Limit at $+\infty$

Example

Definition $D \subseteq \mathbb{R}$ $L \in \mathbb{R}$

Why is X big? because it's big, in particular for small ε

$f : D \to \mathbb{R}$

$$\lim_{x \to +\infty} f(x) = L \iff$$

It says all, but you think $\varepsilon > 0$ as small

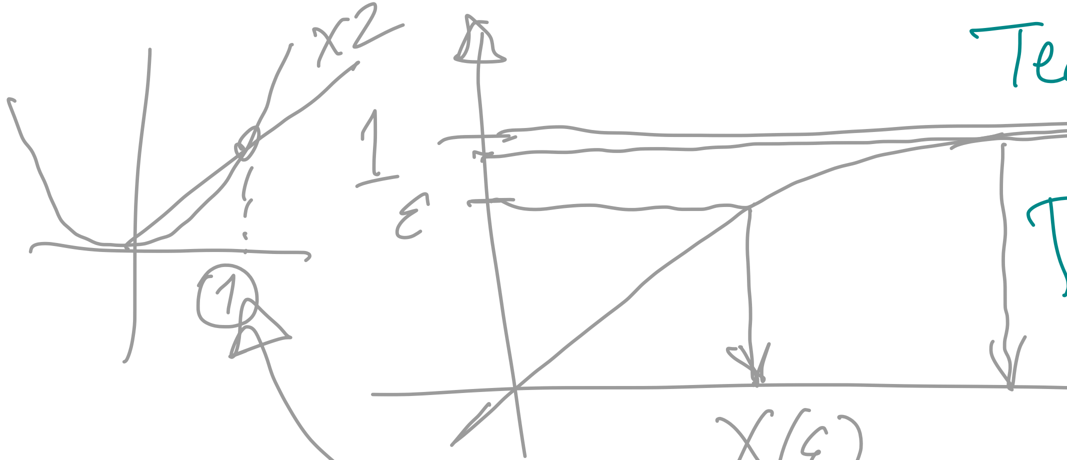
(I) $\exists X_0 \in D$ s.t. $[X_0, +\infty) \subseteq D$

(II) $\forall \varepsilon > 0 \; \exists \cancel{X}(\varepsilon) \geq X_0$ s.t.

$$x > X(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon$$

sol. of

Please DEMAND



Teacher: be rigorous!

$D = [1, +\infty)$

D is the first domain: D

Let us do a complicated example now.

$$\lim_{x\to+\infty}\left(x-\sqrt{x^2-x}\right)=1/2$$

do algebra

$$\left(x-\sqrt{x^2-x}\right)\cdot\frac{x+\sqrt{x^2-x}}{x+\sqrt{x^2-x}} = \frac{x^2-(x^2-x)}{x+\sqrt{x^2-x}} = \frac{x}{x+\sqrt{x^2-x}}$$

$(a-b)(a+b)=a^2-b^2$

provided $x \geq 1$

diff. algebraic expressions of the same function

$\forall x \geq 1$

$D = [1, +\infty)$

$$\boxed{x - \sqrt{x^2 - x} = \frac{x}{x + \sqrt{x^2 - x}}} \to 1/2$$

$\approx 2x$ large x

Prove $\lim_{x \to +\infty} \left(x - \sqrt{x^2 - x}\right) = \frac{1}{2}$. $X_0 = 1$

Let $\varepsilon > 0$ be arbitrary. To find $X(\varepsilon)$

I work backwards from

$$\left| \left(x - \sqrt{x^2 - x}\right) - \frac{1}{2} \right| < \varepsilon$$

to solve this
simplify

Solve this!
for x. Sol.
must look like
$x >$ formula
in ε

$$\left|\left(x-\sqrt{x^2-x}\right)-\frac{1}{2}\right| \overset{BK}{=} \left|\frac{x}{x+\sqrt{x^2-x}}-\frac{1}{2}\right| \overset{BK}{=}$$

$$\boxed{\text{Assume } x \geq 1}$$

$$= \left|\frac{2x-\left(x+\sqrt{x^2-x}\right)}{2\left(x+\sqrt{x^2-x}\right)}\right| = \left|\frac{\overbrace{x-\sqrt{x^2-x}}^{>0}}{\underbrace{2\left(x+\sqrt{x^2-x}\right)}_{>0}}\right| =$$

$$= \frac{x-\sqrt{x^2-x}}{2\left(x+\sqrt{x^2-x}\right)} = \frac{x^2-\left(x^2-x\right)}{2\left(x+\sqrt{x^2-x}\right)^2} = \underbrace{\frac{x}{2\left(x+\sqrt{x^2-x}\right)^2}}_{\text{I am ready for the pizza-party.}}$$

mult. top & bot by $x+\sqrt{x^2-x}$

make the party smaller

$x \geq 1$

$$2(x+\underbrace{\sqrt{x^2-x}}_{\text{go away, we want more pizza.}})^2 \geq 2x^2$$

$$\leq \frac{x}{2x^2} = \frac{1}{2x} < \varepsilon$$

Celebrate: We just proved that $\forall\, x \geq 1$ we have

$$\boxed{\left|\left(x-\sqrt{x^2-x}\right)-\frac{1}{2}\right| \leq \frac{1}{2x}}$$

plot

verify by plot

$X(\varepsilon)$

Solve $\frac{1}{2x} < \varepsilon$

$x \geq 1$

$2x > \frac{1}{\varepsilon}$

$x > \frac{1}{2\varepsilon}$

$\cancel{X(\varepsilon) = \frac{1}{2\varepsilon}}$

(after class notes)

Ooops, This is not a correct $X(\varepsilon)$. We must also have $X(\varepsilon) \geq X_0$.

Let $\varepsilon > 0$. Set $X(\varepsilon) = \max\left\{\frac{1}{2\varepsilon}, 1\right\}$.

We will prove the following inequality:

$$\boxed{x > \max\left\{\frac{1}{2\varepsilon}, 1\right\}} \Longrightarrow \left| x - \sqrt{x^2 - x} - \frac{1}{2} \right| < \varepsilon.$$

<u>Proof.</u> In this proof we will use two inequalities

$$\boxed{\forall x \geq 1 \quad \left| x - \sqrt{x^2 - x} - \frac{1}{2} \right| \leq \frac{1}{2x}} \quad (*)$$

This inequality is proved above. You should reprove it for practice.

$$\boxed{\forall x > 0 \quad \frac{1}{2x} < \varepsilon \Longleftrightarrow x > \frac{1}{2\varepsilon}} \quad (**)$$

This inequality follows from basic algebra (from axioms)

Assume $\boxed{x > \max\left\{\frac{1}{2\varepsilon}, 1\right\}}$. Then $\boxed{x > 1}$. Since $x>1$ inequality (*) holds, that is $\boxed{\left|x - \sqrt{x^2 - x} - \frac{1}{2}\right| < \frac{1}{2x}}$. Since $\boxed{x > \max\left\{\frac{1}{2\varepsilon}, 1\right\}}$ holds we have $\boxed{x > \frac{1}{2\varepsilon}}$. By (**) $\boxed{x > \frac{1}{2\varepsilon}}$ implies $\boxed{\frac{1}{2x} < \varepsilon}$. Finally, we proved two things $\boxed{\left|x - \sqrt{x^2 - x} - \frac{1}{2}\right| < \frac{1}{2x}}$ and $\boxed{\frac{1}{2x} < \varepsilon}$.

By the transitivity property of order, we deduce $\boxed{\left|x - \sqrt{x^2 - x} - \frac{1}{2}\right| < \varepsilon}$. Thus, we proved the implication

$$x > \max\left\{\frac{1}{2\varepsilon}, 1\right\} \Rightarrow \left|x - \sqrt{x^2 - x} - \frac{1}{2}\right| < \varepsilon$$

Previously red $\Rightarrow$ has been greenified.