Hello!

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

Thm $\forall x \in \mathbb{R}$ $|x| = \max\{x, -x\}$

Proof. Case 1 $x < 0$

$|x| = -x$

$-x > 0$

$0 < -x$

trans. Ax

$x < -x$

def. of max def

$\max\{x, -x\} = -x$

This proves $|x| = \max\{x, -x\}$ in this case.

Case 2. $x \geq 0$

(ii) $\forall x \in \mathbb{R}$ $|x| \geq 0$ Cases

(iii) $\forall x \in \mathbb{R}$ $|-x| = |x|$

$|x| = \max\{x, -x\} = \max\{-(-x), -x\}$

$= \max\{-x, -(-x)\} \underset{(i)}{=} |-x|$

(iii) $\forall x \in \mathbb{R} \quad -x \leq |x|$ and $x \leq |x|$

<u>Proof.</u> (i) $\quad |x| = \underline{\max\{x, -x\}} \geq x$

$$\geq -x$$

(iv) $\forall x, y \in \mathbb{R}$

$$|xy| = |x||y|$$

(iii) $\forall x \in \mathbb{R} \quad -x \leq |x|$ and $x \leq |x|$

Proof. (i) $\underline{|x| = \max\{x, -x\}} \geq x$

$\geq -x$

(iv) $\forall x, y \in \mathbb{R}$

$$|xy| = |x||y|$$

Do many cases

C1 $x < 0$, $y < 0$

C2 $\boxed{x < 0}$, $\boxed{y \geq 0}$ $\Rightarrow$ $\boxed{xy \leq 0} \Rightarrow |xy| = -(xy)$

$|x| = -x$, $|y| = y$ $\Rightarrow |x||y| = (-x)y = -(xy)$

C3 $x \geq 0$, $y < 0$

C4 $x \geq 0$, $y \geq 0$

(vi) $\forall x, y \in \mathbb{R}$ with $y \neq 0$ $\quad \left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.

Thm TRIANGLE INEQUALITIES

(i) $\forall a,b \in \mathbb{R}$ $\quad |a+b| \leq |a|+|b|$

(ii) $\forall x,y,z \in \mathbb{R}$ $\quad \underbrace{|x-y|}_{\text{dist.}} \leq |x-z|+|z-y|$

(iii) $\forall x,y \in \mathbb{R}$ $\quad ||x|-|y|| \leq |x-y|$ (reversed TI)

Proof (i) We use $|x| = \max\{x,-x\}$

$$\begin{array}{l} a \leq |a| \\ b \leq |b| \\ \hline a+b \leq |a|+|b| \end{array} \qquad \xrightarrow{\text{axioms}} \qquad \begin{array}{l} -a \leq |a| \\ -b \leq |b| \\ \hline -a-b \leq |a|+|b| \end{array}$$

$-(a+b) \leq |a|+|b|$

$\max\{a+b,-(a+b)\} \leq |a|+|b|$

$\parallel$

$|a+b|$