An example of the Method of Characteristics

$a \neq 0$ $\quad ax^2 + bx + c = 0$

$$\Leftrightarrow \quad x \in \left\{ \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \frac{-b + \sqrt{b^2 - 4ac}}{2a} \right\}$$

$u(x, y)$ is an unknown function of two variables

$A(x, y, z)$, $B(x, y, z)$, $C(x, y, z)$ are given coefficients

We are solving q.l. PDE:

$$A(x, y, u)\, u_x(x, y) + B(x, y, u)\, u_y(x, y) = C(x, y, u)$$

To this PDE we associate the vector field

$$\langle A(x,y,z),\ B(x,y,z),\ C(x,y,z) \rangle$$

On Friday we showed that the sol. surface is tangent to this vector field.

From Math 331 we know how to find curves which are tangent to this vector field.

These curves are called the **characteristics** of the given PDE.

How do we communicate curves in 3-space?

We use vector valued functions:

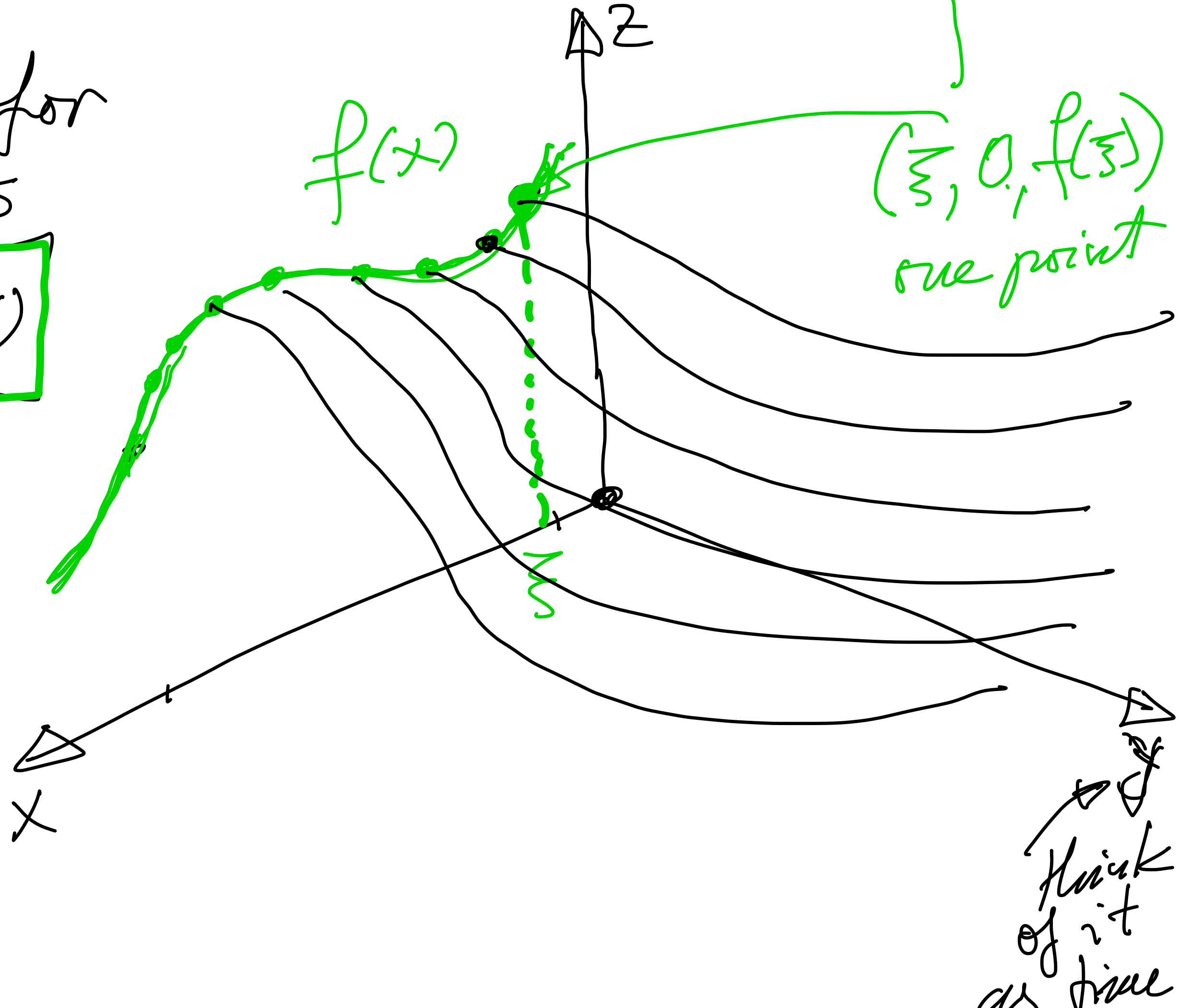
s is a variable $s \longmapsto \langle X(s), Y(s), Z(s) \rangle$ represents a curve in 3-space. For this curve to be tangent to the field $\langle A(\ ), B(\ ), C(\ ) \rangle$ I need to have:

characteristic equations for the PDE

$$\begin{cases} \dfrac{dX}{ds}(s) = A\big(X(s), Y(s), Z(s)\big) \\ \dfrac{dY}{ds}(s) = B\big(X(s), Y(s), Z(s)\big) \\ \dfrac{dZ}{ds}(s) = C\big(X(s), Y(s), Z(s)\big) \end{cases}$$

Given initial Conditions $X(0) = ?$, $Y(0) = ?$, $Z(0) = ?$ I get a specific curve the initial condition for $u(x,y)$ is

$$\boxed{u(x,0)=f(x)}$$



charst. eqs.

$$\begin{cases} \dfrac{dX}{ds} = A(X, Y, Z) \\ \dfrac{dY}{ds} = B(X, Y, Z) \\ \dfrac{dZ}{ds} = C(X, Y, Z) \\ X(0) = \xi \\ Y(0) = 0 \\ Z(0) = f(\xi) \end{cases}$$

IVP

I imagine I can solve this system

I get solutions which depend on two independent variables: s and ξ

$$\left\langle X(s,\xi),\ Y(s,\xi),\ Z(s,\xi) \right\rangle$$

surface, this is the graph of $u(x,y)$ (unknown function)

To find a formula for $z = u(x,y)$ we have to eliminate s, ξ. This can be tricky.

Think of the unit sphere.

Here I explained how I write script $\mathcal{U}$, it is u, like in unit sphere. I differentiate it from the Greek μ, by giving μ a long tail. In LaTeX μ is \mu, ν is \nu, ξ is \xi.

Here I want to illustrate $\langle X(s,\xi), Y(s,\xi), Z(s,\xi)\rangle$ vs. $z = u(x,y)$, that is $\langle x, y, u(x,y)\rangle$ with the unit sphere. The parametric equations of the unit sphere are: $\langle \cos\theta\sin\varphi,\ \sin\theta\sin\varphi,\ \cos\varphi\rangle$.

From these equations, and with $\underline{\cos\varphi \geq 0}$, how would we find the equation $z = \sqrt{1-x^2-y^2}$? . We would need to solve $\boxed{\begin{aligned} x &= \cos\theta\sin\varphi \\ y &= \sin\theta\sin\varphi \end{aligned}}$ for φ in terms of x and y and substitute for φ in formula for z

Here is how it goes: $x^2+y^2 = (\sin\varphi)^2 = 1-(\cos\varphi)^2$. Thus $(\cos\varphi)^2 = 1-x^2-y^2$. Since $\cos\varphi \geq 0$, $z = \sqrt{1-x^2-y^2}$

Now example: Transport Equation.

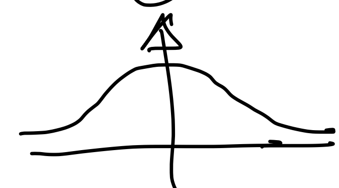
$$a u_x(x,t) + u_t(x,t) = 0$$

$a \in \mathbb{R}$ $a \neq 0$

The Vector field is $\langle a, 1, 0 \rangle$.

The initial condition:

$$u(x,0) = f(x) \qquad e^{-x^2}$$



characterist[ic] equations
$$
\begin{cases}
X'(s) = a \\
T'(s) = 1 \\
Z'(s) = 0
\end{cases}
$$

$$X(0) = \xi$$
$$T(0) = 0$$
$$Z(0) = f(\xi)$$

$Z(s)$ is constant

solve

$$X(s) = as + \xi$$
$$T(s) = s$$
$$Z(s) = f(\xi)$$

$$x = as + \xi$$
$$t = s$$

$$x = at + \xi$$
$$\xi = x - at$$

$$u(x,y) = f(x-at)$$

The sol. is $u(x,t) = f(x - at)$

$$u(x,0) = f(x - a\cdot 0) = f(x)$$

$$u_x(x,t) = f'(x-at)\cdot 1$$

$$u_t(x,t) = f'(x-at)(-a)$$

$$a u_x + u_t = a f'(x-at) + (-a) f'(x-at) = 0$$