More examples of MoC

Rieview of Transport Eq.

$a \in \mathbb{R}$ $\quad a u_x(x,t) + u_t(x,t) = 0$ $\left\{ \begin{array}{l} \text{state of } u \text{ at time } t=0 \text{ given} \\ u(x,0) = f(x) \end{array} \right.$

vect. field. $\vec{F}(x,y,z) = \langle a, 1, 0 \rangle$

solve ChEqs

~~we know this~~ 124 125 337

the Charact. equations

unknown functions $X(s), T(s), Z(s)$

seek curves in xtz-space

$\langle X(s), T(s), Z(s) \rangle$

$X'(s) = a$

$T'(s) = 1$

$Z'(s) = 0$

Z is constant

In. Cond. (Initial Cond.)

$X(0) = \xi$

$T(0) = 0$

$Z(0) = f(\xi)$

$X(s) = as + \xi$

$T(s) = s$

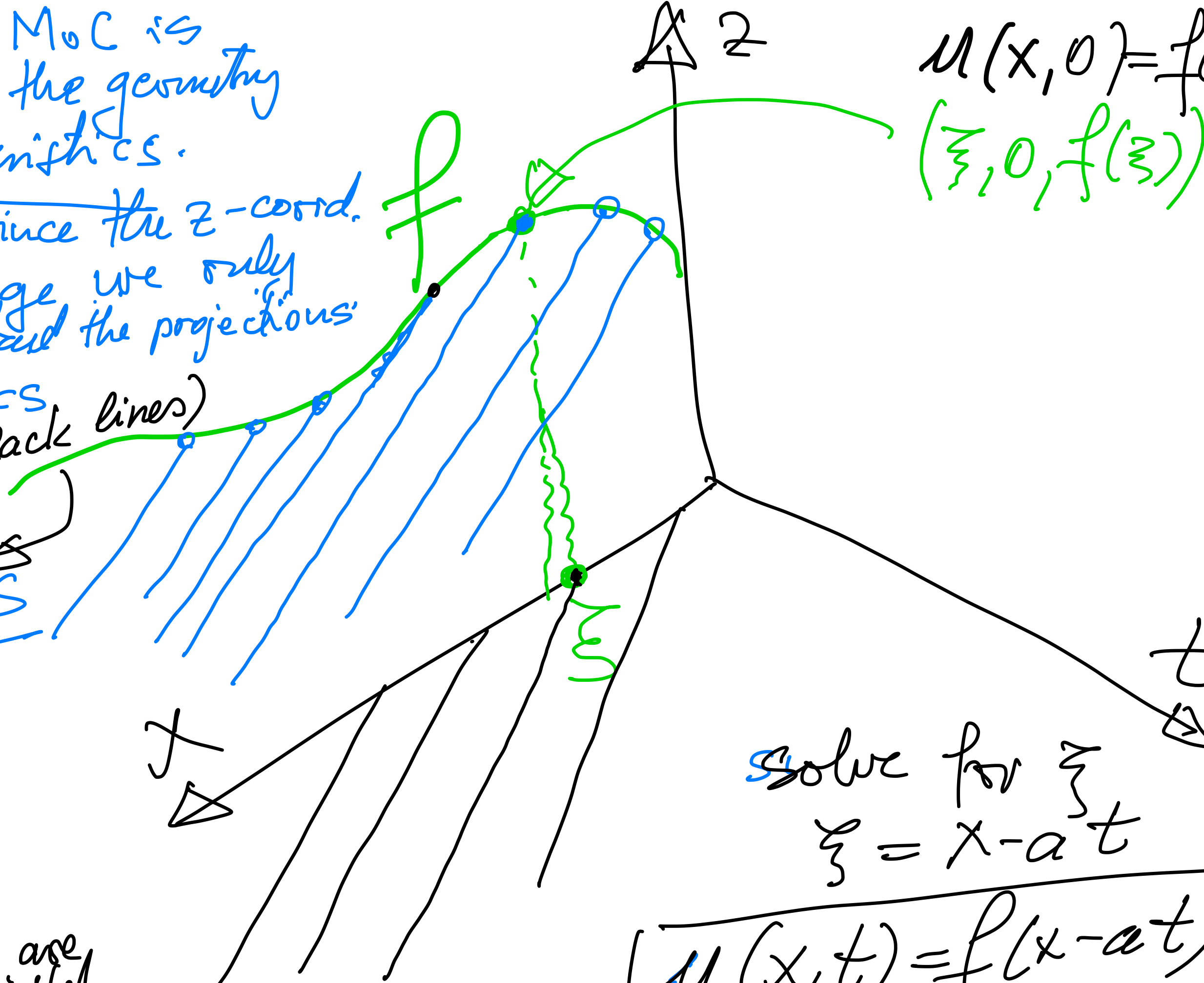
$Z(s) = f(\xi)$

vector equation for a characteristic is:

$\langle \underbrace{as+\xi, s}_{\text{projected characteristic}}, f(\xi) \rangle$

A big part of MoC is understanding the geometry of the characteristics.

In this case, since the z-coord. does not change we only need to understand the projections of characteristics xt-plane (black lines)
called projected characteristics



$u(x,0) = f(x)$

$(\xi, 0, f(\xi))$

$x = as + \xi$

$t = s$

$x = at + \xi$

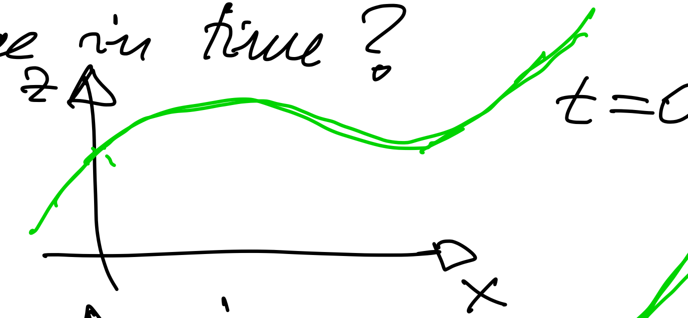
all projected charater. are parallel

solve for ξ

$\xi = x - at$

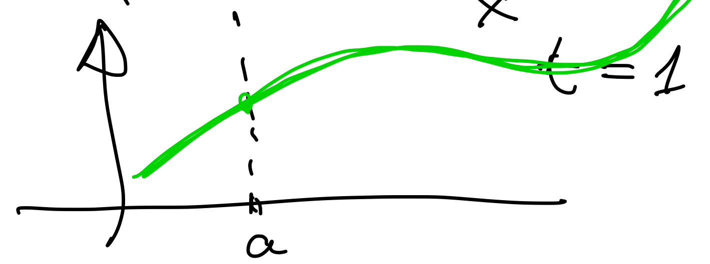
$$\boxed{u(x,t) = f(x-at)}$$

How does $u(x,t)$ change in time?



$t=0 \quad u(x,0) = f(x)$

$t=1 \quad u(x,1) = f(x-a)$



$$u(x,t) = f(x-at)$$

is just a rightward shift; the graph of $f(x)$ moves to the right at speed a units of x/unit of time.

Burgers' equation

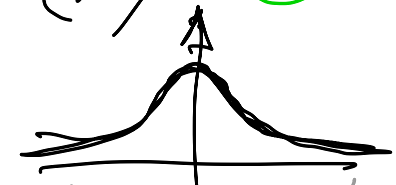
xtz-space

$z = u(x,t)$

"formula" for the graph of u

$$\boxed{u\,u_x + u_t = 0}$$

$$u(x,0) = f(x) \qquad e^{-x^2}$$



vector field for the characteristic equations

$$\langle z, 1, 0 \rangle$$

Initial conditions

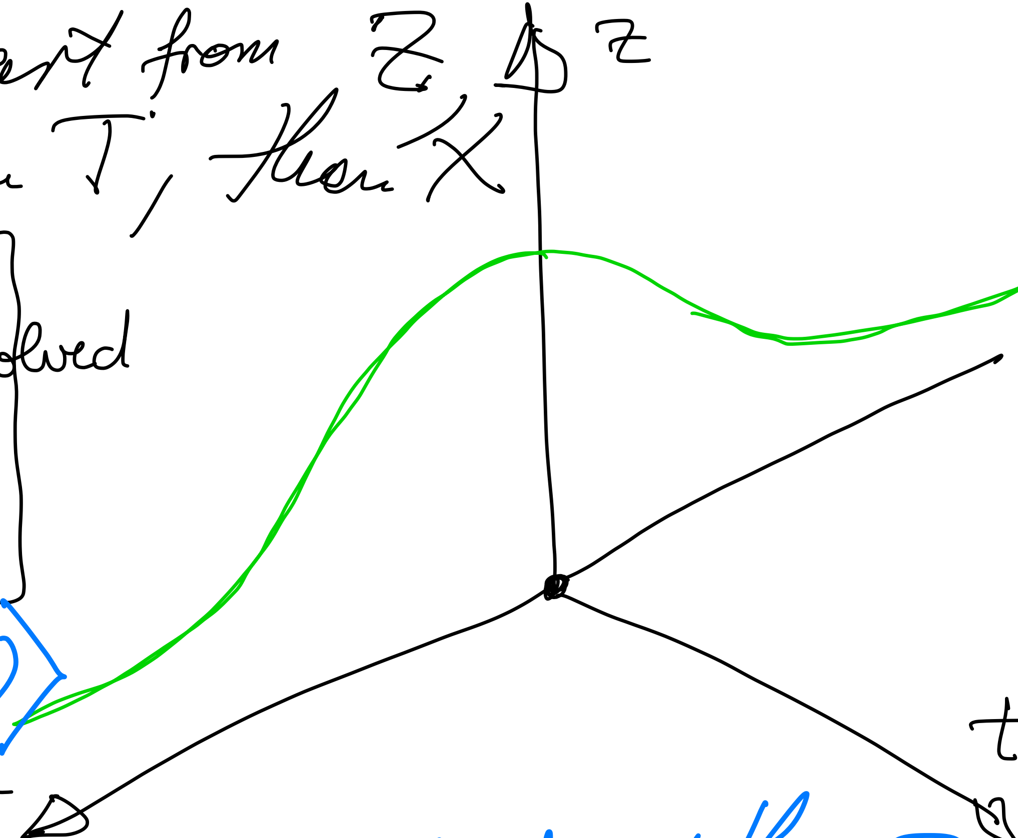
$$X'(s) = Z(s) \qquad X(0) = \xi$$

$$T'(s) = 1 \qquad T(0) = 0$$

$$Z'(s) = 0 \qquad Z(0) = f(\xi)$$

2

Solve! → Start from Z, then T, then X



$$\boxed{\begin{aligned} Z(s) &= f(\xi) \\ T(s) &= s \\ X(s) &= f(\xi)s+\xi \end{aligned}}$$

solved

$$\langle f(\xi)s+\xi,\ s,\ f(\xi)\rangle$$

s is the variable, ξ const.

these are the characteristics

for each ξ we get one characteristic

Understand the projections first! (to xt-plane)

Can you understand the projected characteristics?

$$\langle f(\xi)s+\xi,\ s\rangle \qquad f(\xi)=e^{-\xi^2}$$

these are lines, but how they are positioned? How they change with ξ?
Just plotting many of them in Mathematica would answer that, but it is interesting to play with this question just on paper.