Method of Characteristics

Two examples

Problem 4

$$u_x + x\,u_y = u \quad \text{in } \mathbb{R}^2$$

$$u(0,y) = g(y), \qquad y \in \mathbb{R}$$

$g(y) = y$

$g(y) = \cos y$

$$(x,y,z) \longmapsto \langle 1, x, z \rangle$$

Characteristic Equations

$$\boxed{\begin{aligned} X'(s) &= 1 \\ Y'(s) &= X(s) \end{aligned}}$$

$$Z'(s) = Z(s)$$

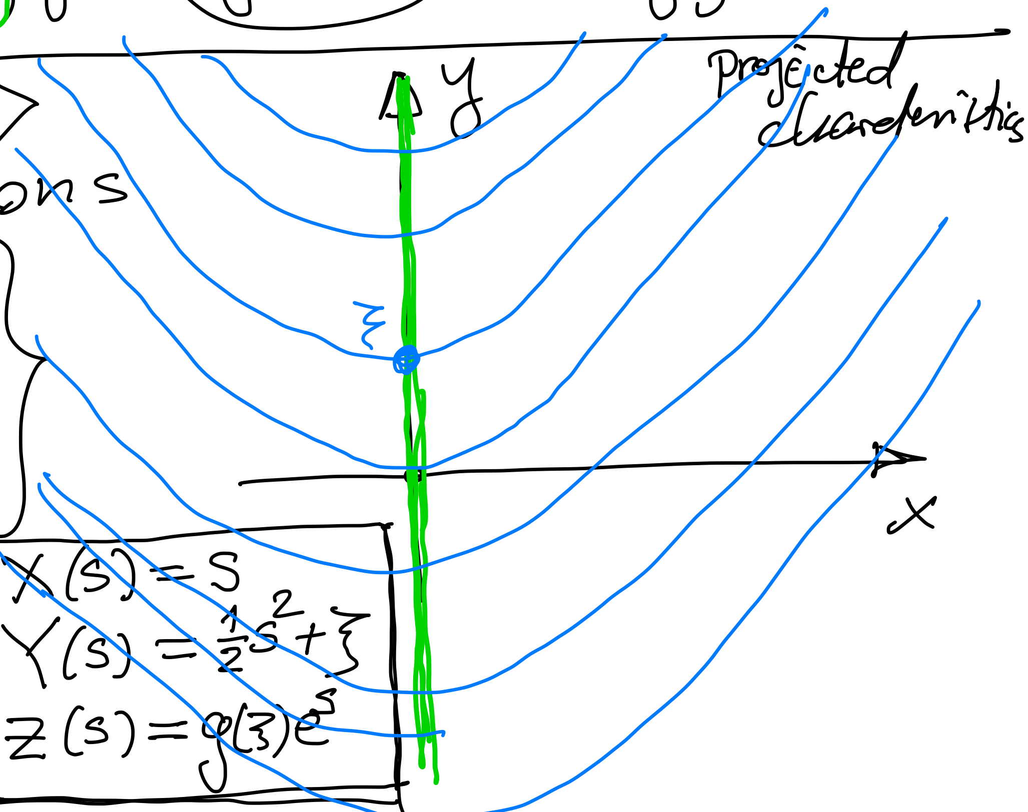
I.C.

$$X(0) = 0 \qquad \xi \in \mathbb{R}$$

$$Y(0) = \xi$$

$$Z(0) = g(\xi)$$

$$\boxed{\begin{aligned} X(s) &= s \\ Y(s) &= \tfrac{1}{2}s^2 + \xi \\ Z(s) &= g(\xi)e^{s} \end{aligned}}$$



The parametric equations of the solution's graph are

$$\left\langle \underbrace{s}_{x},\ \underbrace{\tfrac{1}{2}s^2+\xi}_{y},\ \underbrace{g(\xi)e^{s}}_{z=u(x,y)} \right\rangle = \left\langle x,\ y,\ u(x,y) \right\rangle$$

project. charac.

$$y=\frac{1}{2}x^2+\xi$$

$$\xi=y-\frac{1}{2}x^2$$

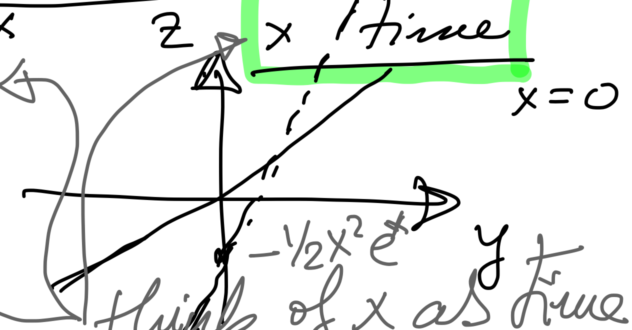
$$\boxed{u(x,y)=g\left(y-\tfrac{1}{2}x^2\right)e^{x}}$$

since g is def on $\mathbb{R}$, this funct. is def on $\mathbb{R}^2$

$g(y)=y$ $\qquad u(x,y)=\left(y-\frac{1}{2}x^2\right)e^{x}$

linear in y

$g(y)=\cos(y)$ $\qquad u(x,y)= e^{x}\cos\left(y-\frac{1}{2}x^2\right)$

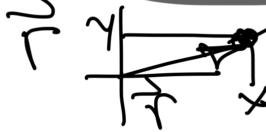


Problem 6

$$x\,u_x(x,y) + y\,u_y(x,y) = u(x,y)$$

$$u(\cos\theta, \sin\theta) = 1$$

$$\theta \in [0, 2\pi)$$



Vect field:

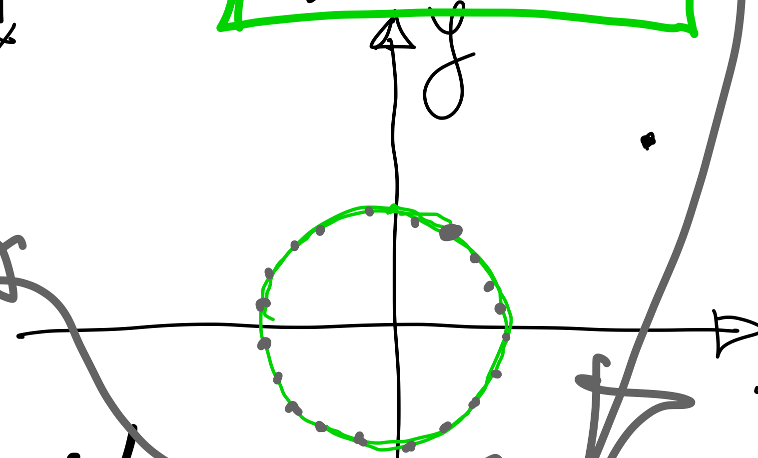
$(x,y,z) \mapsto \langle x, y, z \rangle$

CEs

$$X'(s) = X(s)$$
$$Y'(s) = Y(s)$$
$$Z'(s) = Z(s)$$

$$X(0) = \cos\theta$$
$$Y(0) = \sin\theta$$
$$Z(0) = 1$$

characteristic equations I.C.s



The Surface $z = u(x,y)$ must go through $(\cos\theta, \sin\theta, 1)$

$$\Rightarrow \left\langle \underbrace{(\cos\theta)e^{s}}_{x}, \underbrace{(\sin\theta)e^{s}}_{y}, \underbrace{e^{s}}_{z} \right\rangle$$

$$x^2+y^2=e^{2s} \qquad e^{s}=\sqrt{x^2+y^2} \qquad z=u(x,y)$$

Hence $u(x,y)=\sqrt{x^2+y^2}$, $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$

this function is not differentiable at $(0,0)$, so we need to exclude $(0,0)$ from the domain. A solution of a PDE must be differentiable on its domain.