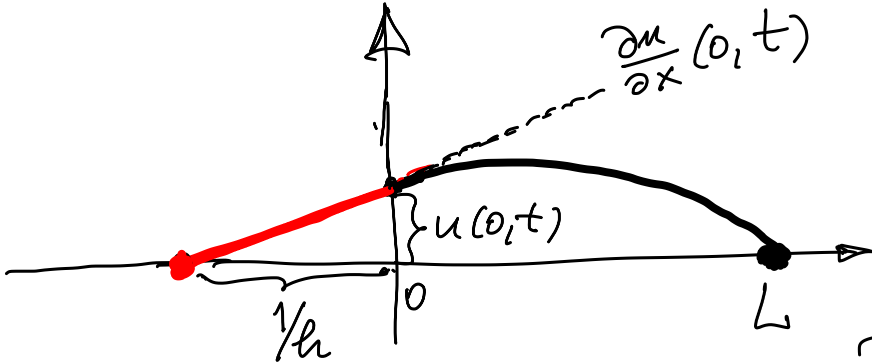
Solving a Sturm-Liouville Problem



1. Separate variables

$$u(x,t) = A(x)B(t)$$

$$A(x)\,B''(t) = c^2 A''(x)\,B(t)$$

$$\frac{B''(t)}{c^2 B(t)} = \frac{A''(x)}{A(x)} = -\lambda$$

$$\boxed{B''(t) = -\lambda c^2 B(t)}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{u(0,t)}{1/h}$$

BCs $\boxed{\begin{aligned} &\frac{\partial u}{\partial x}(0,t) - h\,u(0,t) = 0 \\ &u(L,t) = 0 \end{aligned}}$

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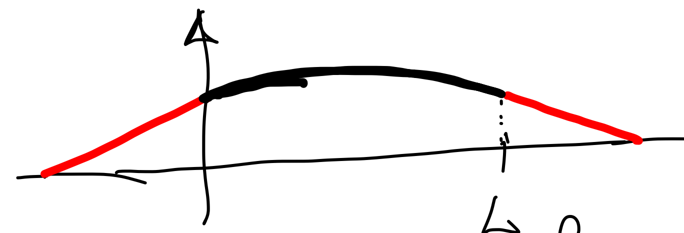


IC: $u(x,0) = f(x)$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

$$0 \le x \le L$$

The Sturm-Liouville Eigenvalue Problem is

$$-A''(x) = \lambda A(x)$$

BCs $A'(0) - h A(0) = 0$

$A(L) = 0$

$$-y''(x) = \lambda y(x) \quad 0 \le x \le L$$

BCs $\boxed{\begin{aligned} y'(0) - h y(0) = 0 \\ y(L) = 0 \end{aligned}}$ $L > 0$

Solve this eigenvalue problem!

Case 1 $\lambda > 0$ my style $\lambda = \mu^2, \mu > 0$

The Fundamental Solution of $y'' + \mu^2 y = 0$ is

all sols

$$y(x) = C_1 c(\mu x) + C_2 s(\mu x)$$

We seek μ-s for which $\exists$ nonzero y which satisfies BCs. So, substitute y into BCs and look for a nontrivial sol for c_1 & c_2

$$y'(x) = -\mu c_1 s(\mu x) + \mu c_2 c(\mu x)$$

Now BCs $\mu c_2 - h c_1 = 0$

$$c_1 c(\mu L) + c_2 s(\mu L) = 0$$

Math 204: Linear system in c_1 & c_2:

$$\underbrace{\begin{bmatrix} -h & \mu \\ c(\mu L) & s(\mu L) \end{bmatrix}}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ homogeneous system

has nontrivial sol. $\Leftrightarrow$ det $= 0$

$$-h\, s(\mu L) - \mu\, c(\mu L) = 0 \quad \Big/ \div \cos(\mu L) \neq 0$$

Assume $h \neq 0$

$$-\mu = h \tan(\mu L) \quad \Big/ * L > 0$$

do not lose any sols for μ

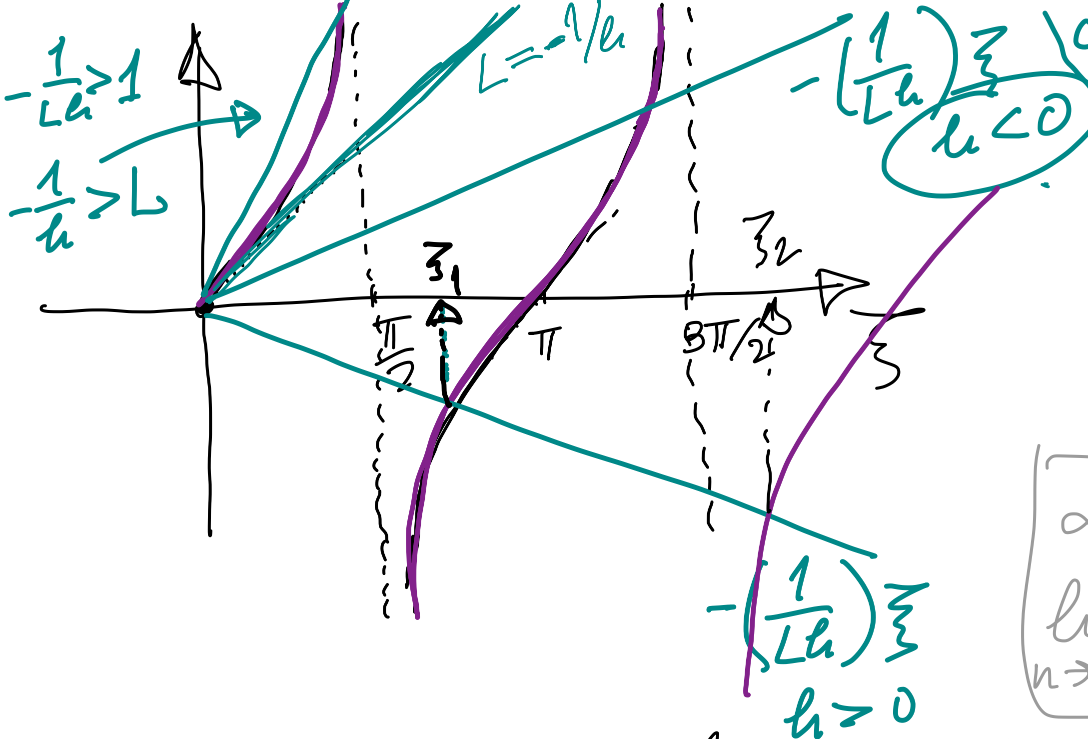
$$-\mu L = L h \tan(\mu L) \quad \Big/ \div L h$$

$$-\underbrace{\left(\frac{1}{Lh}\right)}_{\substack{\text{green number} \\ h \in \mathbb{R}\setminus\{0\}}} \underbrace{\mu L}_{\xi} = \boxed{\tan(\underbrace{\mu L}_{\xi})}$$

slope of ξ

$L > 0 \quad \mu > 0$

$\xi > 0$



$$0 < -\left(\frac{1}{Lh}\right) < 1$$

$$-Lh > 1$$

$$-h > \frac{1}{L}$$

$$L > -1/h$$

$$\alpha_n \sim \beta_n \ (n \to +\infty)$$

$$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 1$$

We can visualize a sequence of solutions $0 < \xi_1 < \xi_2 < \xi_3 < \dots < \xi_n < \dots$

$h > 0$ $\quad \xi_k \in \left((2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\right)$ & $\xi_k \underset{k\to+\infty}{\sim} \overline{(2k-1)\frac{\pi}{2}}$

We obtain a sequence of μs

$$\mu_k = \frac{\xi_k}{L}, \quad k \in \mathbb{N}$$

For these μ-s we have nontrivial sol. for c_1 & c_2. There are many sols for c_1 & c_2. We choose

$$c_1 = \mu \qquad c_2 = h$$

So, ~~the~~ an eigenfunction is:

$$y_k(x) = \mu_k\, c(\mu_k x) + h_k\, s(\mu_k x)$$

Check: $$\int_0^L y_k(x)\, y_m(x)\, dx = 0$$

(by hand this is a chore, but Mathematica can)

$$k \neq m$$

Case 2 $\lambda = 0$

The fund sol. $y(x) = c_1 + c_2 x$

$$y'(x) = c_2$$

BCs
$$c_2 - h c_1 = 0$$
$$c_1 + c_2 L = 0$$

$$\begin{vmatrix} 1 & -h \\ 1 & L \end{vmatrix} = L + h = 0$$

So $\lambda = 0$ is an e-value $\Longleftrightarrow h = -L < 0$

Case 3 Neg. $\lambda < 0$

$$\lambda = -\mu^2$$

The Fund Sol is:

$$y(x) = C_1 \cosh(\mu x) + C_2 \sinh(\mu x)$$

$$y'(x) = \mu C_1 \sinh(\mu x) + \mu C_2 \cosh(\mu x)$$

$$y(0) = C_1 \quad y'(0) = \mu C_2$$

BCs are:
$$\mu c_2 - h c_1 = 0$$
$$c_1 \operatorname{ch}(\mu L) + c_2 \operatorname{sh}(\mu L) = 0$$

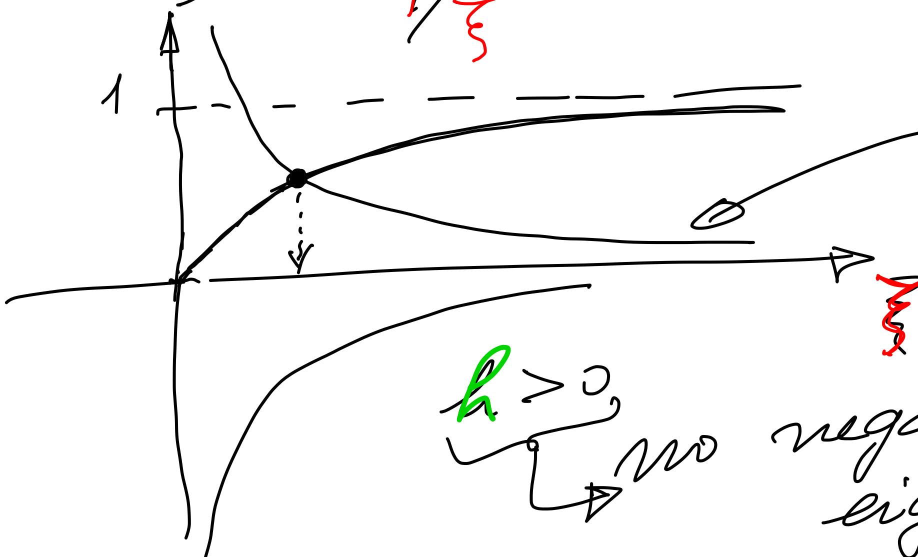
$$\det \begin{vmatrix} \mu & -h \\ \operatorname{ch}(\mu L) & \operatorname{sh}(\mu L) \end{vmatrix} =$$

$$= \mu \operatorname{sh}(\mu L) + h \operatorname{ch}(\mu L) = 0$$

$$\mu \operatorname{th}(\mu L) + h = 0, \quad \ldots$$

$$-\frac{h}{\mu} = \tanh(\mu L)$$

$$\underbrace{(-hL)}_{>0}\,\frac{1}{\underbrace{\mu L}_{1/\xi}} = \mathrm{th}(\underbrace{\mu L}_{\xi})$$



$-hL > 0$

$h < 0$ $\left\{\begin{array}{l}\text{one} \\ \text{negative} \\ \text{eigenvalue}\end{array}\right.$

$h > 0$ → no negative eigenvalues