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Math 312

2.4.4) Let $A$ be a non-empty subset of $\mathbb{R}$.
Prove that $A$ is bounded iff $\exists K > 0 : -K \le x \le K \quad \forall x \in A$

Proof. (1) Let $A$ be a non-empty subset of $\mathbb{R}$.

(2) Assume that $A$ is bounded.

(3) By Def'n of bounded set, $A$ is bounded above and below

(4) By Def'n of bounded below, $\exists a \in \mathbb{R} : a \le x \quad \forall x \in A$.

(5) By Def'n of bounded above, $\exists b \in \mathbb{R} : x \le b \quad \forall x \in A$.

(6) By Axiom O2, $a \le b$.

(7) Case 1: Assume that $a \ge 0$.

(8) By (6), (7), $b \ge 0$.

(9) Set $K = a + b$.

(10) $K = a + b \ge 0 + b = b$

(11) $-K = -(a+b) \le -(a+b) + b = -a \le a$.

(12) By (10), (11), $-K \le a \le x \le b \le K \quad \forall x \in A$.

~~Case 1a: Assume that $a = b = 0$.~~

(13) By (1), (9), $K > 0$.

(14) Case 2: Assume that $a < 0$.

(15) Case 2a: Assume that $b < 0$.

(16) Set $-K = (a+b) + 1$ (a,b may both be zero).

(17) $-K = a + b < a + 0 = a$

(18) $K = -a - b > 0 - b = -b > b$.

(19) By (17), (18), $-K < a \le x \le b < K. \quad \forall x \in A$.

(20) By (1), (18), $K > 0$.

(21) Case 2b: Assume that $b \ge 0$.

(22) Set $K = -a + b$.

(23) $K = -a + b > 0 + b = b$.

(24) $-K = a - b \le a - 0 = a$.

(25) By (23), (24), $-K \le a \le x \le b < K \quad \forall x \in A$.

(26) By ~~(25)~~, (1), (14), (22) $K > 0$.

(27) ← By (12), (13), (19), (20), (25), (26), $\exists K > 0 : -K \le x \le K \quad \forall x \in A$.

↓

~~26~~ Part 2:

$a = -K \longrightarrow$ (28) Assume that $\exists K > 0 : -K \leq x \leq K \quad \forall x \in A$.

(29) By (28), $\exists a \in \mathbb{R} : a \leq x \quad \forall x \in A$.

(30) By Def'n of bounded below, $A$ is bounded below.

$b = K \longrightarrow$ (31) By (28), $\exists b \in \mathbb{R} : x \leq b \quad \forall x \in A$.

(32) By Def'n of bounded above, $A$ is bounded above

(33) By (30), (32), Def'n of bounded set, $A$ is bounded.

$\square$.

This proof is much easier using the absolute value:

$$K = |a| + |b| + 1.$$

Then by the properties of ~~the~~ abs we have

$$-K \leq a \leq b \leq K.$$