Power of the Infinite Series. Proof of Eulers Identity

$\frac{72}{92} = \frac{18}{23}$ fraction in decimal number system

in hex

$18 = 1 * 16 + 2 * 16^0$ in hex 12

$23 = 1 * 16 + 7 * 16^0$ in hex 17

$\left(\frac{18}{23}\right)_{hex} = \frac{12}{17}$

$\frac{1}{2} = 0.5$ $\quad \left(\frac{1}{2}\right)_{hex} = \cancel{\frac{1}{2}}$ $\quad 0.8$

$\boxed{\begin{array}{l} \forall k \in \mathbb{N} \\ h_k \in \{0, 1, \dots, F\} \end{array}}$

$\frac{1}{2} = h_1 * \frac{1}{16}$

$\frac{1}{3} = h_1 \frac{1}{16} + h_2 \frac{1}{16^2} + h_3 \frac{1}{16^3} + \dots = \sum_{k=1}^{\infty} h_k \frac{1}{16^k}$

How can I open my mind?

$\frac{1}{7} = 0.1428\ldots$ until it repeats.

Euler's Identity

$x^2 = -1$ does not have a sol. in $\mathbb{R}$

We introduce a new "number" called imaginary unit

$i = \sqrt{-1}$

All numbers of the form $a + ib$ with $a, b \in \mathbb{R}$ are called COMLEX NUMBERS.

Notation is $\mathbb{C}$ the set of all complex numb.

Addition, mult. division is defined as

$$(a+ib)+(c+id) = a+c+i(b+d)$$

$$(a+ib)(c+id) \underset{i^2=-1}{=} (ac-bd)+i(ad+bc)$$

A remarkable fact is

$$(a+ib)\underbrace{(a-ib)}_{\text{conjugate of } a+ib} = \underbrace{a^2+b^2}_{\geq 0} \in \mathbb{R}_+ \cup \{0\}$$

$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(a+ib)(c-id)}{\underbrace{c^2+d^2}_{\in \mathbb{R}_+}}$$

What is $2^i = ? = \underbrace{\quad}_{\text{real}} + i \underbrace{\quad}_{\text{real}}$

This mistery is resolved by Euler's Identity

$$\underline{\forall x \in \mathbb{R}} \qquad e^{ix} = \underbrace{\cos x}_{\in \mathbb{R}} + i \underbrace{\sin x}_{\in \mathbb{R}}$$

angle

$$\boxed{\forall \theta \in \mathbb{R} \qquad e^{i\theta} = \cos\theta + i \sin\theta}$$

The idea is to "convert" e^x to

multiplication and addition.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$\forall x \in \mathbb{R}$

$$\boxed{e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!}}$$

$x \to \underbrace{i\theta}_{\text{replace}}$ $\notin \mathbb{R}$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^{n}}{n!}$$

$n \in \{0\} \cup \mathbb{N}$, $(i\theta)^n = i^n \theta^n$

$$i^0 = 1$$
$$i^1 = i$$
$$i^2 = -1$$
$$i^3 = i^2 \cdot i = -i$$
$$i^4 = (-i) \cdot i = -i^2 = 1$$

dichotomy

n even, $n = 2k$, $k \in \{0\} \cup \mathbb{N}$

$$i^n = i^{2k} = (i^2)^k = (-1)^k$$

i^n is REAL

n odd, $n = 2k+1$, $k \in \mathbb{N} \cup \{0\}$

$$i^n = i^{(2k+1)} = i^{2k} \cdot i = \underbrace{(-1)^k i}_{\text{imaginary number } ib,\ b \in \mathbb{R}}$$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{i^n \theta^n}{n!}$$

Separate even from odd

$$= \sum_{k=0}^{\infty} \frac{(i)^{2k} \theta^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(i)^{2k+1} \theta^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\theta^{2k} + \sum_{k=0}^{\infty} \frac{i(-1)^k}{(2k+1)!}\theta^{2k+1}$$

$$= \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\theta^{2k}}_{\text{an infinite series}} + i \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\theta^{2k+1}}_{\text{an infinite series}}$$

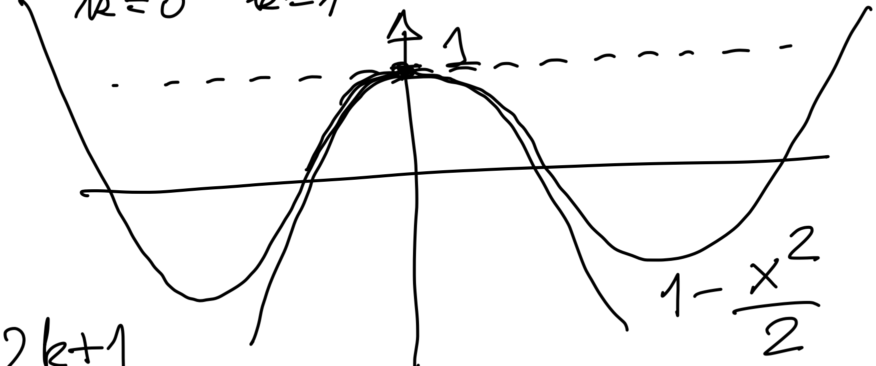
$\in \mathbb{R}$ $\leftarrow$ WHY? $\rightarrow$ $\in \mathbb{R}$

$\cos\theta$ $\qquad\qquad$ $\sin\theta$

What is the sum of

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$k=0$ $k=1$ $k=2$



$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

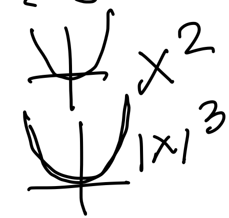
$k=0$

Lemma Let $g:\mathbb{R}\to\mathbb{R}$, CONTINUOUS.

Assume $\exists\, M\in\mathbb{R}_+$ and $m\in\{0\}\cup\mathbb{N}$

Such that $\forall x\in\mathbb{R}$

$$|g(x)|\le M|x|^m.$$



Then

$$\left|\int_0^x g(t)\,dt\right|\le \frac{M}{m+1}|x|^{m+1}$$

Proof $\to$ Tomorrow

Background Knowledge:

$$\int_0^x \sin t\,dt = (-\cos t)\Big|_0^x = 1-\cos x$$

$$\int_0^x \cos t\,dt = (\sin t)\Big|_0^x = \sin x$$