# Monotone Convergence Theorem

## Proof & Examples

# Completeness Axiom for $\mathbb{R}$

If $A, B \subseteq \mathbb{R}$, $A, B \neq \emptyset$ and $\forall a \in A \ \forall b \in B \quad a \leq b$,

then $\exists c \in \mathbb{R}$ s.t. $\forall a \in A \ \forall b \in B \quad a \leq c \leq b$

# Monotone Convergence Theorem

Every bounded monotonic sequence in $\mathbb{R}$ converges.

Proof. First translate English into Mathlish

Let $s:\mathbb{N}\to\mathbb{R}$ be a sequence in $\mathbb{R}$. Assume that $s$ is nondesresing, that is

$$\forall n\in\mathbb{N}\quad s_n\le s_{n+1}, \text{ or } s_1\le s_2\le\cdots\le s_n\le s_{n+1}\le\cdots$$

Assume also that $s$ is bounded above, that is $\exists M\in\mathbb{R}$ such that $\forall n\in\mathbb{N}\ \boxed{s_n\le M}$.

What is red? We need to prove that

$$\exists L\in\mathbb{R} \text{ s.t. } \forall\varepsilon>0\ \exists N(\varepsilon)\in\mathbb{R} \text{ s.t. } \forall n\in\mathbb{N}\quad n>N(\varepsilon)\Rightarrow |s_n-L|<\varepsilon$$

Our tool is CA. So, we need A and B.

Set $A=\{s_n : n\in\mathbb{N}\}$ (range of $s$)

$B=\{b\in\mathbb{R} : \forall n\in\mathbb{N}\ s_n\le b\}$

$M \in B$, therefore $B \neq \emptyset$.

$s_1 \in A$, so $A \neq \emptyset$.

How $B$ is defined we have

$\forall a \in A \ \forall b \in B \quad a \leq b$.

To prove this take $a \in A$ and $b \in B$ arbitrary.

$$\left.\begin{array}{l} a \in A \Rightarrow \exists k \in \mathbb{N} \text{ s.t. } a = s_k \\ b \in B \Rightarrow \forall n \in \mathbb{N} \ s_n \leq b \end{array}\right\} \Rightarrow a \leq b.$$

By CA $\exists \, c \in \mathbb{R}$ such that

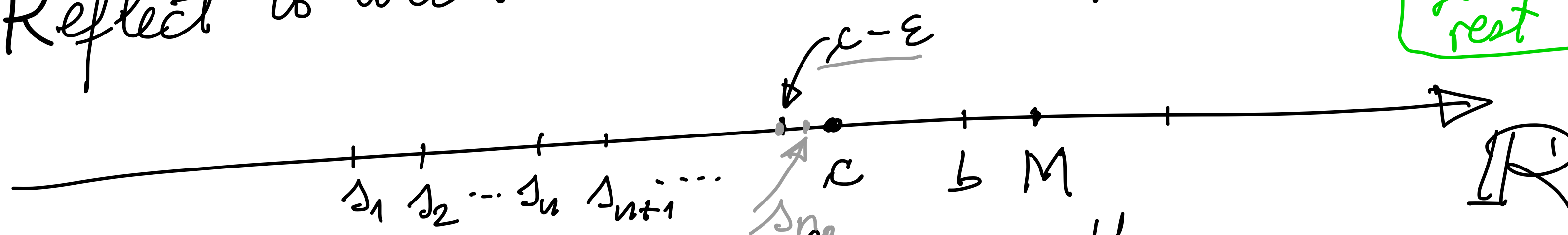
$\forall a \in A \ \forall b \in B \quad a \leq c \leq b$.

Since $A$ is the range of $s$ this means

$\forall n \in \mathbb{N}\ \forall b \in B\ \boxed{s_n \le c \le b}$.

The key for the rest

Reflect to what we have in a picture:




The key here is that below $c$ there are no elements of $B$.

Now we can prove $\lim\limits_{n \to +\infty} s_n = c$.

Let $\varepsilon > 0$ be arbitrary. Then $c - \varepsilon < c$.

Since $\forall b \in B$ we have $c \le b$, we conclude that $c-\varepsilon \notin B$. Recall the def of $B$:

$$B = \{ b \in \mathbb{R} : \forall n \in \mathbb{N} \ s_n \le b \}.$$

Thus $c-\varepsilon \notin B$ implies $\boxed{\forall n \in \mathbb{N} \ s_n \le c-\varepsilon}$ NOT TRUE

the negation is TRUE $\boxed{\exists n_0 \in \mathbb{N} \text{ s.t. } s_{n_0} > c-\varepsilon}$

But now recall that $s$ is nondecreasing, so

for all $n \in \mathbb{N}$ $n \ge n_0$ implies $s_n \ge s_{n_0}$

Therefore $\forall n \in \mathbb{N} \ n \ge n_0 \Rightarrow s_n > c-\varepsilon$.

But, by $(*)$ $\forall n \in \mathbb{N} \ s_n \le c$.

Therefore $\forall n \in \mathbb{N} \ n \ge n_0 \Rightarrow c-\varepsilon < s_n \le c$.

Therefore $\forall n \in \mathbb{N}$ $n \geq n_0 \Rightarrow |s_n - c| < \varepsilon$.

Hence, we can set $N(\varepsilon) = n_0$.

This proves $\lim_{n \to +\infty} s_n = c$.

So $L = c$. This completes the proof.

QED

**Example** Consider the sequence

$$S_n = \sum_{k=0}^{n} \frac{1}{k!} \quad \forall n \in \mathbb{N}.$$

We proved earlier $S_n \leq 3 \ \forall n \in \mathbb{N}$.

$S_1 = \frac{1}{0!} + \frac{1}{1!} = 2 \leq 3$ ✓

$S_2 = 2 + \frac{1}{2!} = \frac{5}{2} \leq 3$ ✓

Pizza-Party

$n > 2$ $\quad S_n = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \dots + \frac{1}{n!} \leq$

$= 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{(n-1)n} =$

$= 1 + 1 + \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \dots + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n}\right)$

$= 1 + 1 + 1 - \frac{1}{n} \leq 3$

$S_{n+1} - S_n = \frac{1}{(n+1)!} > 0$, so $\underline{\underline{S_n \leq S_{n+1}}}$

$\forall n \in \mathbb{N}$

Thus the sequence $\{S_n\}$ is bdd and monotonic.

Thus it converges!

We define the famous number $e$ to be the limit:

$$e = \lim_{n \to +\infty} S_n$$

Example Consider recursively def.

Sequence :

$$x_1 = 2, \quad x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \quad \forall n \in \mathbb{N}$$

$$x_2 = \frac{2}{2} + \frac{1}{2} = \frac{3}{2} < 2$$

$$x_1 > x_2 > x_3 > \dots$$

$$x_3 = \frac{3}{4} + \frac{2}{3} = \frac{17}{12} < \frac{18}{12} = \frac{3}{2} = x_2$$

$x_1 > 0$, $x_2 = \frac{x_1}{2} + \frac{1}{x_1} > 0$, by recursion $\forall n \in \mathbb{N} \; x_n > 0$

We will prove that this sequence is non increasing.

$$n > 1 \quad (x_n)^2 = \left(\frac{x_{n-1}}{2} + \frac{1}{x_{n-1}}\right)^2 = \frac{(x_{n-1})^2}{4} + \underbrace{2\frac{\cancel{x_{n-1}}}{\cancel{2}} \cdot \frac{1}{\cancel{x_{n-1}}}}_{1} + \frac{1}{(x_{n-1})^2}$$

$$1 = 2 - 1$$

$$= 2 + \left(\frac{x_{n-1}}{2}\right)^2 - 1 + \left(\frac{1}{x_{n-1}}\right)^2 = 2 + \left(\frac{x_{n-1}}{2} - \frac{1}{x_{n-1}}\right)^2$$

$\forall n \in \mathbb{N} \quad (x_n)^2 > 2 \ldots$

$\overset{?}{\Longrightarrow} x_{n+1} > x_n$

divide by $x_n$, $x_n > 0$

$$x_n > \frac{2}{x_n}$$

divide by $2 > 0$

$$\frac{x_n}{2} > \frac{1}{x_n}$$

add $\frac{x_n}{2}$

$$\frac{x_n}{2} + \frac{x_n}{2} > \frac{x_n}{2} + \frac{1}{x_n}$$

$$\boxed{x_n > x_{n+1}}$$

Proved!