Consequences of the MCT

Definition of Infinite Series

1. Definition of e

$\forall n \in \mathbb{N}$ we define $S_n = \sum_{k=0}^{n} \frac{1}{k!}$.

→ we prove $\forall n \in \mathbb{N}$ $S_n \leq S_{n+1}$

→ we prove $\forall n \in \mathbb{N}$ $S_n \leq 3$

MTC $\Rightarrow$ $\{S_n\}$ converges. Set

$$e = \lim_{n \to +\infty} S_n$$

e is irrational (English) For a proof see the short note on the class website.

$\forall p \in \mathbb{Z}$ $\forall n \in \mathbb{N}$ $e \neq \frac{p}{q}$

Here I explain the BINOMIAL THEOREM, needed in my short paper about e, see the website

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3$$

$$(a+b)^{4} = a^4 + ?\,a^3b^1 + ?\,a^2b^2 + \cdot \quad b^4$$

$$\underbrace{(a+b)}_{\text{binomial}}{}^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

That is about e.

Now $\sqrt{2}$: ?

$\sqrt{2}$ is irrational !

$\forall p \in \mathbb{Z} \; \forall q \in \mathbb{N} \quad \frac{p^2}{q^2} \neq 2$:

Have you seen this proof ?

Look for it, if not, review it if yes.

But, PROVE

$\exists \alpha \in \mathbb{R}$ s.t. $\alpha^2 = 2$.

(We stated above $\forall \alpha \in \mathbb{Q}\ \alpha^2 \neq 2$.)

We can now PROVE $\exists \alpha \in \mathbb{R}$ s.t. $\alpha^2 = 2$

We studied:

$$x_1 = 2, \quad x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \quad \forall n \in \mathbb{N}$$

We proved $\forall n \in \mathbb{N} \quad x_n > 0$

We proved $\forall n \in \mathbb{N} \quad x_{n+1} \leq x_n$

By MTC $\exists L \in \mathbb{R}$ such that

$$\lim_{n \to \infty} x_n = L$$

We also proved $(x_n)^2 \geq 2$ $\forall n \in \mathbb{N}$

sorry $\forall$ — Sorry, I forgot the quantifier!

You are very important

Now ALGEBRA of Limits:

$a_n \to K \quad b_n \to L$

$\lim_{n\to+\infty} a_n \cdot b_n = K \cdot L$

$X_n \to L$

$(X_n)^2 \to L^2$

$(X_n)^2 \geq 2 \ \forall n \in \mathbb{N} \Rightarrow L^2 \geq 2$

$X_n > 0 \ \forall n \in \mathbb{N} \Rightarrow \underbrace{L \geq 0}$

$\Rightarrow L > 0$

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \quad \forall n \in \mathbb{N}$$

$$\lim_{n\to\infty} \frac{x_n}{2} = \frac{L}{2}, \quad \text{Alg. of limits} \quad \lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{L}$$

$L > 0$

Alg. of Limits: $\lim_{n\to+\infty} \left(\frac{x_n}{2} + \frac{1}{x_n} \right) = \frac{L}{2} + \frac{1}{L}$

Alg. of Limits: $\lim_{n\to+\infty} x_{n+1} = L$

Since $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \Rightarrow L = \frac{L}{2} + \frac{1}{L}$

$\forall n \in \mathbb{N}$

$$\Downarrow$$

$$\frac{L}{2} = \frac{1}{L}$$

$$\Downarrow$$

$$L^2 = 2$$

$$L > 0$$

Here we proved

$\exists L \in \mathbb{R}$ s.t. $L > 0$ and $L^2 = 2$