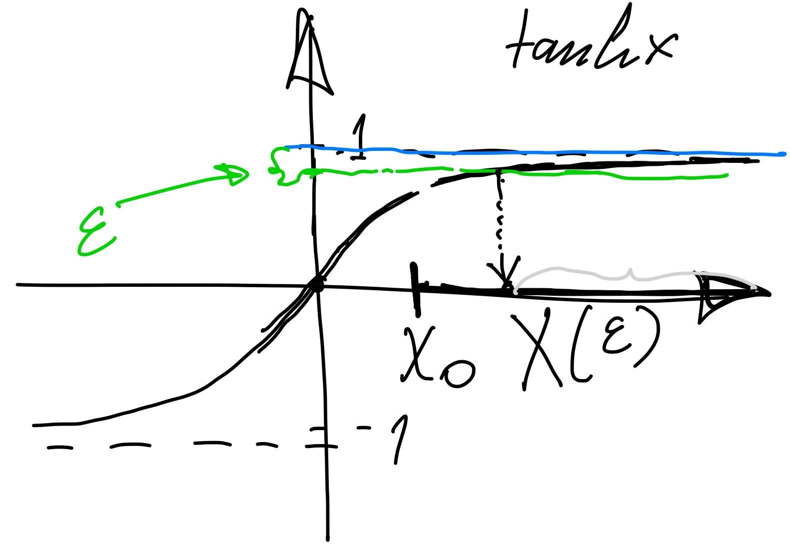
Formal definition of $\lim_{x \to +\infty} f(x) = L$ and an example

$f : D \to \mathbb{R}$

$$\lim_{x\to+\infty} \tanh x = 1$$



<u>Definition</u> Let $D \subseteq \mathbb{R}$ and let $L \in \mathbb{R}$.
A function $f : D \to \mathbb{R}$ has the limit L as x approaches $+\infty$ if the following two conditions are satisfied

(I) $\exists\, \cancel{x_0} \in D$ s.t. $[x_0, +\infty) \subseteq D$

(II) $\forall \varepsilon > 0$ $\exists\, \cancel{X}(\varepsilon) \geq x_0$ s.t.

formula involving ε must be true

essence $x > X(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon$

error less than ε

Example 1

$$\lim_{x\to+\infty} \frac{\lceil x \rceil}{\lfloor x \rfloor} = 1$$

interval

$\lfloor x \rfloor = 0 \Leftrightarrow x \in [0,1)$

(I) What is D?

$D = [1, +\infty)$

$L = 1$

$X_0 = 1$

(II) $\forall \varepsilon > 0 \; \exists \; X(\varepsilon) \geq 1$ s.t.

$$x > X(\varepsilon) \Rightarrow \underbrace{\left| \frac{\lceil x \rceil}{\lfloor x \rfloor} - 1 \right|}_{\text{study}} < \varepsilon$$

do the proof

discover

Let $x \geqslant 1$

$$\left|\frac{\lceil x\rceil}{\lfloor x\rfloor}-1\right| \underset{\substack{\text{algebra}\\ \text{BK}\\ \text{(background knowledge)}}}{=} \left|\frac{\lceil x\rceil-\lfloor x\rfloor}{\lfloor x\rfloor}\right| \underset{\text{BK}}{=} \frac{|\lceil x\rceil-\lfloor x\rfloor|}{\lfloor x\rfloor} =$$

without this my formula might crash.

legality

$x \geqslant 1$

$x \geq 1$

$\lfloor x\rfloor \geqslant 1 > 0$

☺

Pizza-Party

$$\frac{\lceil x\rceil-\lfloor x\rfloor}{\lfloor x\rfloor} \underset{}{\overset{\text{BK}}{=}} \frac{1}{\lfloor x\rfloor} \leq \frac{1}{x-1}$$

have to take $x > 1$

BK → require discovery

Please be rigorous.

$0 \leq \lceil x\rceil - \lfloor x\rfloor \leq 1$

: $0 \leq \lceil x\rceil - \lfloor x\rfloor \leq 1$

BK

$\frac{1}{\lfloor x\rfloor} < \varepsilon$ still difficult to solve

make party smaller $x-1$

We disovered this BIN
$\underbrace{}_{\text{big inequality}}$

$$\boxed{\forall x>1 \text{ we have } \left|\frac{\lceil x\rceil}{\lfloor x\rfloor}-1\right| \leq \frac{1}{x-1}}$$

Solving $\frac{1}{x-1}<\varepsilon$ for x, instead of $\left|\frac{\lceil x\rceil}{\lfloor x\rfloor}-1\right|<\varepsilon$ is much easier.

$x>1$ must be

$$\frac{1}{x-1}<\varepsilon \overset{BK}{\Longleftrightarrow} x-1>\frac{1}{\varepsilon} \overset{BK}{\Longleftrightarrow} x>\underbrace{\frac{1}{\varepsilon}+1}_{X(\varepsilon)}$$

$\boxed{\text{I claim, for } \varepsilon>0}$

$$\underline{x>\frac{1}{\varepsilon}+1} \Longrightarrow \underline{\left|\frac{\lceil x\rceil}{\lfloor x\rfloor}-1\right|<\varepsilon}$$

I will prove it here!

We proved above $\boxed{\forall x>1 \text{ we have } \left|\frac{\lceil x\rceil}{\lfloor x\rfloor}-1\right| \leq \frac{1}{x-1}}$ ✱

We also proved above $\boxed{\forall x>1 \quad \frac{1}{x-1}<\varepsilon \iff x>\frac{1}{\varepsilon}+1}$ ✱✱

Now the proof. Let $\boxed{\varepsilon>0}$. (use $\Leftarrow$)

Assume $\boxed{x>\frac{1}{\varepsilon}+1}$. Since $\varepsilon>0$, we have $x>1$. By ✱✱ we deduce that $\boxed{\frac{1}{x-1}<\varepsilon}$. By ✱ we have $\boxed{\left|\frac{\lceil x\rceil}{\lfloor x\rfloor}-1\right| \leq \frac{1}{x-1}}$

From the last two green boxes, by transitivity of order we deduce $\boxed{\left|\frac{\lceil x\rceil}{\lfloor x\rfloor}-1\right| < \varepsilon}$.

The content of the blue box proves the implication:

$$x > \frac{1}{\varepsilon} + 1 \Longrightarrow \left| \frac{\lceil x \rceil}{\lfloor x \rfloor} - 1 \right| < \varepsilon$$

Together with the proofs of $*$ and $**$ this proves

$$\lim_{x \to +\infty} \frac{\lceil x \rceil}{\lfloor x \rfloor} = 1 .$$ QED