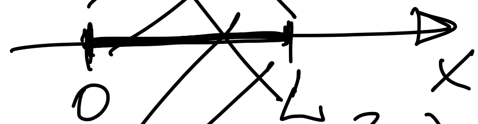
# Laplace's Equation

# in a Rectangle

$$\cancel{\frac{\partial u}{\partial t} = \varkappa \frac{\partial^2 u}{\partial x^2}}$$

Equilibrium solutions

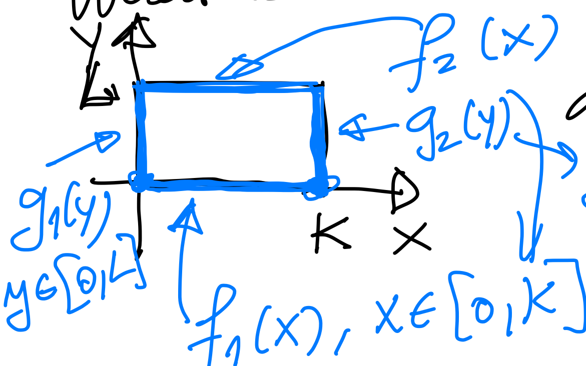



in two dimensions

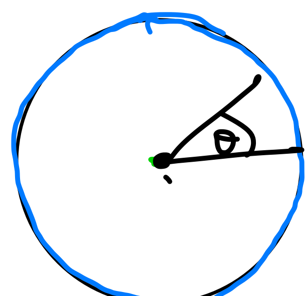
$$\frac{\partial u}{\partial t} = \varkappa \underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}_{=0}$$

+ BCs

What is the shape of our heated plate?




and




circular plate

disk

$f(\theta)$ given

$-\pi \le \theta \le \pi$

# Laplaces Equation in a rectangle:

$$[0,K]\times[0,L] = \{(x,y)\in\mathbb{R}^2 : x\in[0,K],\ y\in[0,L]\}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$u(0,y) = u(K,y) = 0$

$x\in[0,K]$

Subject to BCs $u(x,0) = f_1(x),\ u(x,L) = f_2(x)$

$u(x,0) = 0 = u(x,L)$

$u(0,y) = g_1(y),\ u(K,y) = g_2(y)$

$y\in[0,L]$

We split this problem in two problems:

$$\underbrace{u(x,y)}_{\text{this is the sol. of the original problem.}} = u_1(x,y) + u_2(x,y)$$

# Solving the orange problem.

Use SoV $\quad u(x,y) = A(x)B(y)$

ignore $f_1, f_2$ just use

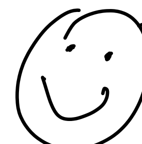
$$\boxed{u(0,y) = u(K,y) = 0 \quad \forall y \in [0,2]}$$

$$\Downarrow$$

$$A(0)B(y) = A(K)B(y) = 0$$

$$\Downarrow$$

$$\boxed{\begin{aligned} A(0) &= 0 \\ A(K) &= 0 \end{aligned}}$$



PDE $\quad A''(x)B(y) + A(x)B''(y) = 0$

SoV $\quad \underbrace{-\frac{A''(x)}{A(x)}}_{\text{BCs avail.}} = \frac{B''(y)}{B(y)} = \lambda$

I have eigenvalue problem for A

$$\boxed{\begin{aligned} \left(-\frac{d^2}{dx^2}\right)A &= \lambda A \\ A(0) &= 0 \\ A(K) &= 0 \end{aligned}}$$

Seen before :

orthogonality comes from the fact that sin-s are eigenfunctions.

Sols are

$\lambda_m = \left(\frac{m\pi}{K}\right)^2, \quad \sin\left(\frac{m\pi}{K} x\right)$

$m \in \mathbb{N}$

The equation for B is

$$B''(y) - \left(\frac{m\pi}{K}\right)^2 B(y) = 0$$

just a ODE + we accept all sol.

The fundamental set of solutions is

$$\left\{ \exp\left(-\frac{m\pi}{K} y\right), \exp\left(\frac{m\pi}{K} y\right) \right\}$$

A calculus student could give

A more advanced calculus trick

$$\left\{ \cosh\left(\frac{m\pi}{K}y\right), \sinh\left(\frac{m\pi}{K}y\right) \right\}$$

we need good behavior at $y = 0$ and $y = L$


at $0 = 1$
at $L = 0$

at $0 = 0$
at $L = 1$

need

$$u(x,y) = A(x)B(y)$$

$$u(x,0) = A(x)\underbrace{B(0)}_{\text{either 0 or 1}}$$

$$u(x,L) \qquad B(L)$$

Based on what we want I choos the fundamental set of solutions to be

$$\left\{ \frac{\operatorname{sh}\left(\frac{m\pi}{K} y\right)}{\operatorname{sh}\left(\frac{m\pi}{K} L\right)}, \frac{\operatorname{sh}\left(\frac{m\pi}{K}(L-y)\right)}{\operatorname{sh}\left(\frac{m\pi}{K} L\right)} \right\}$$

Finally I have the product solutions

$m \in \mathbb{N}$

$$u_{m,1}(x,y) = \frac{\sin\left(\frac{m\pi}{K} x\right) \operatorname{sh}\left(\frac{m\pi}{K} y\right)}{\operatorname{sh}\left(\frac{m\pi}{K} L\right)}, \quad \frac{\sin\left(\frac{m\pi}{K} x\right) \operatorname{sh}\left(\frac{m\pi}{K}(L-y)\right)}{\operatorname{sh}\left(\frac{m\pi}{K} L\right)} = u_{m,2}(x,y)$$

Our orange solution is, by superposition

$$\boxed{u_1(x,y) = \sum_{m=1}^{\infty} a_{m,1}\, u_{m,1}(x,y) + \sum_{m=1}^{\infty} a_{m,2}\, u_{m,2}(x,y)}$$

I bring back $f_1$ & $f_2$

$$f_1(x) = u(x,0) = \sum a_{m,1} \underbrace{u_{m,1}(x,0)}_{=0} + \sum a_{m,2} \underbrace{u_{m,2}(x,0)}$$

$$= \sum a_{m,2} \sin\left(\frac{m\pi}{K}x\right)$$

$$a_{m,2} = \frac{2}{K}\int_0^K f_1(x)\sin\left(\frac{m\pi}{K}x\right)dx$$

$$f_2(x) = u(x,L) = \sum a_{m,1} u_{m,1}(x,L) + \underbrace{\sum a_{m,2} u_{m,2}(x,L)}_{=0}$$

$$f_2(x) = \sum a_{m,1} \sin\left(\frac{m\pi}{K} x\right)$$

$$a_{m,1} = \frac{2}{K} \int_0^K f_2(x) \sin\left(\frac{m\pi}{K} x\right) dx$$

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$$u_2(x,y) = \sum b_{m,1} \sin\left(\frac{m\pi}{L} y\right) \frac{\operatorname{sh}\left(\frac{m\pi}{L} x\right)}{\operatorname{sh}\left(\frac{m\pi}{L} K\right)} + $$

$$+ \sum b_{m,2} \sin\left(\frac{m\pi}{L} y\right) \frac{\operatorname{sh}\left(\frac{m\pi}{L}(K-x)\right)}{\operatorname{sh}\left(\frac{m\pi}{L} K\right)}$$

$$b_{m,2} = \frac{2}{L} \int_0^L g_1(y) \sin\left(\frac{m\pi}{L} y\right) dy, \quad b_{m,1} = \frac{2}{L} \int_0^L g_2(y) \sin\left(\frac{m\pi}{L} y\right) dy$$