Solving the heat equation using the Separation of variables method

$$\frac{\partial u}{\partial t}(x,t) = \varkappa \frac{\partial^2 u}{\partial x^2}(x,t) \qquad 0 \le x \le L$$
$$t \geq 0$$

\kappa $\varkappa > 0$

Dirichlet boundary conditions: $u(0,t) = 0, \; u(L,t) = 0, \quad t \geq 0$

I.C. $u(x,0) = f(x) \qquad 0 \le x \le L$

$$u(x,t) = \boxed{A(x)\, B(t)}$$

one unknown function $u(x,t)$ is exchanged to three unknown objects

$A, B \quad \lambda$

$\in \mathbb{R}$

$$\frac{B'(t)}{\varkappa B(t)} = \frac{A''(x)}{A(x)} = -\lambda$$

separation of variables

$B' = -\lambda \varkappa B \qquad B(t) = b\, e^{-\lambda \varkappa t}$

arb. constant

The equation for A & λ is:

$$-A''(x) = \lambda A(x)$$

B.C. $\quad A(0)=0, A(L)=0$

This is an eigenvalue problem for

$$\left(-\frac{d^2}{dx^2}\right)A = \lambda A$$

subject to the BCs $A(0)=0, A(L)=0$

This is an eigenvalue - eigenfunction problem. Like 204-304

We solved this problem and found out that the eigenvalues are $\lambda_m = \left(\frac{m\pi}{L}\right)^2$ with the corresponding eigenfunctions $\sin\left(\frac{m\pi}{L}x\right)$.

$$-\frac{d^2}{dx^2}\left(\sin\left(\frac{m\pi}{L}x\right)\right) = -\frac{d}{dx}\left(\frac{m\pi}{L}\cos\left(\frac{m\pi}{L}x\right)\right) = \left(\frac{m\pi}{L}\right)^2 \sin\left(\frac{m\pi}{L}x\right) = \lambda_m$$

Here $m \in \mathbb{N} = \{1, 2, \dots\}$

The result of the SOV method is that we have a sequence of "product solutions":

$$u_m(x,t) = b_m e^{-\left(\frac{m\pi}{L}\right)^2 \varkappa t} \sin\left(\frac{m\pi}{L} x\right)$$

$m \in \mathbb{N}$

Since our problem is linear each linear combination of these solutions will also be a solution of the heat equation together with BCs.

$$u(x,t) = \sum_{m=1}^{\infty} b_m e^{-\left(\frac{m\pi}{L}\right)^2 \varkappa t} \sin\left(\frac{m\pi}{L} x\right)$$

arbitrary $b_m \in \mathbb{R}$

How to satisfy the IC.? Can I choose $b_1, b_2, b_3, \ldots$ such that

$$f(x) = u(x,0) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi}{L}x\right)$$

$0 \le x \le L$

initial temperature distribution.

This looks like an impossible task. However, the orthogonality property of $\sin\left(\frac{m\pi}{L}x\right)$, $m \in \mathbb{N}$:

this means:

$$\int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = 0$$

$m \neq n$

$m, n \in \mathbb{N}$

orthogonality

Prove? See the notes of Friday.

This integral plays the role of dot prod.

What we will do next, is very similar to working with symmetric matrices in 304. S $n\times n$ symmetric matrix

$$S\vec{v} = \lambda \vec{v} \quad \text{an eigenvalue-eigenvector problem}$$

there always exist $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and orthogonal set of eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ such that $S\vec{v}_k = \lambda_k \vec{v}_k$, $k = 1, \dots, n$.

$\vec{v}_k \cdot \vec{v}_j = 0$ whenever $j \neq k$.

The big thing in M304 is $\forall \vec{x} \in \mathbb{R}^n$ we have

$$\vec{x} = \sum_{k=1}^{n} \underbrace{\frac{\vec{x}\cdot\vec{v}_k}{\vec{v}_k\cdot\vec{v}_k}}_{\alpha_k} \vec{v}_k$$

$$\vec{x} = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n$$

$$\vec{x}\cdot\vec{v}_k = \alpha_1 \vec{v}_1\cdot\vec{v}_k + \alpha_2 \vec{v}_2\cdot\vec{v}_k + \dots + \alpha_n \vec{v}_n\cdot\vec{v}_k$$

$$j \neq k \qquad \vec{v}_j \cdot \vec{v}_k = 0$$

$$\boxed{\vec{x}\cdot\vec{v}_k = \alpha_k\, \vec{v}_k\cdot\vec{v}_k}$$

Exactly the same logic works for functions $\sin\left(\frac{m\pi}{L}x\right)$, $m \in \mathbb{N}$.

We want

$$f(x) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi}{L}x\right) \qquad * \sin\left(\frac{k\pi}{L}x\right)$$

$$0 \le x \le L$$

$$f(x)\sin\left(\frac{k\pi}{L}x\right)=\sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{k\pi}{L}x\right)$$

$$\int_0^L f(x)\sin\left(\frac{k\pi}{L}x\right)dx=\underbrace{\sum_{m=1}^{\infty} b_m \underbrace{\int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{k\pi}{L}x\right)dx}_{=0 \quad m\neq k}}_{b_k \int_0^L \left(\sin\left(\frac{k\pi}{L}x\right)\right)^2 dx}$$

$$b_k=\frac{\int_0^L f(x)\sin\left(\frac{k\pi}{L}x\right)dx}{\int_0^L \left(\sin\left(\frac{k\pi}{L}x\right)\right)^2 dx \cdots\cdots = \frac{L}{2}}$$

$k \in \mathbb{N}$

$$b_k = \frac{2}{L} \underbrace{\int_0^L f(x) \sin\left(\frac{k\pi}{L} x\right) dx}_{\text{inner product for functions}}$$