Monotone

Convergence Theorem

-Preliminaries-

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$\frac{1}{0!} = 1, \quad \frac{1}{1!}, \quad \frac{1}{2!}, \quad \frac{1}{3!}, \quad \frac{1}{4!}, \dots, \frac{1}{n!}, \dots$ sequence of terms

$$S_0 = \frac{1}{0!} = 1, \quad \forall n \in \mathbb{N} \quad S_n = S_{n-1} + \frac{1}{n!}$$

$$S_1 = \frac{1}{0!} + \frac{1}{1!} = 1 + 1 = 2$$

$$S_2 = 2 + \frac{1}{2!}$$

$$S_3 = 2 + \frac{1}{2!} + \frac{1}{3!}$$

$$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} = \sum_{k=0}^{n} \frac{1}{k!}.$$

Does this sequence converge?

We seek properties of the sequence that would imply convergence.

It turns out that those magic properties are <u>boundedness</u> and <u>monotonicity</u>.

First boundedness. A sequence $s:\mathbb{N}\to\mathbb{R}$ is said to be bounded above if

$$\exists M\in\mathbb{R} \text{ s.t. } \forall n\in\mathbb{N} \quad s_n \le M$$

this M is called an upper bound for s.

A sequence $s:\mathbb{N}\to\mathbb{R}$ is said to be bounded below if

$$\exists m\in\mathbb{R} \text{ s.t. } \forall n\in\mathbb{N} \quad m \le s_n$$

such m is called a lower bound for $s:\mathbb{N}\to\mathbb{R}$

A sequence is bounded if its bounded above and bounded below, that is

$$\exists m, M\in\mathbb{R} \text{ s.t. } \forall n\in\mathbb{N} \quad m \le s_n \le M.$$

In the above example $S_n = \sum_{k=0}^{n} \frac{1}{k!}$ we have $S_n > 0 \ \forall n \in \mathbb{N}$

$\frac{1}{k!} > 0 \ \forall k \in \mathbb{N}$

The sequence $\{S_n\}$ is bdd below.

This sequence is bdd above! Here is a **PROOF**!

$k > 1$, $k \in \mathbb{N}$

$$\boxed{\frac{1}{k!}} = \frac{1}{1\cdot 2 \cdot \ldots \cdot k} \underset{\text{Pizza-Party}}{\boxed{\leq}} \frac{1}{(k-1)k} \underset{\text{partial fractions}}{=} \boxed{\frac{1}{k-1} - \frac{1}{k}}$$

$\boxed{n \in \mathbb{N}, n > 1}$

$$\boxed{S_n =} 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} + \frac{1}{n!}$$

$k=2$, $k=3$, $k=4$, $k=n-1$, $k=n$

$$\boxed{\leq} 1 + 1 + \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) \cancel{+} \cancel{\cdots} \cancel{+} \left(\cancel{\frac{1}{n-2}} - \cancel{\frac{1}{n-1}}\right) + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n}\right)$$

$$= 1 + 1 + 1 - \frac{1}{n} = 3 - \frac{1}{n} \boxed{\leq 3}$$

Theorem If a sequence converges, then it is bounded.

In other words, every convergent sequence is bdd.

Proof. Assume that a sequence $a: \mathbb{N} \to \mathbb{R}$ converges to $L \in \mathbb{R}$.

English, translate to mathish

$\forall \varepsilon > 0 \quad \exists N(\varepsilon) \in \mathbb{R}$ s.t.

the power here is $\forall$

$\forall n \in \mathbb{N} \quad n > N(\varepsilon) \Rightarrow \underbrace{L - \varepsilon < a_n < L + \varepsilon}_{\text{somewhat similar expressions}}$

What is red?

$\exists\, m, M \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N} \quad m \leq a_n \leq M$

Make red objects from green; that is what a proof is.

Let $\varepsilon = 1$. From BK we know $1 > 0$. Then $\square$ reads

$$\exists N(1) \in \mathbb{R} \text{ s.t. } \underline{\forall n \in \mathbb{N} \; n > N(1)} \Rightarrow L - 1 < a_n < L + 1$$

$\mathbb{N} = \{1, 2, 3, \ldots\}$

<u>Case 1</u> $N(1) < 1$

assume

$\forall n \in \mathbb{N} \; n > N(1)$

<u>In this case</u> <u>is true</u>

We have $n \geq 1 \; \forall n \in \mathbb{N}$ $\quad 1 > N(1)$

By transitivity of order $n > N(1) \; \forall n \in \mathbb{N}$

So, in this case we have

$$\boxed{\forall n \in \mathbb{N} \quad L-1 < a_n < L+1}$$

therefore I can set $m = L-1$

and $M = L+1$

Thus $a : \mathbb{N} \to \mathbb{R}$ bounded in this case.

<u>Case 2</u>

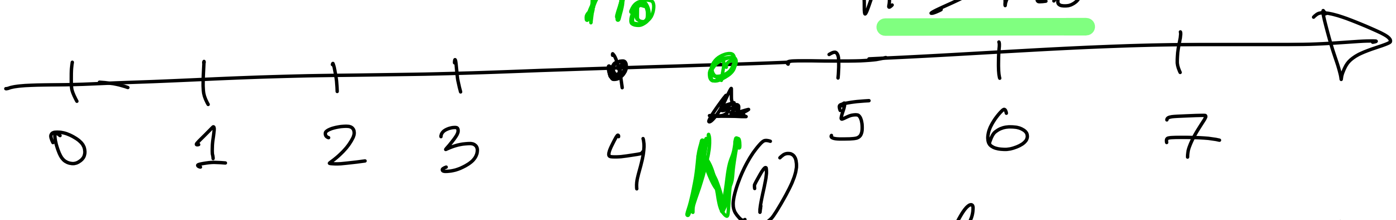
$$\boxed{N(1) \geq 1}$$ ← assume

We want: $\boxed{\forall n \in \mathbb{N} \quad m \leq a_n \leq M}$

We have $\circledast$ $\boxed{\forall n \in \mathbb{N} \quad n > N(1) \quad L-1 < a_n < L+1}$

$$\Updownarrow \ \cancel{\Downarrow}$$

$$n > n_0$$



Set $n_0 = \lfloor N(1) \rfloor$. Then $n_0 \in \mathbb{N}$

$n \in \mathbb{N}$ $n > n_0$ we have $L-1 < a_n < L+1$

$n \leq n_0$ we have no info about

finite set $\rightarrow \{a_1, a_2, a_3, \ldots, a_{n_0}\}$

$$m = \min\{a_1, a_2, \dots, a_{n_0}, L-1\}$$

$$M = \max\{a_1, a_2, \dots, a_{n_0}, L+1\}$$

Then $\forall n \in \mathbb{N} \quad m \leq a_n \leq M$

Can be proved.

Here is a proof.

Case 1. $\boxed{n \in \{1, 2, \dots, n_0\}}$.

Since $\boxed{a_n \in \{a_1, a_2, \dots, a_{n_0}, L-1\}}$, by the definition of minimum $\boxed{m \leq a_n}$.

Since $\boxed{a_n \in \{a_1, a_2, \dots, a_{n_0}, L+1\}}$, by the def. of maximum $\boxed{a_n \leq M}$.

Therefore $\boxed{\forall n \in \{1, 2, \dots, n_0\} \quad m \leq a_n \leq M}$.

Case 2. $\boxed{n \in \mathbb{N} \text{ and } n > n_0}$. Since $n_0 = \lfloor N(1) \rfloor$ we have that $\boxed{n \geq n_0 + 1 > N(1)}$.

By the green box ✱ $n \in \mathbb{N}$ and $n > N(1)$ implies $\boxed{L-1 < a_n < L+1}$. ✱✱

Since $\boxed{L-1 \in \{a_1, a_2, \dots, a_{n_0}, L-1\}}$ and $\boxed{m = \min\{a_1, \dots, a_{n_0}, L-1\}}$
we have $\boxed{m \leq L-1}$. By $(**)$ $\boxed{m < a_n}$.

Since $\boxed{L+1 \in \{a_1, a_2, \dots, a_{n_0}, L+1\}}$ and $\boxed{M = \max\{a_1, \dots, a_{n_0}, L+1\}}$
we have $\boxed{L+1 \leq M}$. By $(**)$ $\boxed{a_n \leq M}$

these two green boxes imply $\boxed{m \leq a_n \leq M}$

Hence, we proved
$n \in \mathbb{N}$ and $n > n_0$ implies $\boxed{m \leq a_n \leq M}$.

By Case 1 and Case 2 $\boxed{\forall n \in \mathbb{N} \quad m < a_n \leq M}$.