Four Big Trig Limits

Sandwich Squeeze Theorem

$f, g, h : D \to \mathbb{R}$

$D \subseteq \mathbb{R}$, $a, L \in \mathbb{R}$

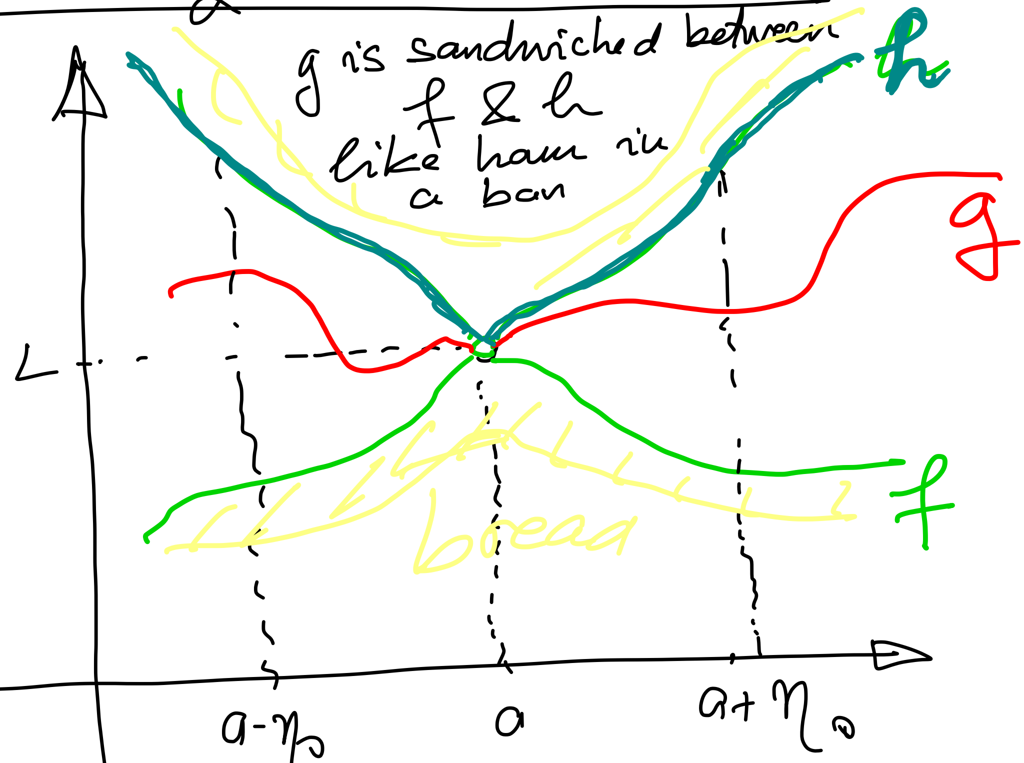
Assumptions:

① $\boxed{\lim_{x \to a} f(x) = L}$

② $\boxed{\lim_{x \to a} h(x) = L}$

③ the inequality HOLDS, see the picture

$\eta_0 > 0$



Conclusion is

$\lim_{x \to a} g(x) = L$

Scissors Squeeze Theorem

① $\lim_{x \to a} f(x) = L$

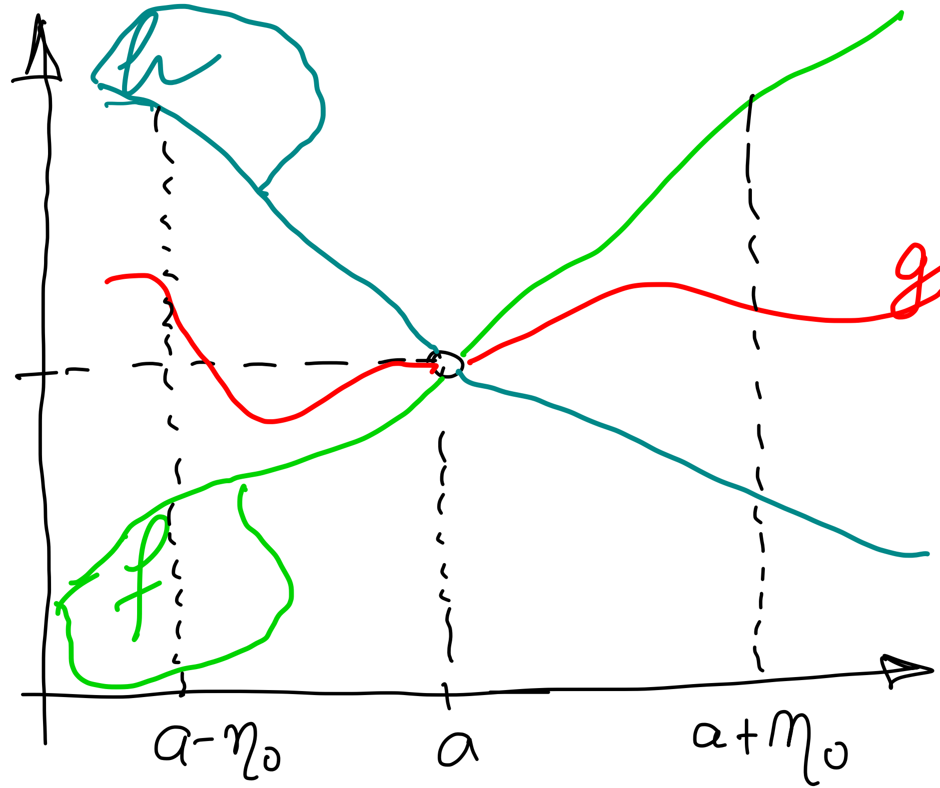
② $\lim_{x \to a} h(x) = L$

③ g is captured between f and h like yarn inbetween Scissors hands

$\eta_0 > 0$

$x \in (a-\eta_0, a)$ $f(x) \leq g(x) \leq h(x)$

$x \in (a, a+\eta_0)$ $h(x) \leq g(x) \leq f(x)$



Conclusion: $\lim_{x \to a} g(x) = L$.

Proofs for four trigonometric Limits — From first principles — that is from the unit circle

① $$\lim_{x \to 0} \cos x = 1$$

u an angle $u \in [0, \pi/3]$

$\overline{OA} = \cos u$, $\boxed{\overline{AC} = 1 - \cos u}$

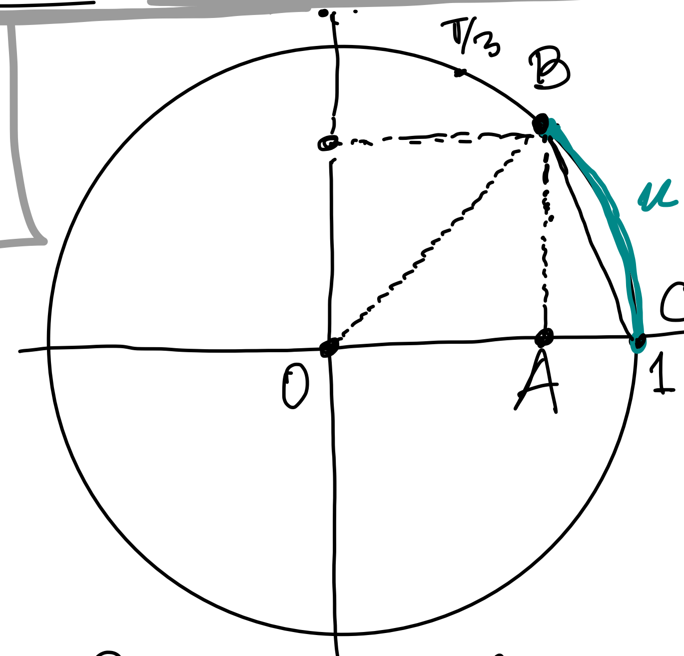
ΔABC is a right triangle

$\overline{BC}$ is the hypothenuse, $\overline{AC}$ is its side. Therefore

$$\boxed{\overline{AC} \leq \overline{BC}.}$$

Since the straight line is the shortest distance between two points

$$\boxed{\overline{BC} \leq \overset{\frown}{BC} = u}$$



The unit circle

$\boxed{1-\cos u \doteq \overline{AC} \leq u}$ TRUE Clearly $0 \leq 1-\cos u$

Thus $\boxed{0 \leq 1-\cos u \leq u}$ for all $u \in [0, \pi/3]$

Let $x \in [-\pi/3, \pi/3]$. Then if $x \in [0, \pi/3]$ set $u = x$

$$0 \leq 1-\cos x \leq x$$

If $x \in [-\pi/3, 0)$ then $-x \in (0, \pi/3]$ so I can set $u = -x = |x|$

Therefore $0 \leq 1-\cos(-x) \leq |x|$. Remember $\cos x$ is even function so $\cos(-x) = \cos x$

TRUE $\boxed{0 \leq 1-\cos x \leq |x|}$ ← thus

true for all $x \in [-\pi/3, \pi/3]$.

This is OUR B.N. Now we can prove

$$\lim_{x \to 0} \cos x = 1$$

(I) we can take $\delta_0 = \pi/3 > 0$. $\cos x$ domain is $\mathbb{R} = D$.

(II) Let $\varepsilon > 0$ be arbitrary. Take $\delta(\varepsilon) = \min\{\varepsilon, \pi/3\}$.

Now prove $\boxed{0 < |x-0| < \min\{\varepsilon, \pi/3\}} \Rightarrow \boxed{|\cos x - 1| < \varepsilon}$

Assume $\boxed{0 < |x-0| < \min\{\varepsilon, \pi/3\}}$. Then $|x| < \pi/3$.

Therefore $x \in [-\pi/3, \pi/3]$. Thus $(*)$ is TRUE. that is

$$\boxed{0 \leq 1 - \cos x \leq |x|}$$

Since $0 \leq 1 - \cos x$, we have $|\cos x - 1| = 1 - \cos x$.

therefore $\boxed{|\cos x - 1| \leq |x|}$

By our assumption $\boxed{|x| < \varepsilon}$

$\Rightarrow \boxed{|\cos x - 1| < \varepsilon}$

② $\boxed{\lim_{x\to 0} \sin x = 0.}$ This is a beautiful exercise!

Just prove $\forall x \in [-\pi/3, \pi/3]$ $|\sin x| \le |x|$

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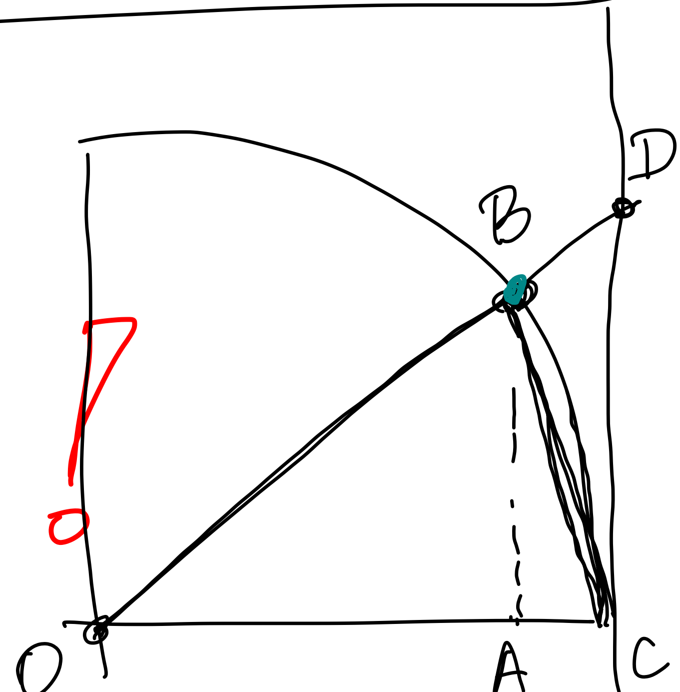
③ $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1.}$

$$\underbrace{1-|x|}_{\lim_{x\to 0}(1-|x|)=1} \le \frac{\sin x}{x} \le 1$$

PROVE IT!

LOOK at Areas



$\triangle OCB$

pizza slice $\triangleleft\ OCB$

$\triangle OCD$

areas

$$\boxed{\triangle OCB \le \triangleleft\, OCB \le \triangle OCD}$$

$$\bigcirc \le \bigcirc \le \bigcirc$$

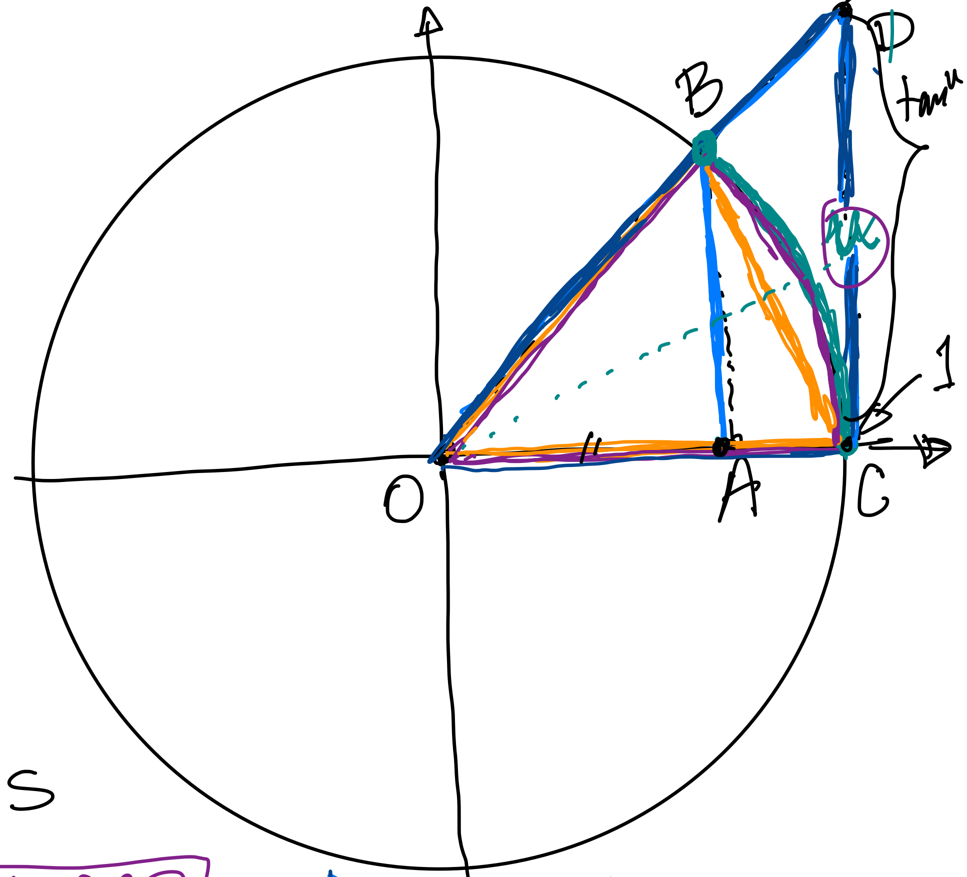
$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

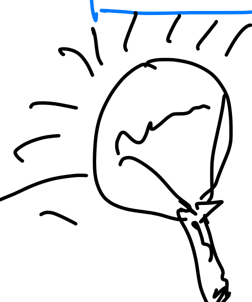
$u\in(0,\pi/3)$

$$\frac{\sin u}{u}$$

$u = \overset{\frown}{BC}$

$\boxed{\sin u} = \overline{AB}$

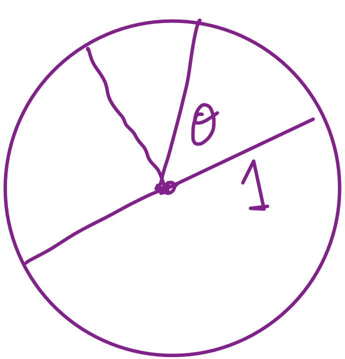




look at areas

$\boxed{\triangle OCB \text{ triangle}} \le \boxed{\text{pizza slice } OCB}$ circle segment $\le \boxed{\triangle OCD}$

$\boxed{\frac{1}{2}\sin u} \le \boxed{\frac{1}{2}u} \le \boxed{\frac{1}{2}\tan u} = \frac{\sin u}{\cos u}$



θ	area
2π	π
π	$\pi/2$
$\pi/2$	$\pi/4$
θ	$\theta/2$

similar △

$\triangle OAB \sim \triangle OCD$

$\cos u = \overline{OA}$, $\overline{AB} = \sin u$

$$\frac{\overline{OA}}{\overline{OC}} = \frac{\overline{AB}}{\overline{CD}} \ ?$$

$\overline{OC} = 1$

$$\boxed{\overline{CD} = \frac{\sin u}{\cos u}}$$

This is a geometric proof of the inequality:

$\forall u \in (0, \frac{\pi}{3})$ we have $\boxed{\frac{1}{2}\sin u \le \frac{1}{2}u \le \frac{1}{2}\frac{\sin u}{\cos u}}$

now we manipulate algebraically:

$\boxed{\sin u \leq u}$

$\boxed{\dfrac{\sin u}{u} \leq 1}$

$\boxed{\cos u \leq \dfrac{\sin u}{u}}$

$$\boxed{\cos u \leq \frac{\sin u}{u} \leq 1} \quad \forall u \in \left(0, \frac{\pi}{3}\right)$$

$x \in \left(-\frac{\pi}{3}, 0\right)$ $\quad -x \in \left(0, \frac{\pi}{3}\right)$ $\quad \cos(-x) \leq \dfrac{\sin(-x)}{-x} \leq 1$

But BK tells me $\cos(-x) = \cos x$

$\sin(-x) = -\sin x$ so $\frac{\sin(-x)}{-x} = \frac{\sin x}{x}$

$$\boxed{\forall x \in \left(-\frac{\pi}{3}, 0\right) \cup \left(0, \frac{\pi}{3}\right), \quad \cos x \leq \frac{\sin x}{x} \leq 1}$$

This is a Squeeze for $\frac{\sin x}{x}$.

We already proved that $\boxed{\lim_{x \to 0} \cos x = 1}$

And, we can prove that $\boxed{\lim_{x \to 0} 1 = 1}$

By **Sandwich Squeeze** Thm we deduce

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

<u>Comment</u> A little bit more work can provide a proof of this limit from the definition:

Yesterday, we proved

$\forall x \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ we have $0 \leq 1-\cos x \leq |x|$ ⟸ equivalent inequalities

$\forall x \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ we have $\boxed{1-|x| \leq \cos x \leq 1}$

Now combine $\cos x \leq \frac{\sin x}{x} \leq 1$ and
to get

$$\boxed{\forall x \in \left(-\frac{\pi}{3}, 0\right) \cup \left(0, \frac{\pi}{3}\right) \quad 1-|x| \leq \frac{\sin x}{x} \leq 1}$$

$$\Rightarrow \boxed{\forall x \in \left(-\frac{\pi}{3}, 0\right) \cup \left(0, \frac{\pi}{3}\right) \text{ we have } \left|\frac{\sin x}{x} - 1\right| \leq |x|} \quad (*)$$

Using the inequality ✱ we can prove

$\forall \varepsilon > 0$

$$0 < |x - 0| < \min\{\varepsilon, \tfrac{\pi}{3}\} \Longrightarrow \left| \frac{\sin x}{x} - 1 \right| < \varepsilon$$

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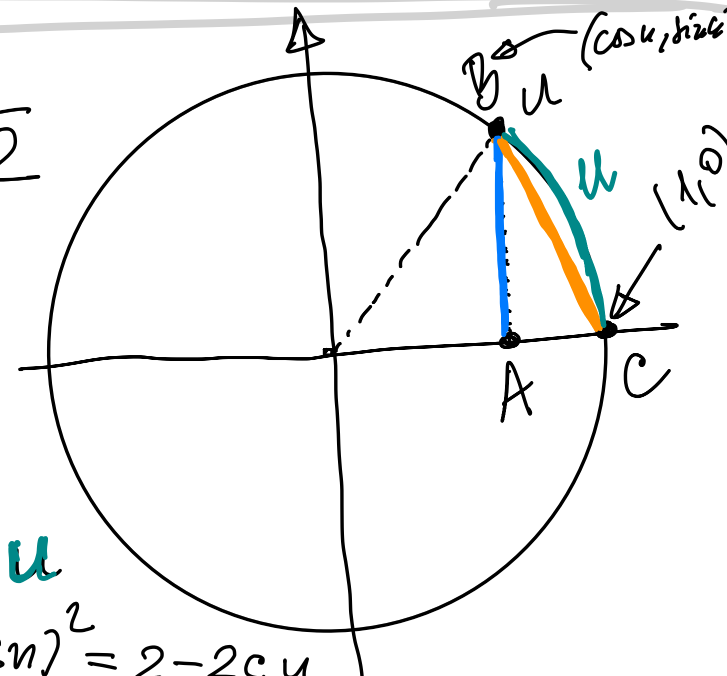
4 $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2} = \dfrac{1}{2}$



Compare lengths

$$\boxed{\overline{AB}} \le \boxed{\overline{BC}} \le \overset{\frown}{BC}$$

$$\boxed{\sin u} \le \boxed{\sqrt{(1-\cos u)^2 + (\sin u)^2}} \le u$$

$$1 - 2cu + (cu)^2 + (su)^2 = 2 - 2cu$$

$$0 \le \boxed{\sin u} \le \boxed{\sqrt{2(1-\cos u)}} \le u \quad \Big/\ {}^2 \text{ (square}$$

$$\frac{(\sin u)^2}{2} \le \cancel{2}(1-\cos u) \le \frac{u^2}{2} \quad \Big/\ \frac{\cancel{2}}{\cancel{2}}\, u^2$$

For all $u \in \left(0, \frac{\pi}{3}\right)$

$$\boxed{\frac{1}{2}\left(\frac{\sin u}{u}\right)^2 \le \frac{1-\cos u}{u^2} \le \frac{1}{2}}$$

for $x \in \left(-\frac{\pi}{3}, 0\right)$ we have $-x \in (0, \pi/3)$

$$\frac{1}{2}\left(\frac{\sin(-x)}{-x}\right)^2 \le \frac{1-\cos(-x)}{(-x)^2} \le \frac{1}{2}$$

By BK $\quad \frac{\sin(-x)}{-x} = \frac{\sin x}{x}$ and $\frac{1-\cos(-x)}{(-x)^2} = \frac{1-\cos x}{x^2}$

therefore

$\forall x \in \left(-\frac{\pi}{3}, 0\right) \cup \left(0, \frac{\pi}{3}\right)$ we have $\quad \frac{1}{2}\left(\frac{\sin x}{x}\right)^2 \leq \frac{1-\cos x}{x^2} \leq \frac{1}{2}$

We proved earlier that

$\forall x \in \left(-\frac{\pi}{3}, 0\right) \cup \left(0, \frac{\pi}{3}\right) \quad 1-|x| \leq \frac{\sin x}{x}$

Since for $x \in (-1, 1)$ we have $1-|x| \geq 0$ we can square the preceding inequality:

$$(1-|x|)^2 \leq \left(\frac{\sin x}{x}\right)^2 \quad \forall x \in (-1,0) \cup (0,1)$$

substitute

Substitute in the green box to get

$\forall x \in (-1,0) \cup (0,1)$ we have $\quad \frac{1}{2}(1-|x|)^2 \leq \frac{1-\cos x}{x^2} \leq \frac{1}{2}$

By BK $\frac{1}{2}(1-|x|)^2 = \frac{1}{2} - |x| + \frac{1}{2}|x|^2 \geq \frac{1}{2} - |x|$. Therefore

$$\boxed{\forall x \in (-1,0) \cup (0,1) \text{ we have } \frac{1}{2} - |x| \leq \frac{1-\cos x}{x^2} \leq \frac{1}{2}}$$

The preceding inequalities tell us that the distance between $\frac{1-\cos x}{x^2}$ and $\frac{1}{2}$ is $\leq |x|$. That is

$$\boxed{\forall x \in (-1,0) \cup (0,1) \text{ we have } \left|\frac{1-\cos x}{x^2} - \frac{1}{2}\right| \leq |x|}$$

Using the last green box we can prove the following implication $\forall \varepsilon > 0$ we have

$$\boxed{0 < |x-0| < \min\{\varepsilon, 1\}} \Rightarrow \boxed{\left|\frac{1-\cos x}{x^2} - \frac{1}{2}\right| < \varepsilon}$$

Here is the proof.

Assume $\boxed{0 < |x - 0| < \min\{\varepsilon, 1\}}$.

Then $\boxed{0 < |x| < 1}$ and $\boxed{|x| < \varepsilon}$.

$0 < |x| < 1$ is equivalent to $\boxed{x \in (-1,0) \cup (0,1)}$. Therefore $\diamondsuit$ is true.

Since $\diamondsuit$ is true and $|x| < \varepsilon$, by transitivity of order we conclude

$$\boxed{\left| \frac{1 - \cos x}{x^2} - \frac{1}{2} \right| < \varepsilon}.$$