Basic Properties of Convergent Series

The Big One:

$$\boxed{\text{If } \sum_{n=1}^{\infty} a_n \text{ converges, then } \lim_{n\to+\infty} a_n = 0.}$$

But remember, the converse is NOT true: the example is the Harmonic Series:

$$\sum_{n=1}^{\infty} \frac{1}{n}$$ diverges but $\lim_{n \to +\infty} \frac{1}{n} = 0$.

The simple story is that one can operate with convergent infinite series as if they were finite sums.

Assume that

$$\sum_{n=1}^{\infty} a_n = A \qquad \sum_{n=1}^{\infty} b_n = B$$

that is these are $\boxed{\text{convergent}}$ series. $c \in \mathbb{R}$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$$

$$\sum_{n=1}^{\infty} \frac{2}{n^2} = \pi^2/3$$

Then $$\underbrace{\sum_{n=1}^{\infty} c\,a_n}_{\text{converges}} = cA$$

$$\underbrace{\sum_{n=1}^{\infty} (a_n + b_n)}_{\text{converges}} = A + B$$

$$\overbrace{\sum_{n=1}^{\infty} (a_n - b_n)} = A - B$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{1}{2^n}\right)=\frac{\pi^2}{6}-1$$

the series $\sum\limits_{1}^{\infty}\frac{1}{n^2}$ converges and $=\pi^2/6$

the series $\sum\limits_{n=1}^{\infty}\frac{1}{2^n}$ converges $=1$

therefore $\sum\left(\frac{1}{n^2}-\frac{1}{2^n}\right)$ converges, and its sum is $\frac{\pi^2}{6}-1$.

But how are we to establish convergence of an infinite series?

We need TOOLS, here are the TOOLS: (they are called the COMPARISON TEST

All series in comparison TESTS must have positive terms. In general, with infinite series, there is a HUGE difference between all positive terms and mixed positive/neg. terms. For example the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} =$

$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ called alternating series

Here we assume all terms are positive.

How did we prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$ CONVERGS

How did we prove it? Yes, we used the telescoping idea, but towards which GOAL?

Towards:

Monotone Convergence Thm

How? What do we need to prove?

We need to prove that the partial sequence of sums

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2} \text{ converges.}$$

$$\left(S_1 = 1,\ S_2 = 1 + \frac{1}{4} = 5/4,\ S_3 = 5/4 + \frac{1}{9} = \frac{46}{36}, \dots \right)$$

converges. We see that

$$S_1 < S_2 < S_3 < \dots < S_n < S_{n+1}$$

$$S_{n+1} - S_n = \frac{1}{(n+1)^2} > 0$$

Increasing!

We need more to establish convergence.

$$S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} \le$$

pizza-party

$n > 1$

$$\le 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{(n-1)n}$$

$$= 1 + \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \cancel{\dots} + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n}\right)$$

$$= 1 + 1 - \frac{1}{n} \le 2$$

I prooved $S_n \le 2 \ \forall n \in \mathbb{N}$.

Only NOW I can state that

MCT $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

Here we did a comparison between two series

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{1}{(n-1)n}$$

$$\frac{1}{n^2} \leq \frac{1}{(n-1)n}$$

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$$\sum_{n=2}^{\infty} \frac{1}{(n-1)n} \text{ Converges to } 1$$

$$S_n = \sum_{k=2}^{n} \frac{1}{(k-1)k} = 1 - \frac{1}{n}$$

telescope

$$\lim_{n \to +\infty} S_n = 1$$

This resoning holds in general:

DIRECT COMPARISON TEST:

If $a_n, b_n > 0$ $\forall n \in \mathbb{N}$ and

$a_n \leq b_n \; \forall n \in \mathbb{N}$

and $\sum_{n=1}^{\infty} b_n$ converges, then

$$\sum_{n=1}^{\infty} a_n$$ also converges!