Differentiating Fourier Series

Math 124

$$\frac{d}{dx}\left(f(x)+g(x)\right)=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

can I put infinitely many functions here?

Assume

① $f:[-L,L]\to\mathbb{R}$ is continuous and piecewise smooth $\left(\forall x\in(-L,L)\ f(x)=\text{its F.S.}\right)$

② f' is piecewise smooth (the F.S. of f' converges to the Fourier periodic ext of f')

Explore the connection between these two Fourier Series.

$$f(x)=a_0+\sum_{k=1}^{\infty}a_k\cos\left(\frac{k\pi}{L}x\right)+\sum_{k=1}^{\infty}b_k\sin\left(\frac{k\pi}{L}x\right)$$

all $x\in(-L,L)$.

Now calculate the F.S. for f'

The coefficients are

$$a_0' = \frac{1}{2L}\int_{-L}^{L} f'(\xi)\,d\xi \overset{FTC}{=} \frac{1}{2L}\big(f(L)-f(-L)\big)$$

Integration by Parts

$$\int_a^b u'v\,d\xi = uv\Big|_a^b - \int_a^b v'u\,d\xi$$

$$a_k' = \frac{1}{L}\int_{-L}^{L} f'(\xi)\cos\left(\frac{k\pi}{L}\xi\right)d\xi =$$

$u = f$

$v = \cos(\ )$

$$= \frac{1}{L}\left(f(\xi)\cos\left(\frac{k\pi}{L}\xi\right)\Big|_{-L}^{L} + \frac{k\pi}{L}\underbrace{\int_{-L}^{L} f(\xi)\sin\left(\frac{k\pi}{L}\xi\right)d\xi}_{b_k}\right)$$

$$= \frac{1}{L}\left(f(L)(-1)^k - f(-L)(-1)^k\right) + \text{-----}$$

$$= \frac{(-1)^k}{L}\big(f(L)-f(-L)\big) + \frac{k\pi}{L}b_k$$

$$b_k' = \frac{1}{L}\left(-k\pi a_k\right)$$

The formal derivative of the F.S. of f is

$$\sum \underbrace{-a_k \frac{k\pi}{L}}_{b_k'} \sin\left(\frac{k\pi}{L}x\right) + \sum \underbrace{b_k \frac{k\pi}{L}}_{a_k'} \cos\left(\frac{k\pi}{L}x\right)$$

We see that this series is identical to the F.S. of f' iff $f(L) = f(-L)$.
That is iff the periodic ext. of f is continuous.