MATH 304 Examination 2 August 2, 2010

Name Key

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.
There are four problems. Each is worth 25 points.

1. A certain experiment produces the following data $(0,8),(1,6),(2,-1)$. Calculate the model that produces a least-square fit of these points by a function of the form

$$A\cos\left(\tfrac{\pi}{2}x\right)+B\sin\left(\tfrac{\pi}{2}x\right)$$

Is the solution of this problem unique?

2. Prove that the matrices A and A^TA have the same null space.

3. Let Q be a $n\times m$ matrix with orthonormal columns. Let $\mathbf{y}\in\mathbb{R}^n$. Prove that the projection of $\mathbf{y}$ onto the column space of Q is given by the formula $QQ^T\mathbf{y}$.

4. Consider the matrix $A=\begin{bmatrix}1&1&1\\1&1&2\\1&0&3\\1&0&4\end{bmatrix}$.

   (a) Find an orthogonal basis for the column space of A.

   (b) Calculate the projection of the vector $\begin{bmatrix}0\\1\\-2\\3\end{bmatrix}$ onto the column space of A.

1

$$\vec{y} = \begin{bmatrix} 8 \\ 6 \\ -1 \end{bmatrix}$$

$$\vec{\beta} = \begin{bmatrix} A \\ B \end{bmatrix} \qquad X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = X^T \begin{bmatrix} 8 \\ 6 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

$$2A = 9 \qquad B = 6$$

The least-square fit model is

$$\boxed{\tfrac{9}{2} \cos\left(\tfrac{\pi}{2} x\right) + 6 \sin\left(\tfrac{\pi}{2} x\right)}$$

The solution is unique since $X^T X$ is an invertible matrix.

2 $\quad \vec{x} \in \text{Nul}\, A \Rightarrow A\vec{x} = \vec{0}$ $\quad \boxed{2}$

$$\Rightarrow A^T A \vec{x} = 0 \Rightarrow x \in \text{Nul}\, A^T A.$$

Conversely, assume $\vec{x} \in \text{Nul}\, A^T A$.

Then $\quad A^T A \vec{x} = \vec{0}$.

Then $\quad \vec{x}^T A^T A \vec{x} = 0$

$$(A\vec{x})^T A\vec{x} = 0$$

$$(A\vec{x}) \cdot (A\vec{x}) = 0$$

$$\|A\vec{x}\|^2 = 0$$

$$A\vec{x} = 0.$$

Thus $\vec{x} \in \text{Nul}\, A^T A \Rightarrow \vec{x} \in \text{Nul}\, A$.

This proves $\text{Nul}\, A = \text{Nul}\,(A^T A)$.

(3) Assume the columns of $\boxed{3}$ Q are orthonormal vectors.

Let $\vec{y} \in \mathbb{R}^n$. Clearly $QQ^T\vec{y}$ is in the column space of Q. We need to show that $\vec{y} - QQ^T\vec{y}$ is orthogonal to the column space of Q.

But $(\mathrm{Col}\, Q)^{\perp} = \mathrm{Nul}(Q^T)$.

So prove

$$Q^T(\vec{y} - QQ^T\vec{y}) = \vec{0}.$$

So calculate

$$Q^T(\vec{y} - QQ^T\vec{y}) = Q^T\vec{y} - \boxed{\underbrace{Q^TQ}_{I}}^{\text{o.n. columns}} Q^T\vec{y}$$

$$= Q^T\vec{y} - Q^T\vec{y} = \vec{0}$$

Hence $\vec{y} - QQ^T\vec{y} \perp (\mathrm{Col}\, Q)$.

$\boxed{4}$

(4) (a)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{a}_2} - \frac{\vec{a}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\vec{v}_3 = \vec{a}_3 - \frac{\vec{a}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\vec{a}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \underbrace{\frac{10}{4}}_{\frac{5}{2}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\frac{1}{2}(-4)}{1}\begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \qquad 1/2$$

$$= \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \qquad \text{So OB is } \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}}_{= \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}}$$

(b) The projection is

$$\hat{y} = \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{\frac{1}{2}\cdot 0}{1}\begin{bmatrix} \\ \\ \\ \end{bmatrix} + \frac{\frac{1}{2}\cdot 6}{1}\begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 - 3/2 \\ 1/2 + 3/2 \\ 1/2 - 3/2 \\ 1/2 + 3/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 2 \end{bmatrix}$$

$\boxed{5}$

verify

$$\vec{y} - \hat{\vec{y}} = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} +1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\perp \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \checkmark \quad \perp \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \checkmark \quad \perp \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \checkmark$$

So $\vec{y} - \hat{y} \perp \text{Col}\, A$.