CHAPTER 3

Continuous functions

In this chapter I will always denote a non-empty subset of $\mathbb{R}$.

3.1. The ϵ-δ definition of a continuous function

Definition 3.1.1. A function $f : I \to \mathbb{R}$ is *continuous at a point* $x_0 \in I$ if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon, x_0) > 0$ such that

$$x \in \bigl(x_0 - \delta, x_0 + \delta\bigr) \cap I \quad \Rightarrow \quad |f(x) - f(x_0)| < \epsilon. \tag{3.1.1}$$

The function f is *continuous on* I if it is continuous at each point of I.

Note that the implication in (3.1.1) can be restated as

$$x \in I \ \text{ and } \ |x - x_0| < \delta(\epsilon, x_0) \quad \Rightarrow \quad |f(x) - f(x_0)| < \epsilon.$$

Next we restate Definition 3.1.1 using the terminology introduced in Section 2.14. For a function $f : I \to \mathbb{R}$ and a subset $A \subseteq I$ we will use the notation $f(A)$ to denote the set $\bigl\{y \in \mathbb{R} : \exists\, x \in A \text{ s.t. } f(x) = y\bigr\} = \bigl\{f(x) : x \in A\bigr\}$.

A function $f : I \to \mathbb{R}$ is continuous at a point $x_0 \in I$ if for each neighborhood V of $f(x_0)$ there exists a neighborhood U of x_0 such that

$$f\bigl(I \cap U\bigr) \subseteq V.$$

3.2. Finding $\delta(\epsilon)$ for a given function at a given point

In this and the next section we will prove that some familiar functions are continuous.

A general strategy for proving that a given function f is continuous at a given point x_0 is as follows:

Step 1. Simplify the expression $|f(x) - f(x_0)|$ and try to establish a simple connection with the expression $|x - x_0|$. The simplest connection is to discover positive constants δ_0 and K such that

$$x \in I \ \text{ and } \ x_0 - \delta_0 < x < x_0 + \delta_0 \quad \Rightarrow \quad |f(x) - f(x_0)| \leq K\,|x - x_0|. \tag{3.2.1}$$

Constants δ_0 and K might depend on x_0. Formulate your discovery as a lemma.

Step 2. Let $\epsilon > 0$ be given. Use the result in Step 1 to define your $\delta(\epsilon, x_0)$. For example, if (3.2.1) holds, then $\delta(\epsilon, x_0) = \min\bigl\{\epsilon/K, \delta_0\bigr\}$.

Step 3. Use the definition of $\delta(\epsilon, x_0)$ from Step 2 and the lemma from Step 1 to prove the implication (3.1.1).

Example 3.2.1. We will show that the function $f(x) = x^2$ is continuous at $x_0 = 3$. Here $I = \mathbb{R}$ and we do not need to worry about the domain of f.

Step 1. First simplify

$$\bigl|f(x) - f(x_0)\bigr| = \bigl|x^2 - 3^2\bigr| = \bigl|(x+3)(x-3)\bigr| = \bigl|x+3\bigr|\,\bigl|x-3\bigr|. \tag{3.2.2}$$

Now we notice that if $2 < x < 4$ we have $\bigl|x+3\bigr| = x+3 \leq 7$. Thus (3.2.1) holds with $\delta_0 = 1$ and $K = 7$. We formulate this result as a lemma.

LEMMA. *Let $f(x) = x^2$ and $x_0 = 3$. Then*

$$|x-3| < 1 \quad \Rightarrow \quad \bigl|x^2 - 3^2\bigr| < 7|x-3|. \tag{3.2.3}$$

PROOF. Let $|x-3| < 1$. Then $2 < x < 4$. Therefore $x+3 > 0$ and $|x+3| = x+3 < 7$. By (3.2.2) we now have $\bigl|x^2 - 3^2\bigr| < 7|x-3|$. □

Step 2. Now we define $\delta(\epsilon) = \min\bigl\{\epsilon/7, 1\bigr\}$.
Step 3. It remains to prove (3.1.1). To this end, assume $|x-3| < \min\bigl\{\epsilon/7, 1\bigr\}$. Then $|x-3| < 1$. Therefore, by Lemma we have $\bigl|x^2 - 3^2\bigr| < 7|x-3|$. Since by the assumption $|x-3| < \epsilon/7$, we have $7|x-3| < 7\,\epsilon/7\epsilon$. Now the inequalities

$$\bigl|x^2 - 3^2\bigr| < 7|x-3| \quad \text{and} \quad 7|x-3| < \epsilon$$

imply that $\bigl|x^2 - 3^2\bigr| < \epsilon$. This proves (3.1.1) and completes the proof that the function $f(x) = x^2$ is continuous at $x_0 = 3$.

EXERCISE 3.2.2. Prove that the reciprocal function $x \mapsto \dfrac{1}{x}$, $x \neq 0$, is continuous at $x_0 = 1/2$.

EXERCISE 3.2.3. State carefully what it means for a function f *not* to be continuous at a point x_0 in its domain. (Express this as a formal mathematical statement.)

EXERCISE 3.2.4. Consider the function $f(x) = \operatorname{sgn} x$. Find a point x_0 at which the function f is not continuous. Provide a formal proof.

EXERCISE 3.2.5. Show that the function $f(x) = x^2$ is continuous on $\mathbb{R}$.

EXERCISE 3.2.6. Prove that $q(x) = 3x^2 + 5$ is continuous on $\mathbb{R}$.

3.3. Familiar continuous functions

EXERCISE 3.3.1. Let $m, k \in \mathbb{R}$ and $m \neq 0$. Prove that the linear function $\ell(x) = m\,x + k$ is continuous on $\mathbb{R}$.

EXERCISE 3.3.2. Let $a, b, c \in \mathbb{R}$ and $a \neq 0$. Prove that the quadratic function $q(x) = a\,x^2 + b\,x + c$ is continuous on $\mathbb{R}$.

EXERCISE 3.3.3. Let $n \in \mathbb{N}$ and let $x, x_0 \in \mathbb{R}$ be such that $x_0 - 1 \leq x \leq x_0 + 1$. Prove the following inequality

$$\bigl|x^n - x_0^n\bigr| \leq n\bigl(|x_0| + 1\bigr)^{n-1}\bigl|x - x_0\bigr|.$$

HINT: First notice that the assumption $x_0 - 1 \leq x \leq x_0 + 1$ implies that $|x| < |x_0| + 1$. Then use the Mathematical Induction and the identity

$$\bigl|x^{n+1} - x_0^{n+1}\bigr| = \bigl|x^{n+1} - x\,x_0^n + x\,x_0^n - x_0^{n+1}\bigr|.$$

EXERCISE 3.3.4. Let $n \in \mathbb{N}$. Prove that the power function $x \mapsto x^n$, $x \in \mathbb{R}$, is continuous on $\mathbb{R}$.

EXERCISE 3.3.5. Let $n \in \mathbb{N}$ and let $a_0, a_1, \dots, a_n \in \mathbb{R}$ with $a_n \neq 0$. Prove that the n-th order polynomial

$$p(x) = a_0 + a_1\,x + \cdots + a_{n-1}\,x^{n-1} + a_n\,x^n$$

is a continuous function on $\mathbb{R}$.

EXERCISE 3.3.6. Prove that the reciprocal function $x \mapsto \dfrac{1}{x}$, $x \neq 0$, is continuous on its domain.

EXERCISE 3.3.7. Prove that the square root function $x \mapsto \sqrt{x}$, $x \geq 0$, is continuous on its domain.

EXERCISE 3.3.8. Let $n \in \mathbb{N}$ and let x and a be positive real numbers. Prove that

$$\left| \sqrt[n]{x} - \sqrt[n]{a} \right| \leq \frac{\sqrt[n]{a}}{a} \left| x - a \right|.$$

HINT: Notice that the given inequality is equivalent to

$$b^{n-1} \left| y - b \right| \leq \left| y^n - b^n \right|, \quad y, b > 0.$$

This inequality can be proved using Exercise 2.7.7 (with $a = 1$ and $x = y/b$).

EXERCISE 3.3.9. Let $n \in \mathbb{N}$. Prove that the n-th root function $x \mapsto \sqrt[n]{x}$, $x \geq 0$, is continuous on its domain.

3.4. Various properties of continuous functions

EXERCISE 3.4.1. Let $f : I \to \mathbb{R}$ be continuous at $x_0 \in I$ and let y be a real number such that $f(x_0) < y$. Then there exists $\alpha > 0$ such that

$$x \in I \cap (x_0 - \alpha, x_0 + \alpha) \quad \Rightarrow \quad f(x) < y.$$

Illustrate with a diagram.

EXERCISE 3.4.2. Let $f : I \to \mathbb{R}$ be a continuous function on I. Let S be a non-empty bounded above subset of I such that $u = \sup S$ belongs to I. Let $y \in \mathbb{R}$. Prove: If $f(x) \leq y$ for each $x \in S$, then $f(u) \leq y$.

3.5. Algebra of continuous functions

All exercises in this section have the same structure. With the exception of Exercise 3.5.3, there are three functions in each exercise: f, g and h. The function h is always related in a simple (green) way to the functions f and g. Based on the given (green) information about f and g you are asked to prove a claim (red) about the function h.

EXERCISE 3.5.1. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be given functions with a common domain. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = f(x) + g(x), \quad x \in I.$$

(a) If f and g are continuous at $x_0 \in I$, then h is continuous at x_0.
(b) If f and g are continuous on I, then h is continuous on I.

EXERCISE 3.5.2. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be given functions with a common domain. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = f(x)g(x), \quad x \in I.$$

(a) If f and g are continuous at $x_0 \in I$, then h is continuous at x_0.
(b) If f and g are continuous on I, then h is continuous on I.

EXERCISE 3.5.3. Let $g : I \to \mathbb{R}$ be a given functions such that $g(x) \neq 0$ for all $x \in I$. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = \frac{1}{g(x)}, \qquad x \in I.$$

(a) If g is continuous at $x_0 \in I$, then h is continuous at x_0.
(b) If g is continuous on I, then h is continuous on I.

EXERCISE 3.5.4. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be given functions with a common domain. Assume that $g(x) \neq 0$ for all $x \in I$. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = \frac{f(x)}{g(x)}, \qquad x \in I.$$

(a) If f and g are continuous at $x_0 \in I$, then h is continuous at x_0.
(b) If f and g are continuous on I, then h is continuous on I.

EXERCISE 3.5.5. Let I and J be non-empty subsets of $\mathbb{R}$. Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be given functions. Assume that the range of f is contained in J. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = g(f(x)), \qquad x \in I.$$

(a) If f is continuous at $x_0 \in I$ and g is continuous at $f(x_0) \in J$, then h is continuous at x_0.
(b) If f is continuous on I and g is continuous on J, then h is continuous on I.

3.6. Continuous functions on a closed bounded interval $[a, b]$

In this section we assume that $a, b \in \mathbb{R}$ and $a < b$.

EXERCISE 3.6.1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f(a) < 0$ and $f(b) > 0$, then there exists $c \in [a, b]$ such that $f(c) = 0$.

HINT: Consider the set

$$W = \bigl\{ w \in [a, b) \, : \, \forall x \in [a, w] \ f(x) < 0 \bigr\}.$$

Prove the following properties of W:

(i) W does not have a maximum.
(ii) W has a supremum. Set $w = \sup W$.
(iii) Review Exercise 3.4.2.
(iv) Connect the dots.

EXERCISE 3.6.2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $c \in [a, b]$ such that $f(x) \leq f(c)$ for all $x \in [a, b]$.

HINT: Consider the set

$$W = \Bigl\{ v \in [a, b) \, : \, \exists \ z \in [a, b] \ \text{ such that } \ \forall \ x \in [a, v] \ \ f(x) < f(z) \Bigr\}.$$

Here $[a, a]$ denotes the set $\{a\}$. Prove the following properties of the set W:

(i) If $a < u$ and $[a, u) \subseteq W$ and there exists $t \in [a, b]$ such that $f(t) > f(u)$, then $u \in W$.
(ii) W does not have a maximum.
(iii) W has a supremum. Set $w = \sup W$ and prove $[a, w) \subseteq W$.
(iv) The items (ii) and (iii) yield information about w.

EXERCISE 3.6.3. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $d \in [a, b]$ such that $f(d) \leq f(x)$ for all $x \in [a, b]$.

HINT: Use Exercise 3.6.2.

EXERCISE 3.6.4. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then the range of f is a closed bounded interval.

HINT: Use Exercises 3.6.2, 3.6.3, and 3.6.1.

EXERCISE 3.6.5. Consider the function $f(x) = x^2$, $x \in \mathbb{R}$.

(a) Prove that 2 is in the range of f.
(b) Prove that the range of f equals $[0, +\infty)$.

DEFINITION 3.6.6. A function f is *increasing* on an interval I if $x, y \in I$ and $x < y$ imply $f(x) < f(y)$. A function f is *decreasing* if $x, y \in I$ and $x < y$ imply $f(x) > f(y)$. A function which is increasing or decreasing is said to be *strictly monotonic*.

EXERCISE 3.6.7. If f is continuous and increasing on $[a, b]$ or continuous and decreasing on $[a, b]$, then for each y between $f(a)$ and $f(b)$ there is exactly one $x \in [a, b]$ such that $f(x) = y$.

EXERCISE 3.6.8. Let $f(x) = x^3 + x$, $x \in \mathbb{R}$. Prove that f has an inverse. That is, prove that for each $y \in \mathbb{R}$ there exists unique $x \in \mathbb{R}$ such that $f(x) = y$.