$$\lim_{x \to a} f(x) = L$$

Definition Let $a, L \in \mathbb{R}$ and $D \subseteq \mathbb{R}$. Let $f : D \to \mathbb{R}$ be a function. We write

$$\lim_{x \to a} f(x) = L$$

if the foll. conds are satisfied:

(I) $\exists\, \delta_0 > 0$ s.t. $\boxed{(a-\delta_0, a) \cup (a, a+\delta_0) \subseteq D}$

(II) $\forall \varepsilon > 0\ \exists\, \delta(\varepsilon)$ s.t. $0 < \delta(\varepsilon) \leq \delta_0$ and

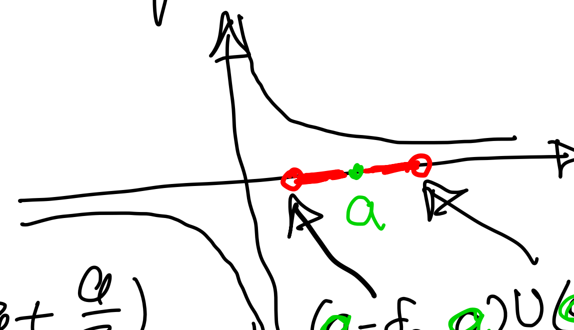
$$0 < |x-a| < \delta(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon$$

Let $a>0$. Prove

$$\lim_{x\to a}\frac{1}{x}=\frac{1}{a}.$$

Here $a>0$ given, $L=\frac{1}{a}$, $f(x)=\frac{1}{x}$, $D=\mathbb{R}\setminus\{0\}$. Now we proceed with the condition (I).

$$\delta_0=\frac{a}{2}>0$$



$(a-\delta_0, a)\cup(a, a+\delta_0)$

$(a-\frac{a}{2}, a)\cup(a, a+\frac{a}{2})$

$= (\frac{a}{2}, a)\cup(a, \frac{3a}{2})$

Here our function f is defined at a, $f(a) = 1/a$, so we can consider one interval $\left(\frac{a}{2}, \frac{3a}{2}\right)$.

(*) Now we can focus only on values of x which are in $\left(\frac{a}{2}, \frac{3a}{2}\right)$.

To do cond. (II) we need to introduce $\varepsilon > 0$ arbitrary and study $\left|\frac{1}{x} - \frac{1}{a}\right| < \varepsilon$.

In fact we need to solve this inequality for $|x-a|$. The first step:

treasure

Simplify

BK is our vast background knowledge

$$\left|\frac{1}{x}-\frac{1}{a}\right| \overset{=}{\text{BK}} \left|\frac{a-x}{xa}\right| \underset{\text{rules for abs}}{=}$$

$|x-a| = |a-x|$

$$= \frac{|x-a|}{|xa|} = \frac{|x-a|}{|x|\,a} =$$

I need to solve $\frac{|x-a|}{xa} < \varepsilon$

x not good here

we can restrict $\boxed{x \in \left(\frac{a}{2}, \frac{3a}{2}\right)}$ good friend

thus $x > 0$

$$= \frac{|x-a|}{xa} \leq \frac{|x-a|}{\frac{a}{2}a} = \frac{2}{a^2}|x-a|$$

bad foe

pizza party

A celebration of this simplification:

We just proved:

BIN

$$\boxed{\forall x \in \left(\frac{a}{2}, \frac{3a}{2}\right) \text{ we have } \left|\frac{1}{x}-\frac{1}{a}\right| \leq \frac{2}{a^2}|x-a|}$$

Who wants to solve $\frac{2}{a^2}|x-a|<\varepsilon$ for $|x-a|$!

$$\frac{2}{a^2}|x-a|<\varepsilon \iff |x-a| < \frac{a^2}{2}\varepsilon$$

$\delta_0 = \frac{a}{2}$ from (I)

For $\varepsilon>0$, we set

$$\delta(\varepsilon) = \min\left\{\frac{a^2}{2}\varepsilon, \frac{a}{2}\right\}$$

Now I have to prove:

$$0<|x-a|<\min\left\{\frac{a^2}{2}\varepsilon, \frac{a}{2}\right\} \Rightarrow \left|\frac{1}{x}-\frac{1}{a}\right|<\varepsilon$$

assumption (hypothesis)

RED

Assume that $\boxed{0<|x-a|<\min\left\{\frac{a^2}{2}\varepsilon, \frac{a}{2}\right\}}$.

As a consequence of our assumption we have

$\boxed{|x-a|<\frac{a}{2}}$ and $\boxed{|x-a|<\frac{a^2}{2}\varepsilon}$ G1

$\Updownarrow$ by BBB

$\boxed{x\in\left(\frac{a}{2},\frac{3a}{2}\right)}$ $\rightarrow$ Since $x\in\left(\frac{a}{2},\frac{3a}{2}\right)$, by

I conclude

$$\boxed{\left|\frac{1}{x}-\frac{1}{a}\right|\le\frac{2}{a^2}|x-a|}$$ G2

By G1 and G2 I deduce $\boxed{\left|\frac{1}{x}-\frac{1}{a}\right|<\varepsilon}$.

Now Green

Example

$$\lim_{x \to a} x^2 = a^2$$

$D = \mathbb{R}$ $\qquad \delta_0 = 1 \qquad (a-1, a+1)$

$$|x^2 - a^2| = |x-a| \underbrace{|x+a|}_{\text{What is the largest if } x \in (a-1,a+1)} \leq \text{?} \; |x-a|$$

Pizza Party Pizza

What is the maximum value of $|x+a|$ when x is restricted to $(a-1, a+1)$?

The value of the function $|x+a|$ at $x=a$ is $2|a|$

Since the graph of the function $|x+a|$ looks like

it is reasonable to concluded that the maximum value is $2|a|+1$. So, I claim

$$\boxed{\forall x\in(a-1,a+1) \text{ we have } |x+a|\le 2|a|+1}$$

Prove this! Assume $\boxed{x\in(a-1,a+1)}$. This means that $a-1<\boxed{x<a+1}$. Multiplying by (-1) we get $-a-1<\boxed{-x<-a+1}$.

Now we find an upper bound for $x+a$ and $-x-a$:

$$x + a < a + 1 + a = 2a + 1 \leq 2|a| + 1$$

$$-x - a < -a + 1 - a = -2a + 1 \leq 2|a| + 1$$

Therefore $|x+a| = \max\{x+a, -x-a\} \leq 2|a|+1.$ QED

proof finished

Another way to prove the red box is to observe the equivalence $x \in (a-1, a+1) \iff |x-a| < 1$ and use the triangle inequality:

$$|x+a| = |x - a + 2a| \leq |x-a| + 2|a| < 2|a| + 1.$$

With the red box inequality proved we can prove $\lim_{x \to a} x^2 = a^2$.

We need to simplify: $|x^2-a^2| = |x-a||x+a| \leq (2|a|+1)|x-a|$

holds only for $x\in(a-1,a+1)$ based on the red boxed inequality.

Hence

BIN

$$\boxed{\forall x\in(a-1,a+1) \text{ we have } |x^2-a^2| \leq (2|a|+1)|x-a|}$$

This inequality has been proved.

Let $\varepsilon>0$ be arbitrary.

Set $\delta(\varepsilon) = \min\left\{\frac{\varepsilon}{2|a|+1}, 1\right\}$.

Now prove: $\boxed{0<|x-a|<\min\left\{\frac{\varepsilon}{2|a|+1}, 1\right\}} \Rightarrow \boxed{|x^2-a^2|<\varepsilon}$

You can do it, You can prove it.