left(\frac{\varepsilon}{2}\right)} \Rightarrow \left|\tanh x - 1\right| < \varepsilon$$

By (**) $\Rightarrow$ $\boxed{2e^{-2x} < \varepsilon}$

(*) $\boxed{|\tanh x - 1| < 2e^{-2x}}$ ✓

By transitivity of order I deduce $|\tanh x - 1| < \varepsilon$

On the following page I finish the proof of $\lim\limits_{x \to +\infty} \tanh x = 1$.
In the proof I use the green statements (*) and (**) which we proved earlier.

It is a good strategy to split a proof in several parts and then combine the parts at the end.

We proved the following two statements (boxed in green)

$$\boxed{\forall x\in\mathbb{R} \quad |\tanh x - 1| = \frac{2e^{-x}}{e^{x}+e^{-x}} < 2e^{-2x}} \qquad (*)$$

Let $\varepsilon>0$.

$$\boxed{\forall x\in\mathbb{R} \quad 2e^{-2x}<\varepsilon \iff x > -\frac{1}{2}\ln\left(\frac{\varepsilon}{2}\right)} \qquad (**)$$

Now we will prove $\boxed{x > -\frac{1}{2}\ln\left(\frac{\varepsilon}{2}\right)} \Rightarrow \boxed{|\tanh x - 1| < \varepsilon}$

Assume $\boxed{x > -\frac{1}{2}\ln\left(\frac{\varepsilon}{2}\right)}$. By the implication $\Leftarrow$ in $(**)$ we conclude that $\boxed{2e^{-2x} < \varepsilon}$. By $(*)$ we have $\boxed{|\tanh x - 1| < 2e^{-2x}}$. From the preceding two green boxes we deduce that $\boxed{|\tanh x - 1| < \varepsilon}$.

We proved