MATH 321 Examination 2
November 3, 2009

Name Key

Give all details of your reasoning.
Each problem is worth 25 points for the total of 100 points.

Problem 1. (a) Express the following complex numbers in the form $x + iy$:

$$\text{(A)} \quad \frac{i}{1+i}, \qquad \text{(B)} \quad (3-2i)^2, \qquad \text{(C)} \quad e^{1+i\frac{\pi}{4}}$$

(b) Write the following numbers in polar form ($|z|\, e^{i\arg(z)}$):

$$\text{(A)} \quad i, \qquad \text{(B)} \quad -1-i, \qquad \text{(C)} \quad -\sqrt{3}+i,$$

Problem 2. A young couple wants to buy a house. They plan to borrow \$250,000 and pay it off in 30 years. The current annual interest rate is 6%. Assume that this interest is compounded continuously. What will be their fixed annual payment? What will be the monthly payment?

Your work must clearly show:

(a) The initial value problem for the loan amount over 30 years.

(b) The equation that was solved to get the annual fixed payment.

Problem 3. Suppose the electrical circuit has a resistor of $R = 5\Omega$ and a capacitor of $C = 1$F. Assume the voltage source is $E(t) = 100e^{-t/5}$V.

(a) If there is no charge on the capacitor at time $t = 0$ find the ensuing charge on the capacitor at time t.

(b) Plot on the same graph the voltage source and the charge on the capacitor. Explain the graph of the charge.

(c) What is the maximum charge on the capacitor? Preferably give the exact answer, or an approximate answer with 5 significant digits.

Problem 4. Consider the following second order homogeneous differential equation with constant coefficients

$$y'' + 2y' + 5y = 0.$$

(a) Find the fundamental set of solutions of the given differential equation. (You do not need to verify that the Wronskian is nonzero.)

(b) Write the general solution of the given differential equation.

(c) Find the particular solution that satisfies the initial conditions

$$y(0) = 1, \qquad y'(0) = 0$$

① ⓐ (A)

$$\frac{i}{1+i} = \frac{i(1-i)}{(1+i)(1-i)} = \frac{i-i^2}{2} = \frac{1+i}{2} = \boxed{\frac{1}{2}+i\frac{1}{2}}$$

$\boxed{1}$

(B) $$(3-2i)^2 = 9-12i+4i^2 = \boxed{5-12i}$$

(C) $$e^{1+i\frac{\pi}{4}} = e\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) =$$
$$= e\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right) = \boxed{\frac{e\sqrt{2}}{2}+i\frac{e\sqrt{2}}{2}}$$

ⓑ (A) $\boxed{e^{i\frac{\pi}{2}}}$

(B) $\boxed{\sqrt{2}\, e^{-i\frac{3\pi}{4}}}$

(C) $\boxed{2\, e^{i\,5\pi/6}}$

②
$$L' = 0.06L - P$$
$$L(0) = 250{,}000 \qquad L(30) = 0$$

$$L' = rL - P \qquad r = 0.06$$
$$L' - rL = -P \;/\; e^{-rt}$$
$$e^{-rt}L' - re^{-rt}L = -Pe^{-rt}$$
$$(e^{-rt}L)' = -Pe^{-rt}$$

$$e^{-rt} L = \frac{P}{r} e^{-rt} + C$$

$$L(t) = \frac{P}{r} + C e^{rt}$$

$$L(0) = 250000 = L_0$$

$$L_0 = L(0) = \frac{P}{r} + C$$

$$C = L_0 - \frac{P}{r}$$

$$L(t) = \frac{P}{r} + \left(L_0 - \frac{P}{r}\right) e^{rt}$$

$$0 = L(30) = \frac{P}{r} + \left(L_0 - \frac{P}{r}\right) e^{30r}$$

Solve for P:

$$\frac{P}{r}\left(1 - e^{30r}\right) + L_0 e^{30r} = 0$$

$$P = \frac{r L_0 e^{30r}}{e^{30r} - 1}$$

$P \approx \boxed{17{,}970.5}$ annual payment

monthly payment $\boxed{\$\,1{,}497.54}$

③ ⓐ

$Q(0)=0$ $\quad \boxed{5Q' + Q = 100e^{-t/5}}$ $\quad \boxed{3}$

$$Q' + \frac{1}{5}Q = 20e^{-t/5}$$

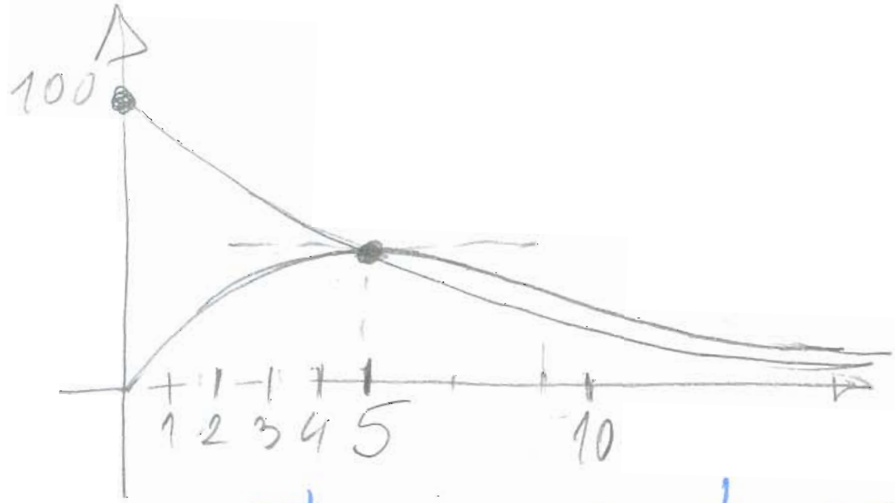
$$e^{t/5}Q' + \frac{1}{5}e^{t/5}Q = 20$$

$$\left(e^{t/5}Q\right)' = 20$$

$$e^{t/5}Q = 20t + C$$

$Q(0)=0$ implies $C=0$

$$\boxed{Q(t) = 20te^{-t/5}}$$



$Q'(t) = 20e^{-t/5} - 4te^{-t/5} = 0$

$\boxed{t=5}$ max

$$\boxed{Q(5) = 100e^{-1} = \frac{100}{e}}$$

$\approx$ $\boxed{36.788}$

The source is strong in the beginning, so it charges capacitor, then the source dies out so the capacitor discharges

(4) $y'' + 2y' + 5y = 0$ $\boxed{4}$

$\lambda^2 + 2\lambda + 5 = 0$

$(\lambda + 1)^2 + 4 = 0$

$(\lambda + 1)^2 = -4$

$\lambda + 1 = \pm 2i$

$\lambda_{1,2} = -1 \pm 2i$

(a) $y_1(t) = e^{-t}\cos(2t),\ y_2(t) = e^{-t}\sin(2t)$

(b) $y(t) = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t)$

$y'(t) = -C_1 e^{-t}\cos(2t) - 2C_1 e^{-t}\sin(2t)$
$\quad - C_2 e^{-t}\sin(2t) + 2C_2 e^{-t}\cos 2t$

$1 = y(0) = C_1$

$0 = y'(0) = -C_1 + 2C_2$

$C_1 = 1,\ C_2 = 1/2$

$y(t) = \underline{e^{-t}\left(\cos(2t) + \tfrac{1}{2}\sin(2t)\right)}$