More on Laplace's Equation in a Disk

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$$

$0 \le r \le R$

$-\pi \le \theta \le \pi$

Subject to B.C. $u(R,\theta)=f(\theta)$

$$\begin{cases} u(r,-\pi)=u(r,\pi), \ \forall r\in[0,R] \\ \frac{\partial u}{\partial \theta}(r,-\pi)=\frac{\partial u}{\partial \theta}(r,\pi), \ \forall r\in[0,R] \end{cases}$$

Ignoring the BC $u(R,\theta)=f(\theta)$ we obtained the sequence of ~~product~~ solutions:

$$1, \ \left(\frac{r}{R}\right)^n\cos(n\theta), \ \left(\frac{r}{R}\right)^n\sin(n\theta)$$

It is a prudent thing to do to verify that these functions truly solve Laplace's Eq..

The "general solution" of Lap. Eq ignoring B.C. is

$$u(r,\theta) = a_0 \cdot 1 + \sum_{n=1}^{\infty} a_n \left(\frac{r}{R}\right)^n \cos(n\theta) + \sum_{n=1}^{\infty} b_n \left(\frac{r}{R}\right)^n \sin(n\theta)$$

For $u(r,\theta)$ to satisfy B.C. $u(R,\theta) = f(\theta)$ we choose a_0, a_n, b_n to be:

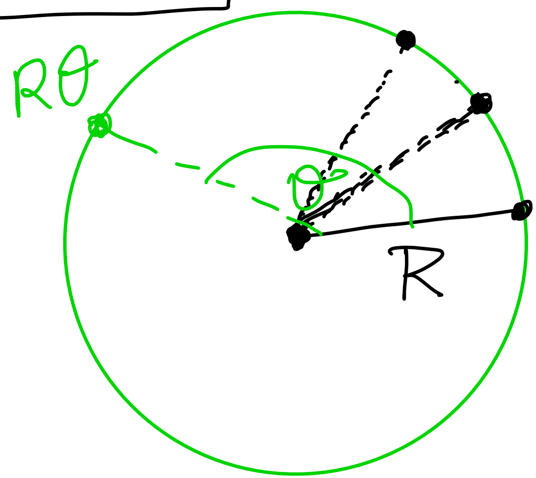
$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta, \quad n \in \mathbb{N}$$

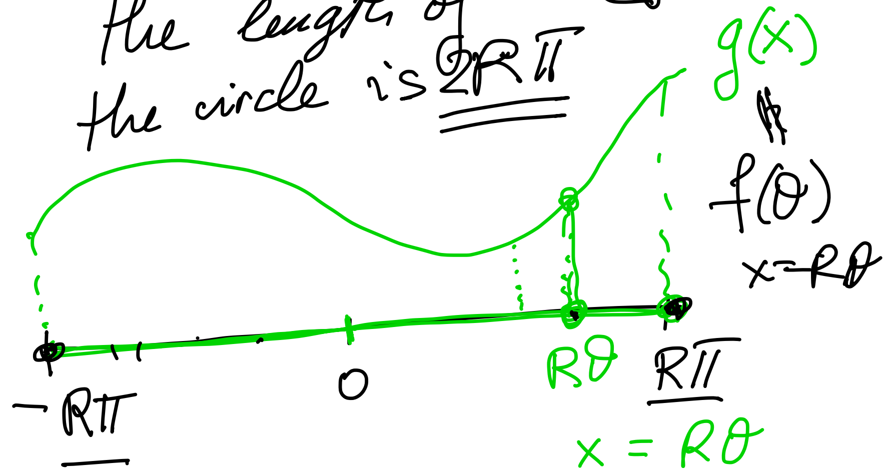
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta, \quad n \in \mathbb{N}$$

A remarkable property of the solution $u(r,\theta)$ is that the value of $u(0,0)$ at the center of the disk is

$$\boxed{u(0,0)} = a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta \overset{\text{proved below}}{=} \left\{ \begin{array}{l} \text{the average} \\ \text{temperature} \\ \text{along the} \\ \text{boundary} \\ \text{of the disk} \end{array} \right.$$



the length of the circle is $\underline{\underline{2R\pi}}$



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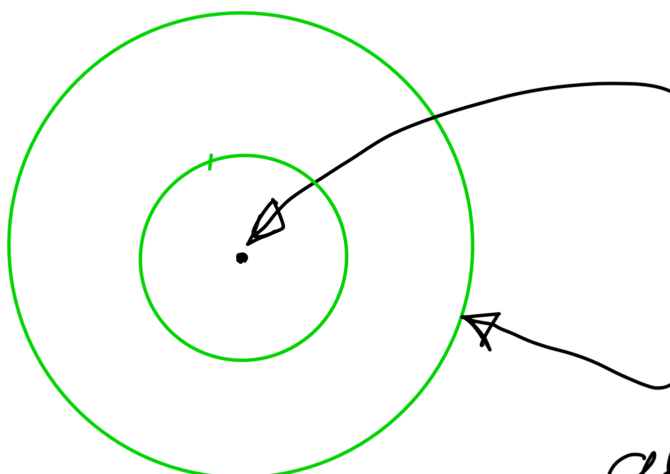
average value of g: $\frac{1}{2R\pi}\int_{-R\pi}^{R\pi} g(x)\,dx = \left| \begin{array}{l} x = R\theta \\ dx = R\,d\theta \\ \begin{array}{c|c} x & \theta \\ \hline -R\pi & -\pi \\ R\pi & \pi \end{array} \\ g(x) = f(\theta) \end{array} \right| = \frac{1}{2\cancel{R}\pi}\int_{-\pi}^{\pi} f(\theta)\cancel{R}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$

$g(x) = f(\theta)$

$g(R\theta) = f(\theta)$

$-\pi \le \theta \le \pi$



the value of $u(0,0)$ is the average of the values of $u(R,\theta)$. True for all R

$$\min\{u(R,\theta) : \theta\in[-\pi,\pi]\} \le u(0,0) \le \max\{u(R,\theta) : \theta\in[-\pi,\pi]\}$$

This property implies that a solution of Laplace's equation cannot take neither maximum nor minimum in any region.