Winter 2019 Math 304 Topics for Exam 1

On an exam you can be asked to prove any of the theorems stated in the summary.

4.7 Change of bases (in fact: Change of coordinates).

- Know that the matrix M with the property $[\mathbf{v}]_{\mathcal{C}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$**. It is denoted $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and it is calculated as

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \Big[\ [\mathbf{b}_1]_{\mathcal{C}}\ \cdots\ [\mathbf{b}_n]_{\mathcal{C}}\ \Big];$$

  here $\mathcal{B} = \{\mathbf{b}, \dots, \mathbf{b}_n\}$ and $\mathcal{C}$ are two bases of an n-dimensional vector space $\mathcal{V}$ and $[\mathbf{v}]_{\mathcal{B}} \in \mathbb{R}^n$ denotes the coordinate vector of $\mathbf{v} \in \mathcal{V}$ relative to the basis $\mathcal{B}$.
- $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$
- Know that there is a special basis of $\mathbb{R}^n$, called the **standard basis**, which consists of the columns of the identify matrix I_n. It is denoted by $\mathcal{E}$. This notattion comes from the fact that the vectors in this basis are commonly denoted by $\vec{e}_1, \cdots, \vec{e}_n$.
- Know that for a basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ for $\mathbb{R}^n$ (this is a special case) we have

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \big[\,\vec{b}_1 \cdots \vec{b}_n\,\big] = P_{\mathcal{B}}$$

  (the matrix $P_{\mathcal{B}}$ was introduced in Section 4.4); this basically says that $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ is a very friendly matrix.
- Know $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{E}}{P}\right)\left(\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}\right) = \left(\underset{\mathcal{E}\leftarrow\mathcal{C}}{P}\right)^{-1} \underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \big[\,\vec{c}_1 \cdots \vec{c}_n\,\big]^{-1} \big[\,\vec{b}_1 \cdots \vec{b}_n\,\big]$
  where $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$ is another basis for $\mathbb{R}^n$; this basically says that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ can be written as a product of a very friendly matrix and the inverse of another very friendly matrix
- Know Exercises 4 - 10, 13, 14.

5.1 Eigenvectors and eigenvalues.

- Know the definition of an eigenvector and eigenvalue. It is a little tricky. Pay attention.
- Know the definition of an eigenspace and how to find an eigenspace corresponding to a given eigenvalue.
- Know that the eigenvalues of a triangular matrix are the entries on its main diagonal.
- **Theorem.** Eigenvectors corresponding to distinct eigenvalues are linearly independent. (Or, in a more formal mathematical language: Let A be an $n{\times}n$ matrix, let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \in \mathbb{R}^n$ and $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$. If $A\vec{v}_k = \lambda_k \vec{v}_k$, $\vec{v}_k \neq \vec{0}$ and $\lambda_j \neq \lambda_k$ for all $j, k = 1, 2, \dots, m$, then $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ are linearly independent.
- Know the proof of the above theorem for $m = 2$ vectors.

5.2 The characteristic equation.

- Know that λ is an eigenvalue of an $n{\times}n$ matrix A if and only if $\det(A - \lambda I) = 0$
- Know how to calculate $\det(A - \lambda I)$ for $2{\times}2$ matrices, how to find eigenvalues and corresponding eigenvectors. Exercises 1-8, but do more and find eigenvectors as well.

5.3 Diagonalization.

- **Theorem.** (The diagonalization theorem) An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
- Know how to decide whether a given 2×2 and 3×3 matrix A, is diagonalizable or not; if it is diagonalizable, how to find invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

5.4 Eigenvectors and linear transformations.

- Know that the matrix M with the property $[T\mathbf{v}]_{\mathcal{C}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **matrix of T relative to the bases $\mathcal{B}$ and $\mathcal{C}$** and it is calculated as

$$M = \Big[\ [T\mathbf{b}_1]_{\mathcal{C}}\ \cdots\ [T\mathbf{b}_n]_{\mathcal{C}}\ \Big],$$

  where T be a linear transformation from an m-dimensional vector space $\mathcal{V}$ to an n-dimensional vector space $\mathcal{W}$, $\mathcal{B} = \{\mathbf{b}, \dots, \mathbf{b}_m\}$ is a basis for $\mathcal{V}$ and $\mathcal{C}$ is a basis for $\mathcal{W}$.

- Know that the matrix M with the property $[T\mathbf{v}]_{\mathcal{B}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **matrix of T relative to the basis $\mathcal{B}$**, it is denoted by $[T]_{\mathcal{B}}$ and it is calculated as

$$[T]_{\mathcal{B}} = \Big[\ [T\mathbf{b}_1]_{\mathcal{B}}\ \cdots\ [T\mathbf{b}_n]_{\mathcal{B}}\ \Big],$$

  where $T : \mathcal{V} \to \mathcal{V}$ is a linear transformation on an n-dimensional vector space $\mathcal{V}$ and $\mathcal{B} = \{\mathbf{b}, \dots, \mathbf{b}_n\}$ is a basis for $\mathcal{V}$.

- **Theorem.** (The diagonal matrix representation) Let A, D, P be $n \times n$ matrices, where P is invertible, D is diagonal and $A = PDP^{-1}$. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ which consists of the columns of P and if $T : \mathbb{R}^n \to \mathbb{R}^n$ is given by $T\vec{v} = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^n$, then $[T]_{\mathcal{B}} = D$.

5.5 Complex eigenvalues.

- Know how to calculate complex eigenvalues and corresponding eigenvectors of 2×2 and 3×3 real matrices

- Know that the most important class of 2×2 matrices with complex eigenvalues are rotation matrices; the matrix of the counterclockwise rotation about the origin by the angle θ measured in radians relative to the standard basis for $\mathbb{R}^2$ is

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix};$$

  the 2×2 matrix R_θ is called the **rotation matrix**.

- **Theorem.** (A "hiding rotation" theorem.) Let A be a real 2×2 matrix with a nonreal eigenvalue. Then there exist an invertible 2×2 matrix P, a positive scalar α and a rotation matrix R_θ such that $A = \alpha P\, R_\theta P^{-1}$.