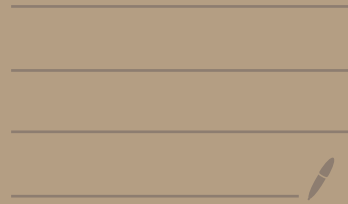


Basic Properties of Convergent Series



The Big One:

If $\sum_{n=1}^{\infty} a_n$ converges, then

$$\lim_{n \rightarrow +\infty} a_n = 0.$$

But remember, the converse is NOT true: the example is the Harmonic Series:

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

The simple story is that one can operate with convergent infinite series as if they were finite sums.

Assume that

$$\sum_{n=1}^{\infty} a_n = A \quad \sum_{n=1}^{\infty} b_n = B$$

that is these are **convergent** series. $c \in \mathbb{R}$

then

$$\underbrace{\sum_{n=1}^{\infty} ca_n}_{\text{converges}} = cA$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$
$$\sum_{n=1}^{\infty} \frac{2}{n^2} = \frac{\pi^2}{3}$$

$$\underbrace{\sum_{n=1}^{\infty} (a_n + b_n)}_{\text{converges}} = A + B$$

$$\underbrace{\sum_{n=1}^{\infty} (a_n - b_n)}_{\text{converges}} = A - B$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{2^n} \right) = \frac{\pi^2}{6} - 1$$

the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and $= \pi^2/6$

the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges $= 1$

therefore $\sum \left(\frac{1}{n^2} - \frac{1}{2^n} \right)$ converges, and
its sum is $\frac{\pi^2}{6} - 1$.

But how are we to establish
convergence of an infinite series?

We need TOOLS, here are the
TOOLS : (they are called
the COMPARISON TEST

All series in comparison TESTS must have
positive terms. In general, with infinite
series, there is a HUGE difference between
all positive terms and mixed positive/neg.
terms. For example the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} =$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

alternating
called series

Here we assume all terms are positive.

How did we prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

CONVERGS

How did we prove it? Yes, we used the telescoping idea, but towards which

GOAL? Towards:

Monotone Convergence Theorem

How? \uparrow What do we need to prove?

We need to prove that the partial sums $S_n = \sum_{k=1}^n \frac{1}{k^2}$ converges.

Sequence of

$$(S_1 = 1, S_2 = 1 + \frac{1}{4} = \frac{5}{4}, S_3 = \frac{5}{4} + \frac{1}{9} = \frac{46}{36}, \dots)$$

converges. We see that

$$S_1 < S_2 < S_3 < \dots < S_n < S_{n+1}$$
$$S_{n+1} - S_n = \frac{1}{(n+1)^2} > 0$$

Increasing \downarrow

We need more to establish convergence.

$$S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} \leq \text{pizza-party}$$

$(n > 1)$

$$\leq 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{(n-1)n}$$

$$= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$= 1 - \frac{1}{n} \leq 2$$

I proved $S_n \leq 2 \forall n \in \mathbb{N}$.

Only NOW I can state that
MCT $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

Here we did a comparison
between two series

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{1}{(n-1)n}$$

$$\frac{1}{n^2} \leq \frac{1}{(n-1)n} \quad (\text{Pizza-Party})$$

$\sum_{n=2}^{\infty} \frac{1}{(n-1)n}$ Converges to 1

$$S_n = \sum_{k=2}^n \frac{1}{(k-1)k} = 1 - \frac{1}{n}$$

↑
telescope

$$\lim_{n \rightarrow +\infty} S_n = 1$$

This reasoning holds in general:

DIRECT COMPARISON TEST:

If $a_n, b_n > 0 \forall n \in \mathbb{N}$ and

$$a_n \leq b_n \quad \forall n \in \mathbb{N}$$

and $\sum_{n=1}^{\infty} b_n$ converges, then

$\sum_{n=1}^{\infty} a_n$ also converges!