Tentitive Solutions

MATH 321 Final Examination December 7, 2009

Name Tentitive Solutions

Give all details of your reasoning. Each problem is worth 12.5 points for the total of 100 points.

Problem 1. Consider the differential equation

$$\frac{dy}{dt} = 2\,t\,y^2. \tag{1}$$

(a) Does equation (1) have any constant solutions? Find them if they exist.

(b) Find the general solution of (1).

(c) Find the particular solution of (1) which satisfies the initial condition $y(0) = 1$. Determine the interval of existence of the solution.

(d) Find the particular solution of (1) which satisfies the initial condition $y(0) = -1$. What is the interval of existence of the solution?

(e) Do you notice a significant difference between the intervals of existence of solutions in (c) and (d)? Explain using the formula for the general solution.

Problem 2. Consider the differential equation

$$y' - 2\,\frac{1}{t}\,y = \frac{1}{t}. \tag{2}$$

(a) Does equation (2) have a constant solution?

(b) Find the particular solution of (2) which satisfies the initial condition $y(1) = -1$.

- (i) Determine the exact interval of existence of the solution. (Notice that (2) is not defined for all t. This has impact of the interval of existence of the solution.)
- (ii) Is this solution increasing, decreasing or neither?

(c) Now consider an initial condition $y(1) = a$. For which numbers a is the solution of the corresponding initial value problem increasing? Explain your answer.

Problem 3. Initially a 100-gallon container is filled with pure water. At time $t = 0$ brine is added to the container at a rate of 1 gallon per minute. The well-stirred mixture is drained from the container at the same rate of 1 gallon per minute. It is not known what is the concentration of the salt in the incoming brine. After 100 minutes it is measured that the amount of salt in the tank is 50 pounds. Determine the concentration of the salt (in pounds per gallon) in the incoming brine.

Problem 4. Consider the initial value problem

$$y''(t) + 2y'(t) + 5y(t) = 0, \qquad y(0) = -\sqrt{3}, \quad y'(0) = \sqrt{3} + 2.$$

(a) Solve the given initial value problem. Denote the solution by $y(t)$.

(b) Write the solution in the form $y(t) = A\,e^{-ct}\cos(\omega\,t - \phi)$. Here A, c, ω, and ϕ should be specific numbers.

(c) Use (b) to find the smallest positive number t_0 for which $y(t_0) = 0$. Give a formula for all numbers t for which $y(t) = 0$.

Problem 5. Consider the initial value problem

$$x''(t) + x(t) = \cos(\omega t), \quad x(0) = 0, \quad x'(0) = 0.$$

(a) Solve the given initial value problem.

(b) Place the answer in the form $A\,\sin(\delta t)\,\sin(\overline{\omega} t)$. Calculate the amplitude and the period of the envelope of the solution as a function of ω.

(c) Find the smallest and the largest value of ω for which the amplitude of the envelope will be bigger than 10. (Note that the smallest such value will be less than 1 and the largest such value will be greater than 1.)

Problem 6. Find the inverse Laplace transform of the following functions

$$\text{(A)}\ \frac{4}{s^3+4s}, \qquad \text{(B)}\ \frac{3s-2}{s^2+4s+3}, \qquad \text{(C)}\ \frac{4s-1}{s^2+2s+5}.$$

Problem 7. Use the Laplace transform to solve the initial value problem

$$y'' + 4y' + 5y = 2\sin(t), \quad y(0) = 1, \quad y'(0) = -1.$$

Problem 8. In Figure 1 below, the solution of

$$x''(t) + 2cx'(t) + \omega_0^2 x(t) = \cos t \tag{3}$$

is drawn in gray. The driving force, $\cos(t)$, is drawn in black. We assume that $c^2 < \omega_0^2$. Thus all solutions of (3) approach the steady-state response. Estimate the gain and the phase from Figure 1. Based on this estimate calculate the values of c and ω_0.

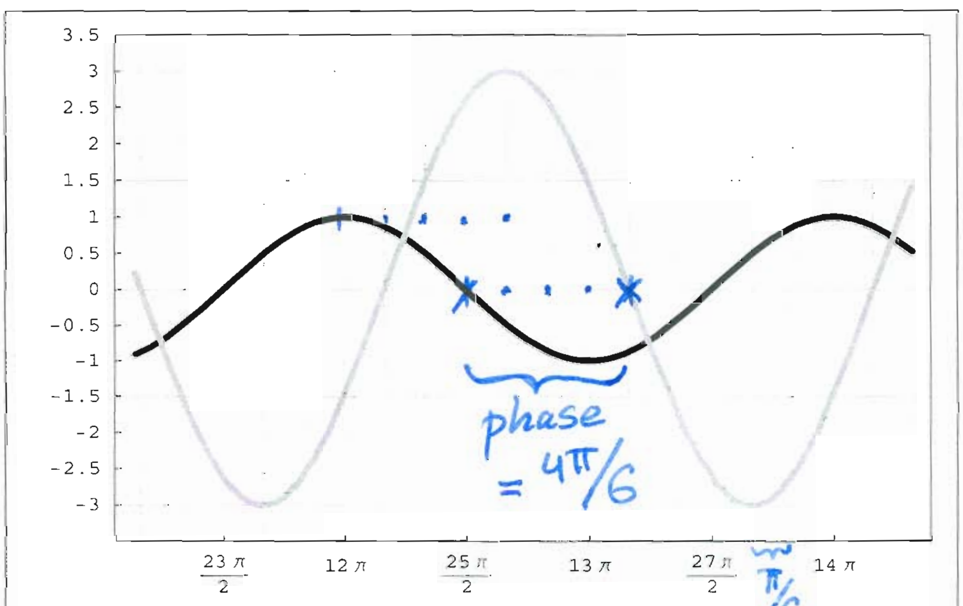

Figure 1: The response to $\cos t$

(1) (a) $y(t)=0$ is a constant solution of this differential equation. $\boxed{1}$

(b) Separate variables

$$\frac{1}{y^2}\,dy = 2t\,dt$$

Integrate

$$-\frac{1}{y} = t^2 + C.$$

Solve for y: $\quad y = \dfrac{-1}{t^2+C}.$

(c) Calculate C from $y(0)=1$, $C=-1$

$$y(t) = \frac{-1}{t^2-1} = \frac{1}{1-t^2}$$

The interval of existence is $(-1, 1)$ (notice that the solution is NOT defined at $t=-1$ and $t=1$. Since $-1<0<1$, $(-1,1)$ is the interval of existence.

(d) Calculate C from $y(0)=-1$: $C=1$

$y(t) = \dfrac{-1}{t^2+1}$. The interval of existence is $(-\infty, +\infty)$.

(e) The reason for a finite interval of existence in (c) is that $y(0)=1$, $\boxed{2}$ a positive number, so y' is positive for positive t, so the solution is increasing. With the increasing ~~solution~~ solution y^2 is increasing as well this contributes that y is increasing faster and faster, resulting in a vertical asymptote.

The reason for the solution in (d) beeing everywhere defined is that $y(0)=-1$, so y starts from a negative value and it is increasing for $t>0$. But increasing y, ~~res~~ since y is negative, results in smaller y^2. Therefore the solution is increasing but the rate of increase will start decreasing as y gets close to 0.

② ⓐ Try $y(t) = c$.

$$0 - 2\frac{1}{t}c = \frac{1}{t}$$

$\boxed{\begin{array}{c} y(t) = -1/2 \\ t > 0 \\ \text{is the const.} \end{array}}\boxed{\text{solut}^{n}}$ $\boxed{3}$

So $c = -1/2$ is a constant solution.

ⓑ First find the general solution.

The integrating factor is

$$\mu(t) = e^{-2\int \frac{1}{t}dt} = e^{-2\ln t} = \frac{1}{t^2}$$

First multiply by $1/t^2$

$$\frac{1}{t^2}y' - 2\frac{1}{t^3}y = \frac{1}{t^3}$$

and write the left hand side as a product (product rule magic)

$$\left(\frac{1}{t^2}y\right)' = \frac{1}{t^3}$$

This is easy to solve:

$$\frac{1}{t^2}y = -\frac{1}{2}\frac{1}{t^2} + C$$

Multiply by t^2:

$$y(t) = -\frac{1}{2} + Ct^2.$$

(2b) Now get C from $y(1) = -1$ $\boxed{4}$

$$-1 = -\frac{1}{2} + C \cdot 1^2, \text{ So } C = -1/2$$

The solution is

$$y(t) = -\frac{1}{2}(1 + t^2)$$

(i) Since in the original equation we have $t \neq 0$, we must choose $t > 0$, that is $(0, +\infty)$ as the interval of existence.

(ii) The solution is decreasing.

(c) Solve $y(0) = a$ for C

$$a = -\frac{1}{2} + C\,1^2, \text{ So } C = a + \frac{1}{2}$$

$$y(t) = -\frac{1}{2} + \left(a + \frac{1}{2}\right)t^2, \quad t > 0$$

This solution is increasing when $a + \frac{1}{2} > 0$, so $a > -1/2$.

3

$\boxed{5}$

$Q(t)$ amount of salt in lb

$$Q(0) = 0$$

$$Q'(t) = k - \frac{1}{100} Q(t)$$

$$Q' + \frac{1}{100} Q = k \quad / \, e^{t/100}$$

$$e^{t/100} Q' + \frac{1}{100} e^{t/100} Q = k e^{t/100}$$

$$\left(e^{t/100} Q\right)' = k e^{t/100}$$

$$e^{t/100} Q = 100 k e^{t/100} + C$$

$$Q(t) = 100k + C e^{-t/100}$$

$$C = -100k$$

$$Q(t) = 100k\left(1 - e^{-t/100}\right)$$

$$Q(100) = 50 = 100k\left(1 - e^{-1}\right)$$

$$k = \frac{1}{2(1-e^{-1})} = \frac{e}{2(e-1)}.$$

④ $\boxed{6}$

$$\lambda^2+2\lambda+5=0$$

$$\lambda_{1,2}=\frac{-2\pm\sqrt{4-20}}{2}=\frac{-2\pm 4i}{2}$$

$$\lambda_{1,2}=-1\pm 2i$$

the general solution is

$$y(t)=C_1e^{-t}\cos(2t)+C_2e^{-t}\sin(2t)$$

$$y'(t)=-C_1e^{-t}\cos(2t)-2C_1e^{-t}\sin 2t$$
$$-C_2e^{-t}\sin(2t)+2C_2e^{-t}\cos(2t)$$

$$-\sqrt{3}=C_1$$

$$\cancel{\sqrt{3}}+2=-\cancel{C_1}+2C_2$$

$$\boxed{\begin{array}{c}C_1=-\sqrt{3}\\ \hline C_2=1\end{array}}$$

$$y(t)=e^{-t}\left(-\sqrt{3}\cos(2t)+\sin(2t)\right)$$

ⓑ $y(t)=e^{-t}\,\mathrm{Re}\left((-\sqrt{3}-i)e^{i2t}\right)$

$$-\sqrt{3}-i=2\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=2e^{i\left(-\frac{5\pi}{6}\right)}$$

$$y(t)=2e^{-t}\cos\left(2t-\frac{5\pi}{6}\right)$$

(c) $\boxed{7}$

$$\cos\left(2t - \frac{\cancel{7}5\pi}{6}\right) = 0$$

$$2t - \frac{\cancel{7}5\pi}{6} = \frac{\pi}{2} + \cancel{2}k\pi$$

$k = 0, \pm 1, \pm 2$

$$2t = \frac{\pi}{2} + \frac{\cancel{7}5\pi}{6} + \cancel{2}k\pi$$

$$2t = \frac{\cancel{10}8\pi}{6} + \cancel{2}k\pi$$

$$2t = \frac{\cancel{5}4\pi}{3} + \cancel{2}k\pi$$

The first positive t is for $k = -1$

$$2t = \frac{\cancel{2}4\pi}{3} - \pi = \frac{\pi}{3}$$

$$\boxed{t_0 = \pi/6}$$

all

$$t = \frac{\cancel{5}4\pi}{6} + k\frac{\pi}{2}$$

$k = 0, \pm 1, \pm 2, \ldots$

$$= \frac{2\pi}{3} + k\frac{\pi}{2}$$

(5) $\boxed{8}$

$$\lambda^2+1=0$$

(a) $x_H(t) = C_1\cos(t) + C_2\sin(t)$

$z_p(t) = a e^{i\omega t}$, $z_p' = a i\omega e^{i\omega t}$

$z_p'' = -\omega^2 a e^{i\omega t}$

$$a(-\omega^2+1)e^{i\omega t} = e^{i\omega t}$$

$$\boxed{a = \frac{1}{1-\omega^2}}$$

$$x_p(t) = \frac{1}{1-\omega^2}\cos(\omega t)$$

The general solution is

$$x(t) = C_1\cos(t) + C_2\sin(t) + \frac{1}{1-\omega^2}\cos(\omega t)$$

$x(0)=0 \Rightarrow C_1 = -\frac{1}{1-\omega^2}\cos(t)$

$x'(0)=0 \Rightarrow C_2 = 0$

So, the solution is

$$x(t) = \frac{1}{1-\omega^2}\left(\cos(\omega t) - \cos(t)\right)$$

(b) Using the given formula we get 9

$$X(t) = \underbrace{\frac{2}{1-\omega^2}\sin\left(\frac{1-\omega}{2}t\right)}_{\text{envelope}}\sin\left(\frac{1+\omega}{2}t\right)$$

$\frac{2}{1-\omega^2}$

Amplitude of the envelope is $\boxed{\left|\frac{2}{1-\omega^2}\right|}$

Period of the envelope is

$$\frac{1-\omega}{2}T = 2\pi, \quad T = \frac{2\pi}{\frac{1-\omega}{2}}$$

$$\boxed{T = \frac{4\pi}{1-\omega}}$$

large for ω close to 1

(c) $\left|\frac{2}{1-\omega^2}\right| = 10$ $\quad \frac{1}{1-\omega^2} = \pm 5$

$1-\omega^2 = \pm\frac{1}{5}$ $\quad \omega^2 = 1-\frac{1}{5} = \frac{4}{5}$

or $\omega^2 = 1+\frac{1}{5} = \frac{6}{5}$ $\quad \boxed{\omega = \frac{2}{\sqrt{5}}}$ smaller than 1

$\boxed{\omega = \sqrt{\frac{6}{5}}}$ greater than 1

so the ampl. > 10 when $\frac{2}{\sqrt{5}} < \omega < \sqrt{\frac{6}{5}}$

$\omega \neq 1$

6 A

10

$$\frac{4}{s^3+4s} = \frac{4}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$$As^2+4A+Bs^2+Cs = 4$$

$$A+B=0 \quad B=-1$$

$$C=0$$

$$4A=4 \quad A=1$$

$$\frac{4}{s^3+4s} = \frac{1}{s} - \frac{s}{s^2+4}$$

Solution $\boxed{1-\cos(2t)}$

B $\frac{3s-2}{s^2+4s+3}$

$$s^2+4s+3=0$$

$$s_{1,2} = \frac{-4\pm\sqrt{16-12}}{2} = \frac{-4\pm2}{2}$$

$$s_1=-3, \quad s_2=-1$$

$$s^2+4s+3 = (s+3)(s+1)$$

$$\frac{3s-2}{s^2+4s+3} = \frac{A}{s+3} + \frac{B}{s+1}$$

$$As+A+Bs+3B = 3s-2$$

$$A+B=3 \qquad A=3-B, \ \boxed{A=11/2}$$

$$A+3B=-2 \qquad 3-B+3B=-2$$

$$2B=-5, \ \boxed{B=-5/2}$$

$\boxed{11}$

$$\frac{3s-2}{(s+3)(s+1)} = \frac{1}{2}\underbrace{\frac{11}{s+3}}_{e^{-3t}} - \frac{1}{2}\underbrace{\frac{5}{s+1}}_{e^{-t}}$$

$$\boxed{\frac{11}{2}e^{-3t} - \frac{\cancel{4}\,5}{2}e^{-t}}$$

(c)

$$\frac{4s-1}{s^2+2s+5} = \frac{4(s+1)-5}{(s+1)^2+4} =$$

$$= 4\,\frac{s+1}{(s+1)^2+4} - \frac{5}{2}\,\frac{2}{(s+1)^2+4}$$

Solution $\boxed{4e^{-t}\cos(2t) - \frac{5}{2}e^{-t}\sin(2t)}$

(7)

$$s^2 Y - s\overset{y(0)}{(+1)}\ \underbrace{\overset{y'(0)}{+1}}\ + 4(sY - 1) + 5Y = 2\,\frac{1}{s^2+1}$$

$$(s^2+4s+5)Y - s - 3 = \frac{2}{s^2+1}$$

$$Y(s) = \frac{2}{(s^2+1)(s^2+4s+5)} + \frac{+s+3}{s^2+4s+5}$$

$$\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4s+5} = \frac{2}{(\ )(\ )}$$ $\boxed{12}$

$$\cancel{As^3} + \cancel{4As^2} + \cancel{5As} + \cancel{Bs^2} + \cancel{4Bs} + 5B$$
$$+ \cancel{Cs^3} + \cancel{Cs} + \cancel{Ds^2} + D = 2$$

$$(A+B) = 0 \qquad \boxed{B = -A} \quad \boxed{D = -3A}$$

$$4A + B + D = 0 \qquad 3A + D = 0$$

$$5A + 4B + C = 0 \qquad A + C = 0$$

$$\boxed{C = -A}$$

$$5B + D = 2$$

$$-5A - 3A = 2$$

$$\boxed{A = -\tfrac{1}{4}} \quad \boxed{B = \tfrac{1}{4}} \quad \boxed{C = +\tfrac{1}{4}}$$

$$\boxed{D = +\frac{3}{4}}$$

$$\frac{2}{(\ )(\ )} = -\frac{1}{4}\,\frac{+s-1}{s^2+1} + \frac{1}{4}\,\frac{1s+3}{s^2+4s+5} =$$

$$(s+2)^2+1$$

$$= +\frac{1}{4}\left(\frac{-s}{s^2+1} + \frac{1}{s^2+1}\right) + \frac{1}{4}\left(\frac{s+2}{(s+2)^2+1} + \frac{1}{(s+2)^2+1}\right)$$

1st part $\boxed{\frac{1}{4}(-\cos t + \sin t) + \frac{1}{4}\left(e^{-2t}\cos t + e^{-2t}\sin t\right)}$

2nd part

$\boxed{13}$

$$\frac{+s+\cancel{3}}{(s+\cancel{2})^2+1} = +\frac{s+2}{(s+2)^2+1} + \cancel{1}\frac{1}{(s+2)^2+1}$$

$$e^{-2t}\cos t + e^{-2t}\underset{\sin}{\cancel{\ }}t$$

So the solution is

$$\frac{1}{4}(-\cos t+\sin t) + \frac{1}{4}e^{-2t}\cos t + \frac{1}{4}e^{-2t}\sin t$$
$$+ e^{-2t}\cos t + e^{-2t}\sin t$$

$$\boxed{\frac{1}{4}(-\cos t+\sin t) + \frac{5}{4}e^{-2t}\cos t + \frac{5}{4}e^{-2t}\sin t}$$

SOLUTION

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8) Phase is $\frac{4\pi}{6} = \frac{2\pi}{3} = \phi$

Gain is 3

$$P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$$

$$P(i) = -1 + 2ci + \omega_0^2 = \frac{1}{3}e^{i\phi}$$

$$\omega_0^2 - 1 = \frac{1}{3}\cos\frac{2\pi}{3} = -\frac{1}{6} \quad \boxed{\omega_0 = \sqrt{5/6}}$$

$$2c = \frac{1}{3}\frac{\sqrt{3}}{2} = \boxed{\frac{\sqrt{3}}{6}}$$