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Exercises

2. The equation to be solved is

$$y''(t) + 5y'(t) + 6y(t) = 0.$$

The characteristic equation is

$$\lambda^2 + 5\lambda + 6 = 0.$$

The solutions are $\lambda_1 = -2$ and $\lambda_2 = -3$. Therefore,

$$y_1(t) = e^{-2t} \qquad \text{and} \qquad y_2(t) = e^{-3t}$$

are solutions of the given equation. Now calculate the Wronskian

$$W\big(e^{-2t}, e^{-3t}\big) = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = e^{-2t}(-3)e^{-3t} - e^{-3t}(-2)e^{-2t} = -e^{-5t} \neq 0.$$

Nonzero Wronskian implies that these solutions are linearly independent. Thus they form a fundamental set of solutions. The general solution is

$$y(t) = C_1e^{-2t} + C_2e^{-3t}.$$

9. The equation to be solved is

$$y''(t) + 4y'(t) + 5y(t) = 0.$$

The characteristic equation is

$$\lambda^2 + 4\lambda + 5 = 0.$$

The solutions are $\lambda_1 = -2 + i$ and $\lambda_2 == -2 - i$. Therefore,

$$y_1(t) = e^{-2t}\cos t \qquad \text{and} \qquad y_2(t) = e^{-2t}\sin t$$

are solutions of the given equation. The Wronskian is not zero, so these solutions are linearly independent. Thus they form a fundamental set of solutions. The general solution is

$$y(t) = C_1e^{-2t}\cos t + C_2e^{-2t}\sin t = e^{-2t}\big(C_1\cos t + C_2\sin t\big) = Ae^{-2t}\cos(t - \phi).$$

21. The initial value problem to be solved is

$$y''(t) + 25y(t) = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

First we find the general solution. The characteristic equation is

$$\lambda^2 + 25 = 0.$$

The solutions are $\lambda_1 = 5i$ and $\lambda_2 = -5i$. Therefore,

$$y_1(t) = e^{5i\,t} \qquad \text{and} \qquad y_2(t) = e^{-5i\,t}$$

are solutions of the given equation. But, we are interested in real solutions. So, use Euler's formula to get the real and imaginary part:

$$y_1 = e^{5i\,t} = \cos(5t) + i\sin(5t) \quad \text{and} \quad y_2 = e^{-5i\,t} = \cos(5t) - i\sin(5t).$$

As we showed in class the real part and the imaginary part of these functions are also solutions of the given equation. Now calculate the Wronskian

$$W\big(\cos(5t), \sin(5t)\big) = \begin{vmatrix} \cos(5t) & \sin(5t) \\ -5\sin(5t) & 5\cos(5t) \end{vmatrix} = 5(\cos(5t))^2 + 5(\sin(5t))^2 = 5 \neq 0.$$

Nonzero Wronskian implies that $\cos(5t)$ and $\sin(5t)$ are linearly independent. Thus they form a fundamental set of solutions of the given equation. The general solution is

$$y(t) = C_1\cos(5t) + C_2\sin(5t).$$

To solve the initial value problem we find the derivative first:

$$y'(t) = -5\,C_1\sin(5t) + 5\,C_2\cos(5t).$$

Now we use the initial conditions:

$$1 = y(0) = C_1 \cdot 1 + C_2 \cdot 0, \qquad -1 = y'(0) = -5\,C_1 \cdot 0 + 5\,C_2 \cdot 1.$$

Thus

$$C_1 = 1 \qquad \text{and} \qquad C_2 = -\frac{1}{5}.$$

The solution of the given initial value problem is

$$y(t) = \cos(5t) - \frac{1}{5}\,\sin(5t).$$

But, much more preferable way to write the solution is in the form $A\,\cos(5t - \phi)$. To do this we use complex numbers:

$$\cos(5t) = \mathrm{Re}\big(e^{5i\,t}\big), \qquad -\frac{1}{5}\,\sin(5t) = \mathrm{Re}\left(\frac{i}{5}\,e^{5i\,t}\right)$$

Thus

$$\cos(5t) - \frac{1}{5}\,\sin(5t) = \operatorname{Re}\left(e^{5i\,t} + \frac{i}{5}\,e^{5i\,t}\right) = \operatorname{Re}\left(\left(1+\frac{i}{5}\right)e^{5i\,t}\right).$$

Now we convert $1 + i/5$ to polar form

$$1 + \frac{i}{5} = \sqrt{1+\frac{1}{25}}\;e^{i\,\theta} = \frac{\sqrt{26}}{5}\;e^{i\,\theta},$$

where $\theta = \arctan\bigl(1/5\bigr)$. Hence,

$$\left(1+\frac{i}{5}\right)e^{5i\,t} = \frac{\sqrt{26}}{5}\;e^{i\,\theta}\;e^{5i\,t} = \frac{\sqrt{26}}{5}\;e^{i(5\,t+\theta)},$$

and therefore

$$\cos(5t) - \frac{1}{5}\,\sin(5t) = \operatorname{Re}\left(\left(1+\frac{i}{5}\right)e^{5i\,t}\right) = \operatorname{Re}\left(\frac{\sqrt{26}}{5}\;e^{i(5\,t+\theta)}\right) = \frac{\sqrt{26}}{5}\cos\bigl(5\,t+\theta\bigr).$$

Thus the solution is

$$y(t) = \cos(5t) - \frac{1}{5}\,\sin(5t) = \frac{\sqrt{26}}{5}\cos\bigl(5\,t-\phi\bigr), \qquad \text{where} \quad \phi = -\arctan\bigl(1/5\bigr).$$

Just to make sure that the last two formulas represent the same function I plot them on the same graph, see Figure 1.

22. The initial value problem to be solved is

$$y''(t) + 10y'(t) + 25y(t) = 0, \quad y(0) = 2, \quad y'(0) = -1.$$

First we find the general solution. The characteristic equation is

$$\lambda^2 + 10\lambda + 25 = 0.$$

There is only one solution $\lambda_1 = -5$. The corresponding solution of the given differential equation is

$$y_1(t) = e^{-5\,t}.$$

As it is explained in the book the other solution is

$$y_2(t) = t\,e^{-5\,t}.$$

The solutions

$$y_1(t) = e^{-5\,t} \qquad \text{and} \qquad y_2(t) = t\,e^{-5\,t}$$

are linearly independent since their Wronskian is nonzero:

$$W\bigl(e^{-5\,t}, t\,e^{-5\,t}\bigr) = \begin{vmatrix} e^{-5\,t} & t\,e^{-5\,t} \\ -5\,e^{-5\,t} & e^{-5\,t} - 5t\,e^{-5\,t} \end{vmatrix} = e^{-10\,t} - 5t\,e^{-10\,t} + 5\,t\,e^{-10\,t} = e^{-10\,t} \neq 0.$$

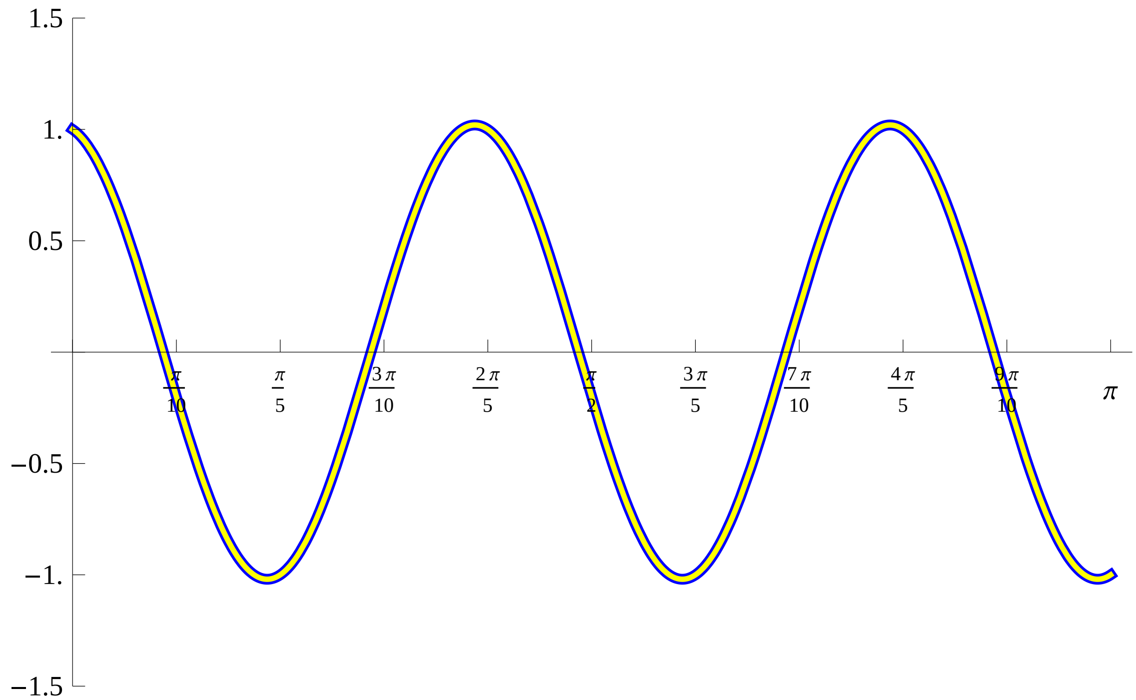

Figure 1: Problem 21

The general solution is

$$y(t) = C_1 \, e^{-5\,t} + C_2 \, t \, e^{-5\,t}.$$

To solve the initial value problem we find the derivative first:

$$y'(t) = -5\,C_1 \, e^{-5\,t} + C_2 e^{-5\,t} - 5\,C_2 t \, e^{-5\,t} = \bigl(-5\,C_1 + C_2 - 5\,C_2 t\bigr)\, e^{-5\,t}.$$

Now we use the initial conditions:

$$2 = y(0) = C_1 \cdot 1 + C_2 \cdot 0, \qquad -1 = y'(0) = -5\,C_1 + C_2.$$

Thus

$$C_1 = 2 \qquad \text{and} \qquad C_2 = 9.$$

The solution of the given initial value problem is, see Figure 2,

$$y(t) = 2\,e^{-5\,t} + 9\,t\,e^{-5\,t} = \bigl(2 + 9\,t\bigr)\, e^{-5\,t}.$$

My comment. The initial value problem that we solved here can be interpreted as describing the motion of a mass attached to a spring with damping. Here the mass is 1kg, damping is 10kg/s and spring constant 25kg/s^2. At time 0 the mass is displaced by 2m from the equilibrium position and it is given the initial velocity -1m/s. The solution that we found

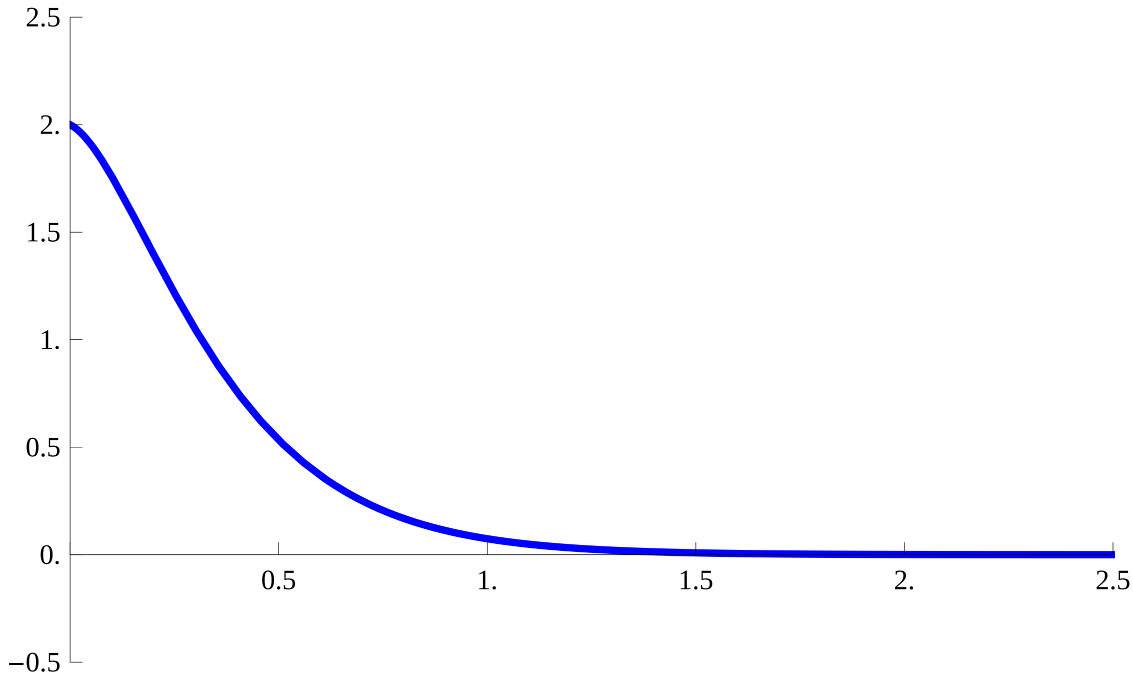

Figure 2: Problem 22

describes the position of the mass at any time $t > 0$. It means that the mass will just creep back to the equilibrium position.

An interesting question that arises here is whether we can give the mass an initial velocity so that it passes through the equilibrium. To solve this problem we need to set the initial velocity as an unknown quantity, call it v_0 and solve the initial value problem:

$$y''(t) + 10y'(t) + 25y(t) = 0, \quad y(0) = 2, \quad y'(0) = v_0.$$

The solution of this problem is

$$y(t) = \big(2 + (10 + v_0)\, t\big)\, e^{-5\,t}.$$

Now the question is: for which v_0 there exists $t > 0$ such that $y(t) = 0$. Since $e^{-5\,t} > 0$, we need a positive t such that

$$2 + (10 + v_0)\, t = 0.$$

Solving for t we get $t = -2/(10 + v_0)$. This quantity will be positive if $v_0 < -10$. For example, for $v_0 = -18$ we have $t = 1/4$ see Figure 3.

Looking at Figure 3, a natural question arises: What is the lowest position that the mass will reach in this situation? Here we assume that $v_0 = -18$ and the solution is

$$y(t) = \big(2 - 8\,t\big)\, e^{-5\,t}.$$

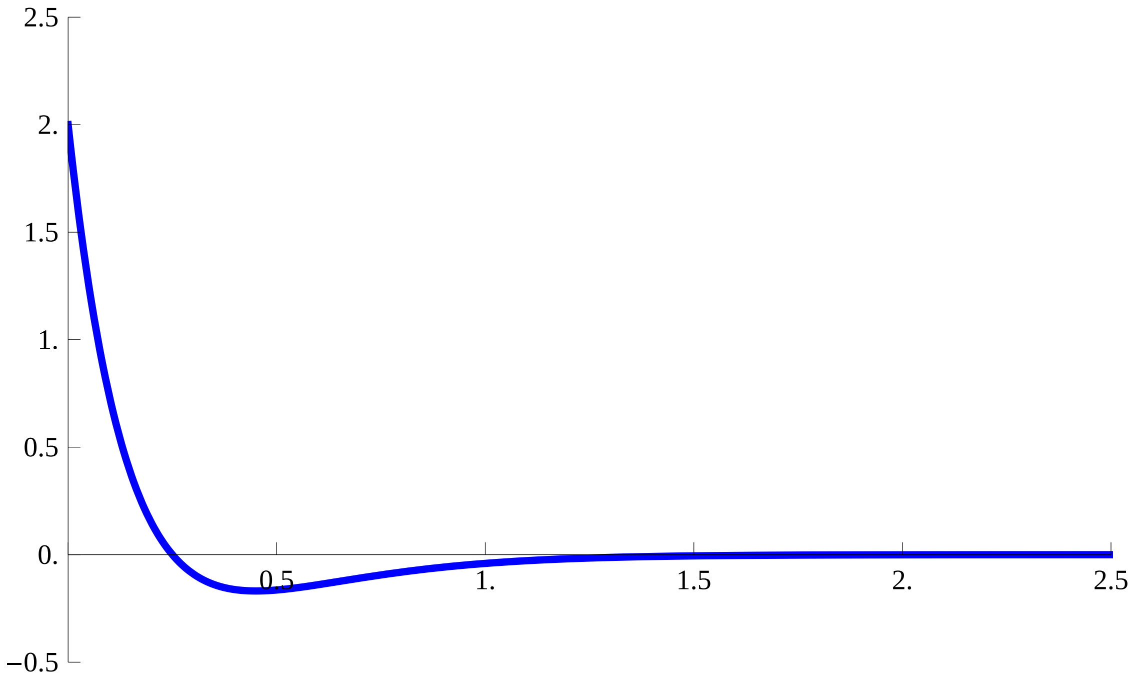

Figure 3: Problem 22, my comments

To answer the question, we take the derivative

$$y'(t) = -5\left(2 - 8\,t\right)e^{-5\,t} - 8e^{-5\,t} = \left(-18 + 40\,t\right)e^{-5\,t}.$$

Solving $y'(t) = 0$ for $t > 0$ we get $t = 9/20$. The lowest position of the mass is

$$y(9/20) = \left(2 - 8\,9/20\right)e^{-5\,9/20} = -(8/5)\,e^{-9/4} \approx -0.168639.$$

24. The initial value problem to be solved is

$$y''(t) - 4y'(t) - 5y(t) = 0, \quad y(1) = -1, \quad y'(1) = -1.$$

First we find the general solution. The characteristic equation is

$$\lambda^2 - 4\lambda - 5 = 0.$$

The solutions are $\lambda_1 = -1$ and $\lambda_2 = 5$. Therefore,

$$y_1(t) = e^{-t} \qquad \text{and} \qquad y_2(t) = e^{5\,t}$$

are solutions of the given equation. These solutions are linearly independent. The general solution is

$$y(t) = C_1e^{-t} + C_2e^{5t}.$$

To solve the initial value problem we find the derivative first:

$$y'(t) = -\,C_1 e^{-t} + 5\,C_2 e^{5t}.$$

Now we use the initial conditions:

$$-1 = y(1) = C_1 \cdot e^{-1} + C_2 \cdot e^5, \qquad -1 = y'(1) = -\,C_1 \cdot e^{-1} + 5\,C_2 \cdot e^5.$$

Thus we need to solve

$$\begin{aligned} C_1 e^{-1} + C_2 e^5 &= -1 \\ -C_1 e^{-1} + 5\,C_2 e^5 &= -1 \end{aligned}$$

We can always solve one equation and substitute the solution into the other equation. But, here it is easier to add two equations to get

$$6C_2 e^5 = -2.$$

Thus $C_2 = -e^{-5}/3$. Substituting this in the first equation we get $C_1 e^{-1} - 1/3 = -1$. Thus $C_1 = -2e/3$. Finally, the solution is

$$y(t) = -\frac{2e}{3} e^{-t} - \frac{e^{-5}}{3} e^{5t} = -\frac{1}{3}\Big(2e^{1-t} + e^{5(t-1)}\Big).$$

26. The initial value problem to be solved is

$$4y''(t) + y(t) = 0, \quad y(1) = 0, \quad y'(1) = -2.$$

First we find the general solution. The characteristic equation is

$$4\lambda^2 + 1 = 0.$$

The solutions are $\lambda_1 = i/2$ and $\lambda_2 = -i/2$. The complex solutions are

$$e^{it/2} = \cos\big(t/2\big) + i\sin\big(t/2\big), \qquad e^{-it/2} = \cos\big(t/2\big) - i\sin\big(t/2\big).$$

The real solutions are

$$\cos\big(t/2\big), \qquad \sin\big(t/2\big)$$

These solutions are linearly independent. The general solution is

$$y(t) = C_1 \cos\big(t/2\big) + C_2 \sin\big(t/2\big).$$

To solve the initial value problem we find the derivative first:

$$y'(t) = -\frac{C_1}{2} \sin\big(t/2\big) + \frac{C_2}{2} \cos\big(t/2\big).$$

Now we use the initial conditions:

$$0 = y(1) = C_1 \cos(1/2) + C_2 \sin(1/2), \qquad -2 = y'(1) = -\frac{C_1}{2}\sin\bigl(1/2\bigr) + \frac{C_2}{2}\cos\bigl(1/2\bigr).$$

Thus we need to solve

$$\begin{aligned} C_1 \cos(1/2) + C_2 \sin(1/2) &= 0 \\ -\frac{C_1}{2}\sin\bigl(1/2\bigr) + \frac{C_2}{2}\cos\bigl(1/2\bigr) &= -2 \end{aligned}$$

Solve the first equation for C_1 and substitute the solution into the second equation:

$$C_1 = -C_2 \sin(1/2)/\cos(1/2), \qquad C_2\,\frac{1}{2}\,\frac{\sin(1/2)}{\cos(1/2)}\sin(1/2) + C_2\,\frac{1}{2}\cos(1/2) = -2.$$

Solve the last equation for C_2:

$$\begin{aligned} C_2\frac{1}{2}\ \frac{(\sin(1/2))^2 + (\cos(1/2))^2}{\cos(1/2)} &= -2 \\ C_2\frac{1}{2}\ \frac{1}{\cos(1/2)} &= -2 \\ C_2 &= -4\cos(1/2) \end{aligned}$$

Now

$$C_1 = -C_2\frac{\sin(1/2)}{\cos(1/2)} = 4\ \cos(1/2)\,\frac{\sin(1/2)}{\cos(1/2)} = 4\sin(1/2).$$

Finally, the solution is (see Figure 4)

$$y(t) = 4\sin(1/2)\cos\bigl(t/2\bigr) - 4\cos(1/2)\sin\bigl(t/2\bigr).$$

My comment. It is interesting to determine the amplitude and the phase of this solution.

$$\begin{aligned} y(t) &= \mathrm{Re}\Bigl(4\sin(1/2)e^{it/2} + 4\cos(1/2)\,i\,e^{it/2}\Bigr) \\ &= 4\,\mathrm{Re}\Bigl(\bigl(\sin(1/2) + i\ \cos(1/2)\bigr)e^{it/2}\Bigr) \\ &= 4\,\mathrm{Re}\Bigl(i\bigl(\cos(1/2) - i\sin(1/2)\bigr)e^{it/2}\Bigr) \\ &= 4\,\mathrm{Re}\Bigl(ie^{-i\,1/2}e^{it/2}\Bigr) \\ &= 4\,\mathrm{Re}\Bigl(e^{i\pi/2}e^{-i1/2}e^{it/2}\Bigr) \\ &= 4\,\mathrm{Re}\bigl(e^{i(t-1+\pi)/2}\bigr) \\ &= 4\cos\bigl((t-1+\pi)/2\bigr) \end{aligned}$$

Thus the amplitude and the phase are

$$A = 4 \qquad \text{and} \qquad \phi = \frac{1-\pi}{2}.$$

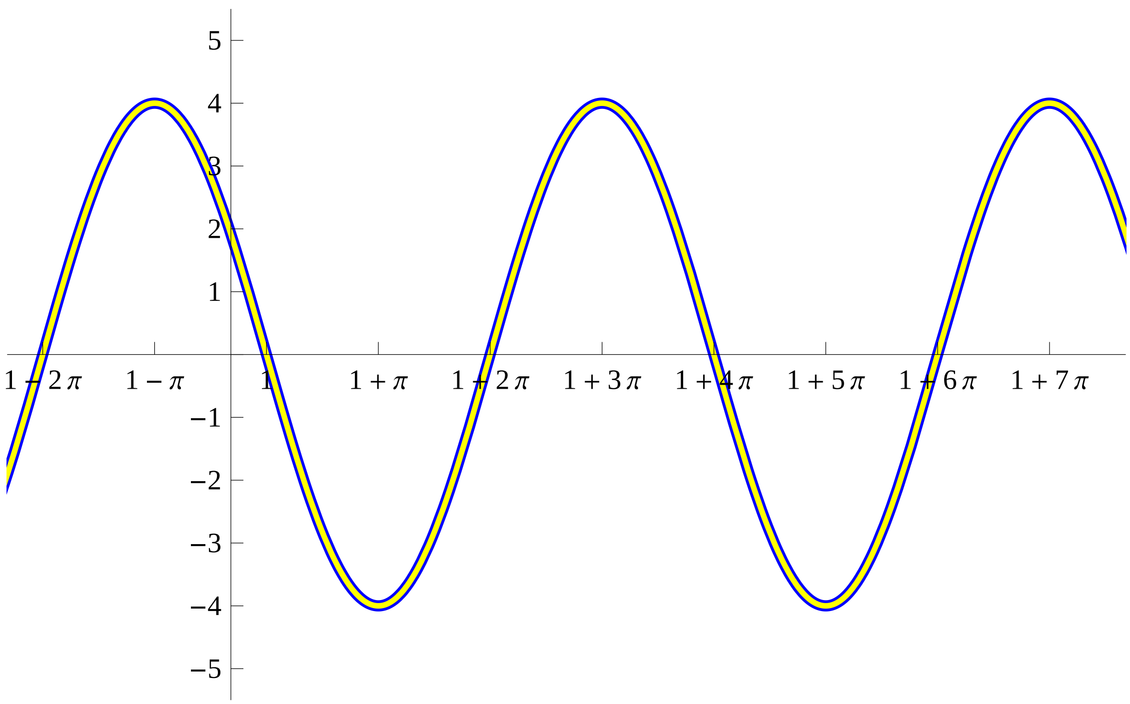

Figure 4: Problem 26