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Exercises

9. We need to calculate the improper integral

$$\int_0^\infty e^{at}\cos(\omega t)e^{-st}dt = \int_0^\infty \cos(\omega t)e^{-(s-a)t}dt$$

For that we need to calculate the indefinite integral

$$\int \cos(\omega t)e^{-(s-a)t}dt.$$

Let us calculate the indefinite integral

$$\int \cos(\omega t)e^{bt}dt.$$

That is done by calculating the complex integral

$$\begin{aligned}\int_0^\infty e^{(b+i\omega)t}dt &= \frac{1}{b+i\omega}e^{(b+i\omega)t}\\ &= \frac{b-i\omega}{b^2+\omega^2}\big(e^{bt}\cos(\omega t)+ie^{bt}\sin(\omega t)\big)\\ &= \frac{e^{bt}}{b^2+\omega^2}\Big(\big(b\cos(\omega t)+\omega\sin(\omega t)\big)+i\big(-\omega\cos(\omega t)+b\sin(\omega t)\big)\Big)\end{aligned}$$

Thus

$$\int \cos(\omega t)e^{bt}dt = \frac{e^{bt}}{b^2+\omega^2}\ \big(b\cos(\omega t)+\omega\sin(\omega t)\big).$$

Consequently, by the Fundamental Theorem of Calculus,

$$\int_0^T \cos(\omega t)e^{bt}dt = \frac{e^{bT}}{b^2+\omega^2}\ \big(b\cos(\omega T)+\omega\sin(\omega T)\big) - \frac{b}{b^2+\omega^2}.$$

Provided that $b<0$ we have

$$\lim_{T\to\infty}\frac{e^{bT}}{b^2+\omega^2}\ \big(b\cos(\omega T)+\omega\sin(\omega T)\big) = 0.$$

Therefore

$$\int_0^\infty \cos(\omega t)e^{bt}dt = -\frac{b}{b^2+\omega^2}.$$

Now we can just substitute $b=-(s-a)$ to get

$$\mathcal{L}\big(e^{at}\cos(\omega t)\big)(s) = \int_0^\infty \cos(\omega t)e^{-(s-a)t}dt = \frac{s-a}{(s-a)^2+\omega^2}.$$