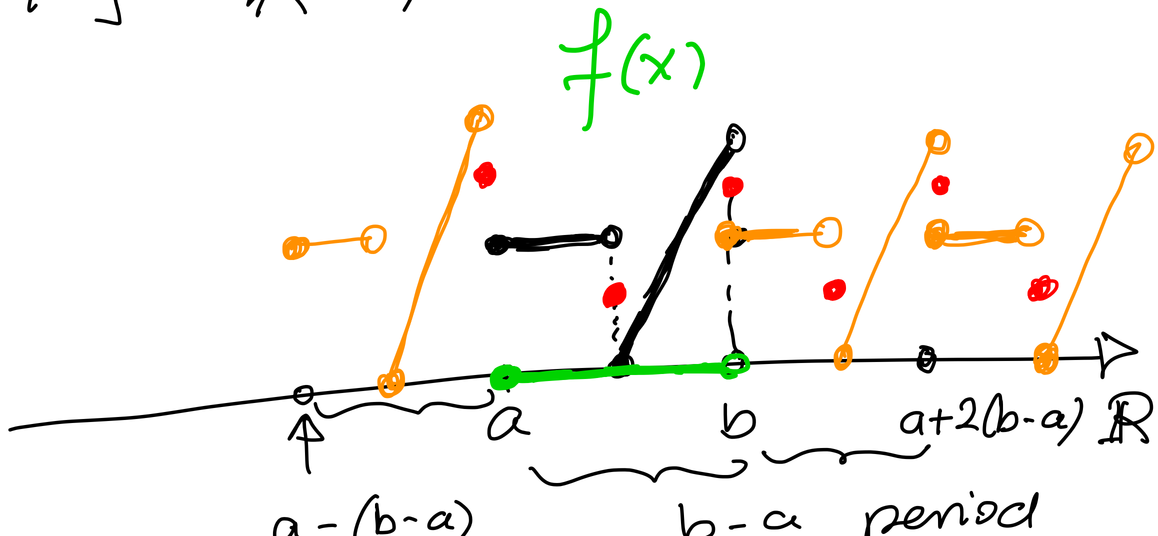
Fourier Periodic Extension and Convergence Theorems

We defined a piecewise continuous and piecewise smooth function.

$f:[a,b]\to\mathbb{R}$, where $a,b\in\mathbb{R}$, $a<b$



The first observation is $\forall x\in\mathbb{R}\ \exists k\in\mathbb{Z}$ such that

$$\boxed{a + k(b-a) \le x < a + (k+1)(b-a)}$$

$k = 0$ $\quad a \le x < a + (b-a) = b$

Given $x \in \mathbb{R}$ calculate k:

$$k(b-a) \le x - a < (k+1)(b-a) \quad / \div\; b-a>0$$

$$k \le \frac{x-a}{b-a} < k+1$$

$\mathbb{Z}$

$v \in \mathbb{R}$
$\lfloor v \rfloor$ is the largest integer which is $\le v$
$\lfloor \pi \rfloor = 3$

By definition of the floor function

$$k = \left\lfloor \frac{x-a}{b-a} \right\rfloor$$

$$\boxed{a \le x - \left\lfloor \frac{x-a}{b-a} \right\rfloor (b-a) < a + (b-a) = b}$$

$\forall x \in \mathbb{R} \quad \underbrace{x - \left\lfloor \frac{x-a}{b-a} \right\rfloor (b-a)} \in [a, b)$ $\qquad f : [a,b) \to \mathbb{R}$

Define a new function

$$\forall x \in \mathbb{R} \quad \underbrace{\tilde{f}(x)} = f\left(x - \left\lfloor \frac{x-a}{b-a} \right\rfloor (b-a) \right)$$

this function is called the periodic extension of $f : [a,b) \to \mathbb{R}$

If $\quad f : (a,b] \to \mathbb{R} \qquad (-L, L]$

$\forall x \in \mathbb{R} \ \exists k \in \mathbb{Z} \quad a+(k-1)(b-a) < x \leq a + k(b-a)$

$k=1 \quad a < x \leq b$

$k=0 \quad a-(b-a) < x \leq a$

Calculate $k \in \mathbb{Z}$ given $x \in \mathbb{R}$

$$k-1 < \frac{x-a}{b-a} \leq k$$

By the def. of ceiling $\quad k = \left\lceil \frac{x-a}{b-a} \right\rceil$

$\forall v \in \mathbb{R}$ $\lceil v \rceil$ is the smallest integer which is $\geq v$

for ceiling I look up

The periodic extension of $f:(a,b] \to \mathbb{R}$ is

$$\tilde{f}(x) = f\left(x - \left(\left\lceil \frac{x-a}{b-a} \right\rceil - 1\right)(b-a)\right)$$

$$\left\lceil \frac{x-a}{b-a} \right\rceil - 1 = \left\lceil \frac{x-a}{b-a} - 1 \right\rceil = \left\lceil \frac{x-a-b+a}{b-a} \right\rceil = \left\lceil \frac{x-b}{b-a} \right\rceil$$

It seems that all at this page is correct and the final formula is the same as

$$\forall x \in \mathbb{R} \quad \exists k \in \mathbb{Z} \quad b+(k-1)(b-a) < x \leq b + k(b-a)$$

$k=0 \quad b+(-1)(b-a)$

$$\overset{\shortparallel}{a} < x \leq b$$

by each $k \in \mathbb{Z}$ we are moving the interval by $b-a$ units

$$k-1 < \frac{x-b}{b-a} \leq k$$

$$k = \left\lceil \frac{x-b}{b-a} \right\rceil$$

For $f: (a,b] \to \mathbb{R}$ the periodic extension is

$$\tilde{f}(x) = f\left(x - \left\lceil \frac{x-b}{b-a} \right\rceil (b-a)\right)$$

We define the FOURIER PERIODIC EXTENSION of $f:[a,b]\to\mathbb{R}$ to be the function defined as follows:

$$\tilde{f}_{\text{Fourier}}(x) = \begin{cases} \tilde{f}(x) & \text{if } \tilde{f} \text{ is continuous at } x \\ \frac{1}{2}\left(\underbrace{\tilde{f}(x+)}_{=\lim\limits_{t\downarrow 0}\tilde{f}(x+t)} + \underbrace{\tilde{f}(x-)}_{=\lim\limits_{t\uparrow 0}\tilde{f}(x+t)}\right) & \text{if } \tilde{f} \text{ has discont. at } x \end{cases}$$

<u>Fourier Convergence Theorem</u> Let $f:[-L,L]\to\mathbb{R}$ be a <u>piecewise SMOOTH</u> function. Let $n\in\mathbb{N}$ and

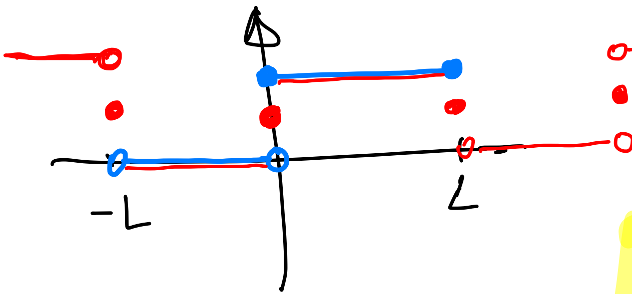
$$S_n^f(x) = a_0 + \sum_{k=1}^{n} a_k \cos\left(\frac{k\pi}{L}x\right) + \sum_{k=1}^{n} b_k \sin\left(\frac{k\pi}{L}x\right)$$

where $$a_0 = \frac{1}{2L}\int_{-L}^{L} f(\xi)\,d\xi$$

$\forall k\in\mathbb{N}$ $$a_k = \frac{1}{L}\int_{-L}^{L} f(\xi)\cos\left(\frac{k\pi}{L}\xi\right)d\xi, \quad b_k = \frac{1}{L}\int_{-L}^{L} f(\xi)\sin\left(\frac{k\pi}{L}\xi\right)d\xi.$$

then $\forall x\in\mathbb{R}$ $$\lim_{n\to+\infty} S_n^f(x) = \tilde{f}_{\text{Fourier}}(x)$$

$L>0$ unit step function $f(x) = \begin{cases} 0, & x \in (-L,0) \\ 1, & x \in [0,L] \end{cases}$



$$\boxed{a_0 = \frac{1}{2L} L = \frac{1}{2}}$$

$$\boxed{a_k} = \frac{1}{L}\int_0^L \cos\left(\frac{k\pi}{L}\xi\right) d\xi = $$

$$= \frac{1}{L}\,\frac{1}{\frac{k\pi}{L}} \sin\left(\frac{k\pi}{L}\xi\right)\Big|_0^L = \boxed{0}$$

$$\boxed{b_k} = \frac{1}{L}\int_0^L \sin\left(\frac{k\pi}{L}\xi\right) d\xi = -\frac{1}{\cancel{L}}\,\frac{1}{\frac{k\pi}{\cancel{L}}} \cos\left(\frac{k\pi}{L}\xi\right)\Big|_0^L$$

$$= +\frac{1}{k\pi}\left(1 - \cos(k\pi)\right) = \boxed{\frac{1}{k\pi}\left(1-(-1)^k\right)}$$

Thus $a_0 = \frac{1}{2}$, $\underset{k \in \mathbb{N}}{a_k = 0}$, $b_{2j-1} = \frac{2}{(2j-1)\pi}$, $j \in \mathbb{N}$

Hence the Fourier series of the unit step function on $[-L, L]$ if

$$f(x) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{j=1}^{\infty} \frac{1}{2j-1} \sin\left((2j-1)\frac{\pi}{L} x\right)$$

$$\sim \frac{1}{2} + \frac{2}{\pi} \sin\left(\frac{\pi}{L} x\right) + \frac{2}{\pi} \frac{1}{3} \sin\left(3\frac{\pi}{L} x\right) + \frac{2}{\pi} \frac{1}{5} \sin\left(5\frac{\pi}{L} x\right) + \vdots$$

	value
$x = 0$	$1/2$
$x = L$	$1/2$
$x = L/2$	$\frac{1}{2} + \frac{2}{\pi} \sum_{j=1}^{\infty} \frac{1}{2j-1} \overbrace{\sin\left((2j-1)\frac{\pi}{2}\right)}^{= (-1)^{j-1}}$

$$= \frac{1}{2} + \frac{2}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2j-1}$$

By the Fourier Convergence Theorem we conclude

$$\frac{1}{2} + \frac{2}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2j-1} = 1$$

That is $$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2j-1} = \frac{\pi}{4}$$

We get the same result by evaluating the value of the Fourier series at $x = -L/2$.