Infinite Series

- Geometric Series

- Harmonic Series
- Telescopic Series

Given a sequence $\{a_n\}$ (or $a: \mathbb{N} \to \mathbb{R}$) terms of a series

the expression $a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k$

is called an infinite series.

The basic **question** is whether $\sum_{k=1}^{\infty} a_k$ converges or diverges. What this asks is whether the sequence of partial sums

$$\underbrace{S_n}_{n\text{-th partial sum}} = \sum_{k=1}^{n} a_k$$

converges or diverges.

The most important example is the GOMETRIC SERIES

The terms of a geometric series are ($a \in \mathbb{R}$, $r \in \mathbb{R}$)

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}, ar^n, ar^{n+1}, \ldots$$

$r^0 = 1$

G.S is $\displaystyle\sum_{k=0}^{\infty} ar^k$

We proved on Friday

$$S_n = \sum_{k=0}^{n} ar^k = a\,\frac{1-r^{n+1}}{1-r}$$

(provided $r \neq 1$)

Therefore $\displaystyle\lim_{n \to \infty} S_n = \frac{a}{1-r}$, provided $|r| < 1$

If $|r| \geq 1$, $a \neq 0$, then $\displaystyle\sum_{k=0}^{\infty} ar^k$

$$\boxed{0.\underbrace{d_1}_{\text{digit}}\underbrace{d_2}_{\text{digit}}d_3d_4\cdots d_n\cdots}$$ where $d_n \in \{0,1,\dots,9\}$

$\longrightarrow$ The meaning of the decimal expansion is in fact the following INFINITE SERIES:

$$\sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

(this is NOT a geometric series)

A big question is: Why does this series converge? The answer is : The Monotone convergence Thm!

Set $S_n = \sum_{k=1}^{n} \frac{d_k}{10^k}$ $\forall n \in \mathbb{N}$

then (1) S_n is Non decreasing !

$\forall n \in \mathbb{N}$ $S_{n+1} - S_n = \frac{d_{n+1}}{10^{n+1}} \geq 0$, $d_{n+1} \in \{0, 1, \dots, 9\}$

therefore $S_n \leq S_{n+1}$ $\forall n \in \mathbb{N}$

(2) The sequence $\{S_n\}$ is bdd above by 1

$\forall k \in \mathbb{N}$ $\frac{d_k}{10^k} \leq \frac{9}{10^k}$. Therefore $S_n \leq \sum_{k=1}^{n} \frac{9}{10^k}$

Now calculate $\displaystyle\sum_{k=1}^{n} \frac{9}{10^k} = \frac{9}{10}\sum_{k=1}^{n} \frac{1}{10^{k-1}} =$

$$= \frac{9}{10}\sum_{k=0}^{n-1} \frac{1}{10^k} = \frac{9}{10}\,\frac{1-\frac{1}{10^n}}{1-\frac{1}{10}}$$

$$\le \frac{9}{10}\,\frac{1}{\frac{9}{10}} = 1$$

thus $\boxed{S_n = \displaystyle\sum_{k=1}^{n} \frac{d_k}{10^k} \le 1}$

thus $\displaystyle\sum_{n=1}^{\infty} \frac{d_n}{10^n}$ converges to $\underline{\underline{0.d_1d_2d_3\ldots d_n\ldots}}$

$\pi \approx 3.1415926536$ ← approximation, rounded

$3.14 < \pi < 3.142$

The famous rational approximation of π is $\frac{22}{7}$

(next one is $\frac{355}{113}$)

Exercise: Find the rational expression for

$3.14141414\ldots = 3.\overline{14} = \frac{p}{q}, \; p, q \in \mathbb{N}.$

Solve using Geometric Series.

$$0.\overline{14} = \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{4}{10^4} + \cdots = \frac{14}{10^2} + \frac{14}{10^4} + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{14}{(10^2)^k} = \sum_{K=1}^{\infty} 14 \left(\frac{1}{10^2}\right)^k = \sum_{k=1}^{\infty} \frac{14}{10^2}\left(\frac{1}{10^2}\right)^{k-1}$$

$$= \sum_{n=0}^{\infty} \frac{14}{10^2}\left(\frac{1}{10^2}\right)^n \overset{GS.}{=}$$

Based on $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $|r|<1$

we conclude

$$0.\overline{14} = \frac{14}{10^2} \frac{1}{1-\frac{1}{10^2}} = \frac{14}{10^2} \frac{1}{\frac{99}{100}} = \frac{14}{99}$$

add 3

$$3+\frac{14}{99} = \frac{297+14}{99} = \frac{311}{99}$$

Is this better than $\frac{22}{7}$?

Sometimes it is taught

$$0.\overline{d_1 d_2 \ldots d_n} = \frac{d_1 d_2 \ldots d_n}{99 \ldots 9}$$

$\boxed{\text{Why?}}$ G.S.

There is no math without Why?

Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Theorem The Harmonic Series diverges.

Study $H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}, n \in \mathbb{N}$

Harmonic numbers

① Since $H_{n+1} - H_n = \frac{1}{n+1} > 0$

The sequence of Harmonic numbers is increasing.

(2) The sequence of Harmonic numbers is unbounded.

<u>Proof</u>. (Pizza-Party)

$$H_1 = 1$$

$$H_2 = 1+\frac{1}{2} = \frac{3}{2}$$

$$H_3$$

m=2 $\quad H_4 = \left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right) \geq \frac{3}{2}+\frac{1}{4}+\frac{1}{4} = \frac{3}{2}+\frac{1}{2} = \frac{4}{2}$

$\vdots$

m=3 $\quad H_8 = \underbrace{\left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)}_{H_4}+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) \geq \frac{4}{2}+\frac{4}{8} = \frac{5}{2}$

$m \geq 1 \quad H_{2^m} = \left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\cdots+\left(\frac{1}{2^{m-1}+1}+\frac{1}{2^{m-1}+2}+\cdots+\frac{1}{2^{m-1}+2^{m-1}}\right)$

$2^{m-1}+2^{m-1} = 2^m$

$$\geq \left(1+\frac{1}{2}\right)+\underbrace{\frac{1}{2}+\cdots+\frac{1}{2}}_{m-1 \text{ terms}} = \frac{3}{2}+\frac{m-1}{2} = \frac{m+2}{2}$$

$2^2 \quad\quad 2^m$

The sequence of H_n diverges since it is unbounded.

Above we proved that $H_{2^m} \geq \frac{m+2}{2} \ \forall m \in \mathbb{N}$

Since $1=2^0, 2^1, 4=2^2, 8=2^3, 16=2^4, \dots, 2^m, \dots$ is an increasing sequence of positive integers the following statement holds:

$$\forall n \in \mathbb{N} \quad \exists m \in \mathbb{N} \cup \{0\} \text{ s.t. } 2^m \leq n < 2^{m+1}$$

Since ln is an increasing function we have

$$2^m \leq n < 2^{m+1} \iff \ln(2^m) \leq \ln n < \ln(2^{m+1})$$

$$\iff m \ln 2 \leq \ln n < (m+1) \ln 2$$

$$(\text{since } \ln 2 > 0) \iff m \leq \frac{\ln n}{\ln 2} < m+1$$

Hence $2^m \leq n < 2^{m+1} \iff m = \left\lfloor \frac{\ln n}{\ln 2} \right\rfloor$

Since $\{H_n\}$ is an increasing sequence, we have

$$H_n \geq H_{2^m} \quad \text{with} \quad m = \left\lfloor \frac{\ln n}{\ln 2} \right\rfloor$$

Since we proved $H_{2^m} \geq \frac{m+2}{2}$ we have that

$$\forall n \in \mathbb{N} \quad H_n \geq \frac{1}{2}\left(\left\lfloor \frac{\ln n}{2} \right\rfloor + 2\right).$$

This inequality shold allow us to answer the following question: Given $M \in \mathbb{R}_+ = (0, +\infty)$, for which $n \in \mathbb{N}$ we have $H_n \geq M$?

The easiest way to solve this is to solve for $m \in \mathbb{N} \cup \{0\}$ such that $\frac{m+2}{2} = \lceil M \rceil \geq M$

Then $m = 2\lceil M \rceil - 2 = 2(\lceil M \rceil - 1)$.

Then we know that

$$H_{2^{2(\lceil M \rceil - 1)}} \geq \frac{2(\lceil M \rceil - 1) + 2}{2} = \lceil M \rceil .$$

In general, for $\forall n \in \mathbb{N}$ such that $n \geq 2^{2(\lceil M \rceil - 1)}$ we have

$$H_n \geq \lceil M \rceil \geq M .$$

This shows that the Harmonic Sequence is not bounded.

Exercise Prove that the infinite series $\sum_{n=1}^{\infty} \frac{1}{H_n}$ diverges.