Fourier's Method of Separation of Variables

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le L \qquad t \ge 0$$

boundary points

T_1

T_2

<u>BCs</u>

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}\right\} \text{Dirichlet BCs}$$

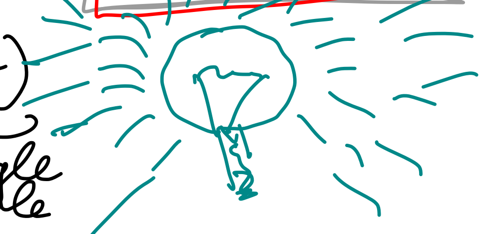
IC $\quad u(x,0) = f(x) \qquad 0 \le x \le L$

We understand that $u(x,t)$ with $x \in [0,L]$, $t \ge 0$

$\boxed{\text{two variables}}$

Try to solve PDE with special

$$\underset{\text{1 unknown}}{u}(x,t) = \underbrace{A(x)}_{\text{single variable}} * \underbrace{B(t)}_{\text{single variable}}$$



We hope to reduce PDE too two ODEs involving A and B separately.

$$\left.\begin{aligned}\frac{\partial u}{\partial t}(x,t) &= A(x)\,B'(t)\\ \frac{\partial^2 u}{\partial x^2}(x,t) &= A''(x)\,B(t)\end{aligned}\right\}\ \uparrow$$

We do not need zero solution, so

$$\left.\begin{aligned}A(x)&\neq 0\\ B(t)&\neq 0\end{aligned}\right\}\ \text{at many } x,t$$

$$A(x)\,B'(t) = \underline{k}\,A''(x)\,B(t)$$

SEPARATE THE VARIABLES:

$$\boxed{\boxed{\frac{B'(t)}{k\,B(t)}} = \boxed{\frac{A''(x)}{A(x)}} = -\lambda}$$

A drama is happening here

there is no change in the ratios

$$* \quad a(x) = b(y) \quad *$$

$$B'(t) = -\lambda k B(t)$$

$$B(t) = C_0 e^{-\lambda k t}$$

B is not a big deal once I get λ and A

$$A''(x) = -\lambda A(x)$$

$$A''(x) + \lambda A(x) = 0$$

We discussed this yesterday

$\lambda > 0$
$\lambda = 0$
$\lambda < 0$

$\lambda = 0$ $\quad \boxed{A(x) = C_1 x + C_2}$

$\lambda > 0$ set $\lambda = \mu^2$
$\mu > 0$

$$\boxed{A(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x)}$$

$\mu > 0$ $\lambda < 0$
$\lambda = -\mu^2$

$$\boxed{\begin{aligned} A(x) &= C_1 e^{-\mu x} + C_2 e^{\mu x} \\ &= C_1 \operatorname{ch}(\mu x) + C_2 \operatorname{sh}(\mu x) \end{aligned}}$$

λ will come from the BCs

BCs are $u(0,t) = A(0)\underbrace{B(t)}_{} = 0 \quad \forall t$

$$\boxed{A(0) = 0}$$

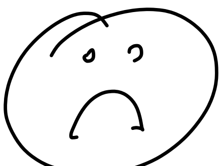
$u(L,t) = 0 \Rightarrow \boxed{A(L) = 0}$

Case 1 $\lambda = 0$

$A(x) = c_1 x + c_2$

$A(0) = c_2 = 0$ $\qquad c_2 = 0$

$A(L) = c_1 L = 0, \; L \neq 0 \Rightarrow c_1 = 0$



not a solution

C means cos

S means sin

Case 2 $\quad \lambda > 0 \quad A(x) = C_1 c(\mu x) + C_2 s(\mu x)$

$\mu > 0 \quad \lambda = \mu^2 \quad 0 = A(0) = C_1 c(\mu 0) = \boxed{C_1 = 0}$

$0 = A(L) = C_2 \boxed{\sin(\mu L) = 0}$

sol. for μ

$\sin(\mu L) = 0 \iff \underbrace{\mu}_{>0} L = m\pi \quad m \in \mathbb{N} = \{1, 2, 3, \ldots\}$

$$\boxed{\mu_m = \frac{m\pi}{L} \text{ with } m \in \mathbb{N}}$$

$$\lambda = \left(\frac{m\pi}{L}\right)^2 \text{ with } m \in \mathbb{N}$$

$$A_m(x) = \sin\left(\frac{m\pi}{L} x\right) \quad m \in \mathbb{N}$$

$$B_m(t) = C_0 e^{-\left(\frac{m\pi}{L}\right)^2 k t}$$

$$\boxed{u_m(x,t) = C_0 e^{-\left(\frac{m\pi}{L}\right)^2 k t} \sin\left(\frac{m\pi}{L} x\right)}$$

$m \in \mathbb{N}$

So, the first few solutions that we obtained are:

$$e^{-\left(\frac{\pi}{L}\right)^2 k t} \sin\left(\frac{\pi}{L} x\right), \quad e^{-\left(\frac{2\pi}{L}\right)^2 k t} \sin\left(\frac{2\pi}{L} x\right), \ldots$$

$m=1$ $\qquad\qquad$ $m=2$