MATH 321 Examination 1 October 17, 2011

Name Key

Give all details of your reasoning.
Each problem is worth 25 points for the total of 100 points.

1. Solve the initial value problem

$$\frac{dy}{dt} = -2y + e^{-t}, \qquad y(0) = 0.$$

Find the interval of existence of the solution.

2. Consider the initial value problem

$$\frac{dy}{dt} = -\frac{1}{y}\frac{1}{(1+t)^2}, \qquad y(0) = a, \qquad a > 0.$$

In this problem we are interested only in $t > 0$.

(a) Solve the given initial value problem.

(b) The interval of existence of the solution depends on a. (We assume $a > 0$.) Is there a significant difference in the interval of existence for different values of a? Explain.

(c) Find the value of $a > 0$ for which the transition from one type of behavior to another occurs.

3. A tank contains 400 gallons of brine with 100 lb of salt. Fresh water is pumped into the tank at the rate of 2 gal/min, and the well-stirred brine leaves at the same rate.

(a) Set up an initial value problem which models this tank.

(b) Solve the initial value problem in (4a).

(c) How long does it take for the amount of salt in the tank to drop to 10 % of its original value?

4. A young couple has taken out a mortgage of 100,000 dollars. They are paying 5% annual interest compounded continuously. The couple has decided to make monthly payments of 1,000 dollars until they pay off the mortgage.

(a) Set up an initial value problem which models this mortgage.

(b) Solve the initial value problem in (4a).

(c) How long will it take for the couple to pay off the mortgage?

(d) How much money has the couple payed to the bank during this period?

1

$$y' + 2y = e^{-t} \Big/ e^{2t}$$

$$e^{2t} y' + 2e^{2t} y = e^{t}$$

$$(e^{2t} y)' = e^{t}$$

$$e^{2t} y = e^{t} + C \,, \quad y(0) = 0$$

$$e^{2t} y = e^{t} - 1$$

$$y(t) = e^{-t} - e^{-2t} = \underline{\underline{e^{-t}(1 - e^{-t})}}$$

The interval of existence of solution is $(-\infty, +\infty)$, all real numbers.

2

$$y' = -\frac{1}{y}\,\frac{1}{(1+t)^2}\,, \quad y(0) = a, \; a > 0.$$

$$y\,y' = -\frac{1}{(1+t)^2} = (1+t)^{-2}$$

$$\frac{d}{dt}\left(\frac{1}{2} y^2\right) = -\frac{1}{(1+t)^2}$$

$$\frac{1}{2} y^2 = + \frac{1}{1+t} + C \,, \quad y(0) = a$$

$$\frac{1}{2} a^2 = + 1 + c \,, \quad C = -1 + \frac{1}{2} a^2$$

$$\frac{1}{2} y^2 = + \frac{1}{1+t} - 1 + \frac{1}{2} a^2$$

$\boxed{2}$

$$y^2 = +\frac{2}{1+t} - 2 + a^2$$

The interval of existence is

$$+\frac{2}{1+t} - 2 + a^2 > 0$$

$$\frac{2}{1+t} > 2 - a^2$$

If $2 - a^2 \geq 0$ then $\frac{2}{1+t}$ is defined for all t. If $2 - a^2 < 0$, then the interval of existence is finite. Solve

$$\frac{2}{2-a^2} = 1+t, \quad t = \frac{2}{2-a^2} - 1 = \frac{2-2+a^2}{2-a^2}$$

$$t = \frac{a^2}{2-a^2}$$

The interval of existence is

$$0 \leq t < \frac{a^2}{2-a^2} \text{ if } a^2 < 2.$$

(c) The value is $\sqrt{2}$.

(3) 3

$$S(0) = 100$$

$$S'(t) = -\frac{2}{400} S(t)$$

$$S' = -\frac{1}{200} S(t)$$

$$S(t) = Ce^{-\frac{1}{200}t}$$

$$\boxed{S(t) = 100\, e^{-\frac{1}{200}t}}$$

(c) Solve $\cancel{100}\, e^{-\frac{1}{200}t} = \frac{1}{10} \cancel{100}$

$$e^{-\frac{1}{200}t} = \frac{1}{10}$$

$$-\frac{1}{200} t = \ln \frac{1}{10} = -\ln 10$$

$$\boxed{t = 200 \ln 10}$$

4 (a)

$$L' = rL - p$$
$$L(0) = L_0$$

$\boxed{4}$

$r = \frac{1}{20}$, $p = 12000$

$L_0 = 100{,}000$

(b)

$$L' - rL = -p$$
$$(e^{-rt}L)' = -pe^{-rt}$$
$$e^{-rt}L = +\frac{p}{r}e^{-rt} + C$$
$$L_0 = \frac{p}{r} + C, \quad C = L_0 - \frac{p}{r}$$
$$e^{-rt}L = \frac{p}{r}e^{-rt} + (L_0 - \frac{p}{r})$$
$$L(t) = \frac{p}{r} + (L_0 - \frac{p}{r})e^{rt}$$

Solve $\frac{p}{r} + (L_0 - \frac{p}{r})e^{rT} = 0$

$$(L_0 - \frac{p}{r})e^{rT} = -\frac{p}{r}$$
$$e^{rT} = \frac{\frac{p}{r}}{\frac{p}{r} - L_0} = \frac{p}{p - rL_0}$$

5

$$rT = \ln \frac{p}{p - rL_0}$$

$$T = \frac{1}{r} \ln \frac{p}{p - rL_0}$$

$$T = 20 \ln \frac{12000}{12000 - \frac{1}{20} 100,000}$$

(c) $$T = 20 \ln \frac{12,000}{12,000 - 5000} = 20 \ln \frac{12}{7}$$

(d) They payed $\left(20 \ln \frac{12}{7}\right) 12000$

$\approx$