MATH 321 Examination 3 November 24, 2009 Name Key

Give all details of your reasoning.
Each problem is worth 25 points for the total of 100 points.

Problem 1. An object of mass $m = 18\,\text{kg}$ is attached to a spring with unknown spring constant k. There is no damping present. Let $x(t)$ be the distance of the mass from the equilibrium position at time t. The mass was initially displaced by $1\,\text{m}$ from its equilibrium position and released without any initial velocity. It is observed that it took 3 seconds for the mass to reach the equilibrium for the first time. Calculate the spring constant k. (Give the exact value for k.)

Problem 2. Consider the initial value problem

$$\frac{1}{4}\,x'' + 16\,x = 10\,\cos(7t), \qquad x(0) = 0, \quad x'(0) = 0.$$

(a) Solve the given initial value problem.

(b) Place the answer in the form $A\,\sin(\delta t)\,\sin(\overline{\omega} t)$. Calculate the amplitude and the period of the envelope of the solution.

Problem 3. Consider the equation

$$x'' + \frac{1}{3}\,x' + \frac{2}{3}\,x = \cos(t).$$

(a) Find the steady-state response in the form $A\,\cos(t - \phi)$.

(b) Find the amplitude of the steady-state response.

(c) Now consider the forcing term $\cos(\omega t)$ with a variable frequency ω. Find the transfer function and the gain.

(d) Plot the gain as a function of ω. From your plot estimate for which frequencies ω the resulting gain is larger than the gain obtained for $\omega = 1$. Estimate the maximum possible gain and the frequency at which it occurs.

(1) $\boxed{1}$

$$18x'' + kx = 0$$

$$\omega_0^2 = \frac{k}{18}$$

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{18}}\, t\right) + c_2 \sin\left(\sqrt{\frac{k}{18}}\, t\right)$$

$x(0) = 1 \quad x'(0) = 0$

$c_1 = 1\ ,\ c_2 = 0$

$x(t) = \cos\left(\sqrt{\frac{k}{18}}\, t\right)$ is the solution

It is given that $\cos\left(\sqrt{\frac{k}{18}} \cdot 3\right) = 0$.

Hence $\sqrt{\frac{k}{18}} \cdot 3 = \frac{\pi}{2}$ or $\frac{k}{\cancel{18}\,2} \cdot \cancel{9} = \frac{\pi^2}{4}$

So $\boxed{k = \frac{\pi^2}{2}} \approx 4.9348$

(2) $\frac{1}{4}\lambda^2 + 16 = 0 \quad \lambda_{1,2} = \pm 8i$

$x_h(t) = C_1 \cos(8t) + C_2 \sin(8t)$.

$z_p(t) = a e^{i7t}\ ,\ z_p''(t) = a(-49) e^{i7t}$

$+\frac{1}{4} a(-49)\cancel{e^{i7t}} + 16a\cancel{e^{i7t}} = 10\cancel{e^{i7t}}$

$a\left(\frac{64}{4} - \cancel{12}\frac{49}{4}\right) = 10,\ a\frac{15}{4} = 10$

$a = \frac{40}{15}$

$\boxed{2}$

$$z_p(t) = \frac{40}{15} e^{i7t}$$

$$x_p(t) = \frac{40}{15} \cos(7t)$$

The general solution is

$$x(t) = c_1 \cos(8t) + c_2 \sin(8t) + \frac{40}{15}\cos(7t)$$

$$x(0) = 0, \quad x'(0) = 0$$

$$c_1 = -\frac{40}{15} \quad c_2 = 0$$

$$x(t) = \frac{40}{15}\left(\cos(7t) - \cos(8t)\right)$$

$$= \frac{40}{15} \operatorname{Re}\Big(\underbrace{e^{i\frac{15}{2}t}}_{i\sin(\frac{15}{2}t)}\underbrace{\left(e^{-\frac{1}{2}ti} - e^{\frac{1}{2}ti}\right)}_{(-2i)\sin(\frac{1}{2}t)}\Big)$$

$$= \frac{80}{15} \sin\left(\frac{1}{2}t\right) \sin\left(\frac{15}{2}t\right)$$

The amplitude of the envelope is $\frac{80}{15}$, the period is $\frac{2\pi}{1/2} = 4\pi$.

(3) (a) $x'' + \frac{1}{3}x' + \frac{2}{3}x = \cos t$

$\boxed{3}$

$z_p(t) = a e^{it}$, $z_p'(t) = a i e^{it}$

$z_p''(t) = -a e^{it}$

$-a \cancel{e^{it}} + \frac{1}{3} a i \cancel{e^{it}} + \frac{2}{3} a \cancel{e^{it}} = \cancel{e^{it}} \, 1$

$\left(-\frac{1}{3} + \frac{1}{3} i\right) a = 1$

$\frac{1}{3}\sqrt{2}\, e^{i\frac{3\pi}{4}} a = 1$

$a = \frac{3}{\sqrt{2}} e^{-i\frac{3\pi}{4}}$

$z_p(t) = \frac{3}{\sqrt{2}} e^{\left(t - \frac{3\pi}{4}\right) i}$.

$x_p(t) = \frac{3}{\sqrt{2}} \cos\left(t - \frac{3\pi}{4}\right)$

(b) This is the steady state response.

The amplitude is $\frac{3}{\sqrt{2}}$

(c) $\boxed{4}$

$$P(\lambda) = \lambda^2 + \frac{1}{3}\lambda + \frac{2}{3}$$

$$P(i\omega) = -\omega^2 + \frac{1}{3} i\omega + \frac{2}{3}$$

$$= \left(\frac{2}{3} - \omega^2\right) + \frac{1}{3}\omega i$$

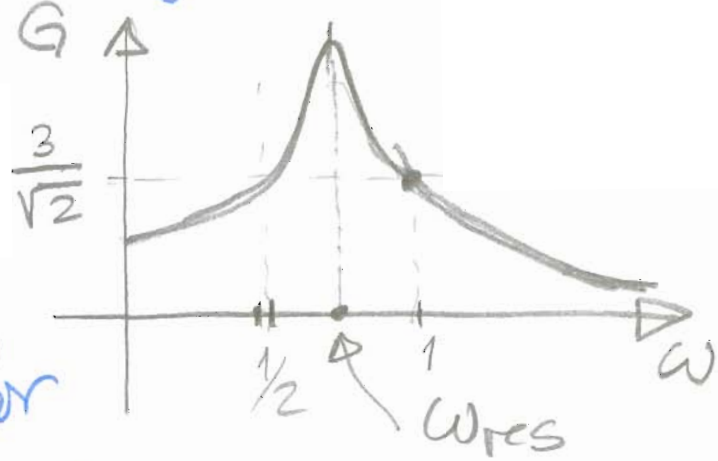
The transfer function is

$$H(i\omega) = \frac{1}{P(i\omega)} = \frac{\left(\frac{2}{3}-\omega^2\right) - \frac{1}{3}\omega i}{\left(\frac{2}{3}-\omega^2\right)^2 + \frac{1}{9}\omega^2}$$

The gain is

$$G(\omega) = \frac{1}{\sqrt{\left(\frac{2}{3}-\omega^2\right)^2 + \frac{1}{9}\omega^2}}$$

$$G(1) = \frac{3}{\sqrt{2}}$$



The gain will be greater than $3/\sqrt{2}$ for $0.4714 \simeq \frac{\sqrt{2}}{3} < \omega < 1$. The maximum gain is at $\omega_{res} = \frac{1}{3}\sqrt{\frac{11}{2}} \simeq 0.78174$ and it is $\frac{18}{\sqrt{23}} \simeq 3.7533$