Problem 3 on Assignment 3

$$\sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

end with even

one before the last is odd

$$= 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n} \right)$$

$$= \sum_{j=1}^{n} \frac{1}{2j-1} - \sum_{j=1}^{n} \frac{1}{2j}$$

$$= \sum_{j=1}^{n} \frac{1}{2j-1} + \sum_{j=1}^{n} \frac{1}{2j} - 2 \sum_{j=1}^{n} \frac{1}{2j}$$

$$= \left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)+\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)-\sum_{j=1}^{n}\frac{2}{2j}$$

$$= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}-\left(\sum_{j=1}^{n}\frac{1}{j}\right)$$

$$= \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$$

distributive law

$$= \sum_{k=1}^{n}\frac{1}{n+k} = \sum_{k=1}^{n}\frac{1}{n}\,\frac{1}{1+\frac{k}{n}} =$$

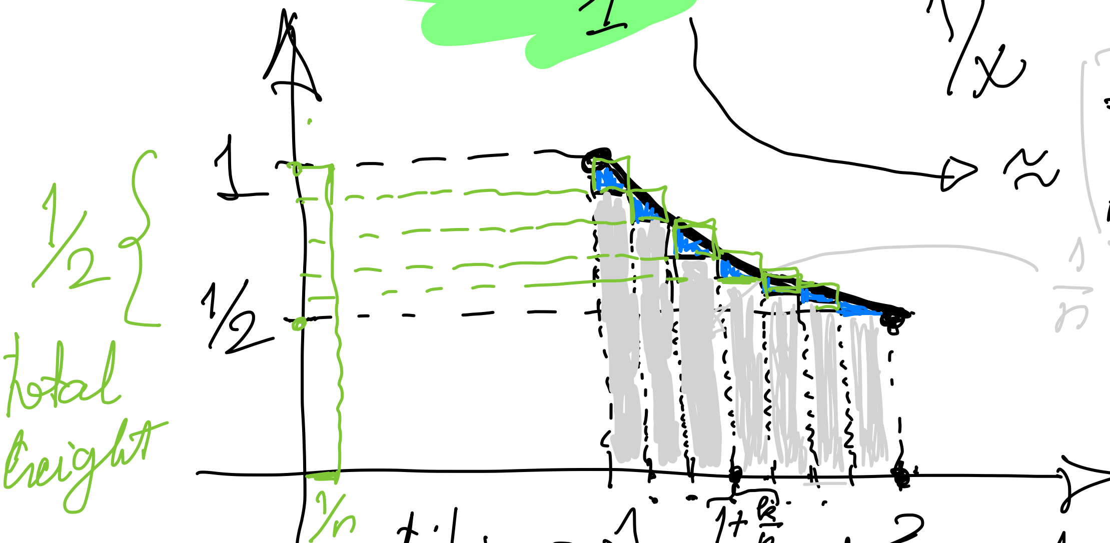
index of summation

constant

$$= \frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}$$

$$\ln 2 = \int_1^2 \frac{1}{x}\,dt$$

Background Knowledge



$1/x$

$$\approx \boxed{\sum_{k=1}^{n}\frac{1}{n}\,\frac{1}{1+\frac{k}{n}}}$$ total area of gray rectang

$\frac{1}{n}\,\frac{1}{1+\frac{k}{n}}$ one gray rectangle

partition $[1,2]$ in n parts

$1+\frac{1}{n},\ 1+\frac{2}{n},\ 1+\frac{3}{n},\ \dots,\ 1+\frac{n-1}{n},\ 2$

For large n $\underbrace{\sum_{k=1}^{n} \frac{1}{n} \frac{1}{1+\frac{k}{n}}}_{<\ln 2} \approx \ln 2$

the sum

From the picture we see

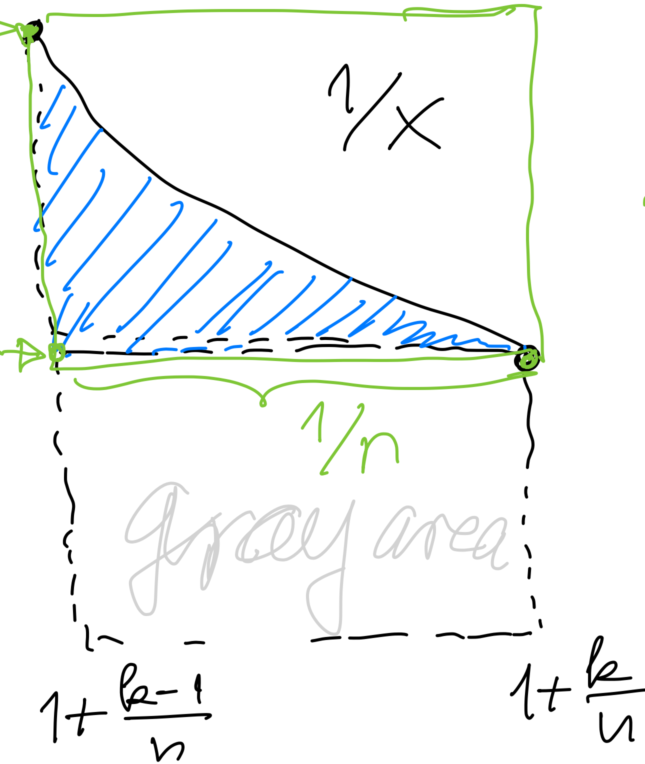
$$\underbrace{\ln 2 - \sum_{k=1}^{n} \frac{1}{n} \frac{1}{1+\frac{k}{n}}}_{\text{is colored blue}} < \frac{1}{2n}$$

Proved below from the pict.

Math 125

$$\frac{1}{1+\frac{k-1}{n}} = \frac{n}{n+k-1}$$

$$\underbrace{\frac{1}{1+\frac{k}{n}}}_{\frac{n}{n+k}}$$



green area is

the green area at point k is $\left(\frac{n}{n+k-1} - \frac{n}{n+k}\right)\frac{1}{n}$

$$= \frac{1}{n+k-1} - \frac{1}{n+k}$$

Total green area is

$$\underbrace{\sum_{k=1}^{n}\left(\frac{1}{n+k-1}-\frac{1}{n+k}\right)}_{\text{should be } \frac{1}{2n}}$$

from the picture $\left(\frac{1}{2}\right)\frac{1}{n}$

$$= \frac{1}{2n}$$

$$=\left(\frac{1}{n}-\cancel{\frac{1}{n+1}}\right)+\left(\cancel{\frac{1}{n+1}}-\cancel{\frac{1}{n+2}}\right)+\dots+\left(\cancel{\frac{1}{n+n-1}}-\frac{1}{2n}\right)$$

$$=\frac{1}{n}-\frac{1}{2n}=\frac{1}{2n}$$

Yes!

$$\underbrace{\ln 2 - \sum_{k=1}^{n} \frac{1}{n} \frac{1}{1+\frac{k}{n}}}_{\text{is colored blue}} < \frac{1}{2n}$$

This is a geometric proof that

$$\lim_{n \to +\infty} \boxed{\sum_{k=1}^{n} \frac{n}{n+k}} = \ln 2$$

$$= S_{2n}$$ ← partial sum of alternating harmonic series