Theorems about Convergent Sequences

$$s : \mathbb{N} \to \mathbb{R} \quad \text{or} \quad s : \mathbb{N}_0 \to \mathbb{R}$$

Definition of Limit

Let $L \in \mathbb{R}$. A sequence $s : \mathbb{N} \to \mathbb{R}$ has the limit L as $n \to +\infty$ if the following condition is satisfied:

$$\forall \varepsilon > 0 \quad \exists N(\varepsilon) \in \mathbb{R} \text{ such that}$$

$$\forall n \in \mathbb{N} \quad n > N(\varepsilon) \Rightarrow |s_n - L| < \varepsilon$$

The notation for the limit is $\lim\limits_{n \to +\infty} s_n = L$

Theorem Let $f : [1, +\infty) \to \mathbb{R}$ and define the sequence $a : \mathbb{N} \to \mathbb{R}$ by

$\forall n \in \mathbb{N} \quad \boxed{a_n = f(n)}$. $\triangle$

Then, if $\boxed{\lim_{x \to +\infty} f(x) = L}$, then $\boxed{\lim_{n \to +\infty} a_n = L}$.

everything is green in this definition

Proof.

Here $D = [1, +\infty)$, $L \in \mathbb{R}$.

(I) $\exists X_0 \in D$ s.t. $[X_0, +\infty) \subseteq D$. Here $X_0 = 1$.

(II) $\forall \varepsilon > 0 \;\; \exists X(\varepsilon) \geq 1$ such that

$\boxed{x > X(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon}$. $\otimes$

Let $\varepsilon > 0$. What is $N(\varepsilon)$? We need:

$$\forall n \in \mathbb{N} \quad n > N(\varepsilon) \underset{\text{must yield}}{\Longrightarrow} |\underset{f(n)}{\boxed{a_n}} - L| < \varepsilon$$

Set $N(\varepsilon) = X(\varepsilon)$.

Assume $\boxed{n \in \mathbb{N} \text{ and } n > N(\varepsilon)}$. Then $\boxed{n > X(\varepsilon)}$.

By $\circledast$ $\boxed{|f(n) - L| < \varepsilon}$

By $\triangle$ the last green BOX can be rewritten as

$$\boxed{|a_n - L| < \varepsilon}$$

Thus I proved:

$n \in \mathbb{N}$ and $n > N(\varepsilon) \Rightarrow |a_n - L| < \varepsilon$.

By def. this proves $\lim_{n \to +\infty} a_n = L$.

Theorem Let $f:(0,1] \to \mathbb{R}$ and let
$\forall n \in \mathbb{N}$ $a_n = f(1/n)$.
If $\lim\limits_{x \downarrow 0} f(x) = L$, then $\lim\limits_{n \to +\infty} a_n = L$.

Theorem (Algebra of Limits) $K, L \in \mathbb{R}$.
Let $a: \mathbb{N} \to \mathbb{R}$, $b: \mathbb{N} \to \mathbb{R}$, $c: \mathbb{N} \to \mathbb{R}$ be sequences.
Assume (1) $\lim\limits_{n \to +\infty} a_n = K$ and (2) $\lim\limits_{n \to +\infty} b_n = L$.

A) If $\forall n \in \mathbb{N}$ $\boxed{c_n = a_n + b_n}$, 3 then 4 $\boxed{\lim\limits_{n\to+\infty} c_n = K+L}$

B) If $\forall n \in \mathbb{N}$ $c_n = a_n \cdot b_n$, then $\lim\limits_{n\to+\infty} c_n = K \cdot L$

C) If $\forall n \in \mathbb{N}$, $b_n \neq 0$ and $c_n = \dfrac{a_n}{b_n}$ and $L \neq 0$,

then $$\lim_{n\to+\infty} \frac{a_n}{b_n} = \frac{K}{L}.$$

Proof. A) prove it!

1 means $\forall \varepsilon > 0$ $\exists N_a(\varepsilon) \in \mathbb{R}$ such that

$\boxed{\forall n \in \mathbb{N} \;\; n > N_a(\varepsilon) \Rightarrow |a_n - K| < \varepsilon}$

2 means $\forall \varepsilon > 0 \ \exists N_b(\varepsilon) \in \mathbb{R}$ such that

$$\forall n \in \mathbb{N} \quad n > N_b(\varepsilon) \Rightarrow |b_n - L| < \varepsilon$$

4 demands

$\forall \varepsilon > 0 \ \exists N_c(\varepsilon)$ such that

$$\forall n \in \mathbb{N} \quad n > N_c(\varepsilon) \not\Rightarrow \underbrace{|\underset{a_n+b_n}{\overset{c_n}{\shortparallel}} - (K+L)| < \varepsilon}$$

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study this
and connect it to

$$|a_n + b_n - (K+L)| = |a_n + b_n - K - L| = |a_n - K + b_n - L|$$

by triangle inequality

$$\leq |a_n - K| + |b_n - L|$$