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Assignment 1

Proof of Prop 4.1:

Let $a \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then $\exists$ unique q & r s.t. that $\boxed{a = nq + r}$ and $\boxed{0 \le r \le n-1}$.

Let $S = \{x \in \mathbb{Z} : n \mid x \text{ and } x \le a\}$.

$S \subseteq \mathbb{Z}$ $(x \in \mathbb{Z})$

$S \ne \emptyset$ $(-n|a| \in S)$ → prove

S is bounded above $(x \le a)$.

Therefore, by p. 1.3.5 S has a maximum.

So, $b = \max S$ exists, with $b = \ell n$ and $\ell \in \mathbb{Z}$.

$b = \ell n$

$b + n = \ell n + n$

$= n(\ell + 1)$

Note $b + n > b$, so $b + n \notin S$.

Since $n \mid (b+n)$ (by p. 1.2.3) and $b + n \notin S$, then $b + n \not\le a$, so $b + n > a$.

Now, we have $b + n > a$ and $b \le a$. Therefore we have $b \le a < b + n$ and $0 \le a - b < n$.

Let $q = \ell$ and $r = a - b$.

$nq + r = n\ell + (a - b) = b + (a - b) = a$

so $\boxed{a = nq + r}$ and $\boxed{0 \le r \le n-1}$.

Proof of uniqueness of q and r:

Let ① $a = nq_1 + r_1$

② $a = nq_2 + r_2$.

①-②: $0 = n(q_1 - q_2) + (r_1 - r_2)$, with $-n + 1 \le r_2 - r_1 \le n - 1$

$r_2 - r_1 = n(q_1 - q_2)$ with $-n < r_2 - r_1 < n$

So $-n < n(q_1 - q_2) < n$

$-1 < q_1 - q_2 < 1$ ⟶ $q_1 - q_2 = 0$, so $\boxed{q_1 = q_2}$.

Now, since $q_1 = q_2$:

$n q_1 = n q_2$

$a - n q_1 = a - n q_2$

$r_1 = a - n q_1 = a - n q_2 = r_2$

So $\boxed{r_1 = r_2.}$