More Convergence Theorems

Theorem Let $K, L \in \mathbb{R}$. Let $a: \mathbb{N} \to \mathbb{R}$, $b: \mathbb{N} \to \mathbb{R}$ and $c: \mathbb{N} \to \mathbb{R}$ be sequences. Assume $\boxed{\lim_{n\to+\infty} a_n = K}$ and $\boxed{\lim_{n\to+\infty} b_n = L}$.

If $\boxed{\forall n \in \mathbb{N} \quad c_n = a_n + b_n}$, then $\boxed{\lim_{n\to+\infty} c_n = K + L}$.

Proof. Assume: $\underbrace{\boxed{\lim_{n\to+\infty} a_n = K}}_{G1}$ and $\underbrace{\boxed{\lim_{n\to+\infty} b_n = L}}_{G2}$ and $\forall n \in \mathbb{N}$ $\underbrace{\boxed{c_n = a_n + b_n}}_{G3}$

The content of G3 is clear. But the content of G1 and G2 comes from the definition of Limit.

$\varepsilon > 0$ is a variable
$\varepsilon = 1$

You can read this box with ε as an empty space ~~in which you can~~ select any > 0.

G1 is

$$\boxed{\forall \varepsilon > 0 \ \exists N_a(\varepsilon) \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N} \quad n > N_a(\varepsilon) \Rightarrow |a_n - K| < \varepsilon}$$

G2 is

$$\boxed{\forall \varepsilon > 0 \ \exists N_b(\varepsilon) \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N} \quad n > N_b(\varepsilon) \Rightarrow |b_n - L| < \varepsilon}$$

The red box in this theorem is

$$\boxed{\forall \varepsilon > 0 \quad \exists N_c(\varepsilon) \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N} \quad n > N_c(\varepsilon) \Rightarrow |c_n - (K+L)| < \varepsilon}$$

$c_n = a_n + b_n$ (G3)

Let $\varepsilon > 0$ be arbitrary.

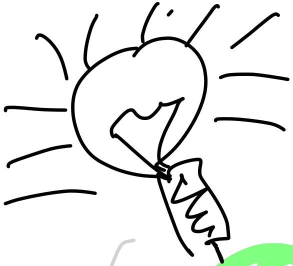
We have establish the connection between

$$\boxed{|a_n + b_n - (K+L)| < \varepsilon}$$

and the green stuff in G1 - G2

the core is $\boxed{|a_n - K| < \varepsilon}$

the core is $\boxed{|b_n - L| < \varepsilon}$



simplify triangle ineq.

BK

$$\left|\underbrace{a_n+b_n}_{c_n}-K-L\right| = |a_n-K+b_n-L| \le |a_n-K|+|b_n-L|$$

B/N

Can I make $|a_n-K|+|b_n-L|<\varepsilon$?

using the green boxes G1 & G2

The big idea is: make $|a_n-K|<\frac{\varepsilon}{2}$

and make $|b_n-L|<\varepsilon/2$.

By G1 $\boxed{\forall n \in \mathbb{N} \quad n > N_a\left(\frac{\varepsilon}{2}\right) \Rightarrow |a_n - K| < \frac{\varepsilon}{2}}$

G2 $\boxed{\forall n \in \mathbb{N} \quad n > N_b\left(\frac{\varepsilon}{2}\right) \Rightarrow |b_n - L| < \frac{\varepsilon}{2}}$

Can you define $N_c(\varepsilon)$?

Let $\varepsilon > 0$ be arbitrary.

$$N_c(\varepsilon) = \max\left\{ N_a\left(\frac{\varepsilon}{2}\right), N_b\left(\frac{\varepsilon}{2}\right) \right\}$$

Now we can prove

$$\boxed{\forall n \in \mathbb{N} \quad n > N_c(\varepsilon)} \Rightarrow \boxed{\left|(a_n + b_n) - (K + L)\right| < \varepsilon}$$

PROOF

Assume $n \in \mathbb{N}$ and $n > N_c(\varepsilon) = \max\{N_a(\frac{\varepsilon}{2}), N_b(\frac{\varepsilon}{2})\}$

By the def of max $N_c(\varepsilon) \geq N_a(\frac{\varepsilon}{2})$

and $N_c(\varepsilon) \geq N_b(\frac{\varepsilon}{2})$

Therefore by (BK) $n > N_c(\varepsilon)$ implies $n > N_a(\frac{\varepsilon}{2})$

and $n > N_b(\frac{\varepsilon}{2})$

By G1 I deduce $|a_n - K| < \frac{\varepsilon}{2}$

By G2 I deduce that $|b_n - L| < \frac{\varepsilon}{2}$

Therefore $\boxed{|a_n - K| + |b_n - L| < \varepsilon}$.

Since $\boxed{|c_n - (K+L)| \leq |a_n - K| + |b_n - L|}$

I conclude $\boxed{|c_n - (K+L)| < \varepsilon}$

I greenified the red box!

THAT IS A PROOF.

Theorem Let $K, L \in \mathbb{R}$, $a: \mathbb{N} \to \mathbb{R}$ and $b: \mathbb{N} \to \mathbb{R}$ sequences. Assume

G1 $\boxed{\lim_{n \to +\infty} a_n = K}$, $\boxed{\lim_{n \to \infty} b_n = L}$ and G2

$\exists n_0 \in \mathbb{N}$ such that G3

$\forall n \in \mathbb{N} \quad n \geq n_0 \Rightarrow a_n \leq b_n$.

then $\boxed{K \leq L}$.

Proof. Remember (BBB principle)

$u, v, w \in \mathbb{R}$ $w > 0$

$$\underbrace{|u - v| < w}_{\text{Bell}} \quad \text{2 vie}$$

some place

$$\Updownarrow$$

$$\underbrace{v - w}_{\text{Brat}} < u < \underbrace{v + w}_{\text{Blaine}}$$

G1 $\forall \varepsilon > 0 \; \exists N_a(\varepsilon) \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$ $n > N_a(\varepsilon) \Rightarrow K - \varepsilon < a_n < K + \varepsilon$

G2 $\forall \varepsilon > 0 \; \exists N_b(\varepsilon) \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$ $n > N_b(\varepsilon) \Rightarrow L - \varepsilon < b_n < L + \varepsilon$

G3 $\boxed{\forall n \in \mathbb{N}\quad n > n_0 \Rightarrow \quad a_n \le b_n}$

Let $\varepsilon > 0$ be arbitrary. I can achive all three to be true:

$$\boxed{\begin{gathered} K-\varepsilon < a_n < K+\varepsilon \\ L-\varepsilon < b_n < L+\varepsilon \\ a_n \le b_n \end{gathered}}$$

Just take

$$\boxed{n > \max\left\{N_a(\varepsilon), N_b(\varepsilon), n_0\right\}}$$

Then all

$$K-\varepsilon < a_n < K+\varepsilon$$
$$L-\varepsilon < b_n < L+\varepsilon$$
$$a_n \leq b_n$$

$$\boxed{K-\varepsilon < a_n \le b_n < L+\varepsilon}$$

$$\boxed{K-\varepsilon < L+\varepsilon}$$

$$\boxed{K-L < 2\varepsilon}$$

$\forall \varepsilon > 0$ is true

this implies $\boxed{K-L \le 0}$

We proved $\forall \varepsilon > 0 \ K-L < 2\varepsilon$.

I claim

$$\forall \varepsilon > 0 \ K-L < 2\varepsilon \Rightarrow K-L \leq 0$$

this is an implication. It is easier to prove the contrapositive -

$$K-L > 0 \Rightarrow \exists \varepsilon > 0 \text{ s.t. } K-L \geq 2\varepsilon.$$

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Just take $\varepsilon = \dfrac{K-L}{4}$ $\quad K-L \geq \dfrac{K-L}{2}$ TRUE

We could have formulated a Lemma.

LEMMA: Let $\alpha \in \mathbb{R}$. The following implication holds

$$\forall_{v>0}\ \alpha < v \Rightarrow \alpha \leq 0.$$

PROOF: We will prove the contrapositive.

$$\boxed{\alpha > 0} \Rightarrow \boxed{\exists\, v > 0 \text{ s.t. } \alpha \geq v}.$$

Assume $\boxed{\alpha > 0}$: Then $\frac{\alpha}{2} > 0$ and $\alpha \geq \frac{\alpha}{2}$.

Therefore we can take $v = \frac{\alpha}{2}$. This proves $\square$