Vibrations of a Rectangular Membrane

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial^2 y}\right)$$

$u(x,y,t)$ $\qquad (x,y) \in [0,K] \times [0,L]$

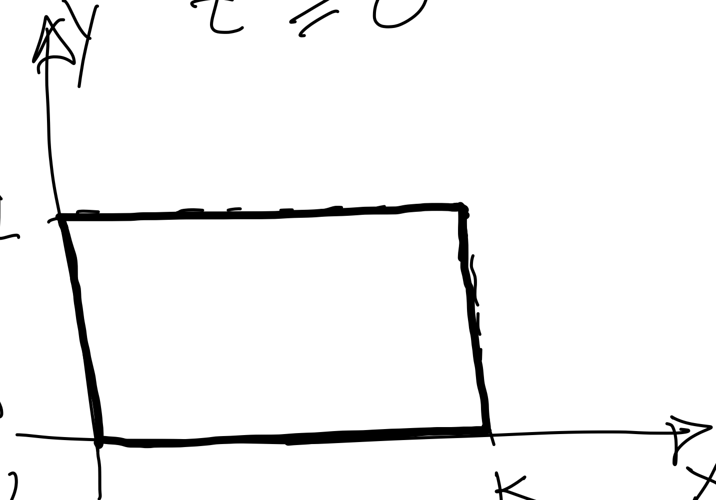
$t \geq 0$

$u(x,0,t) = 0, \quad x \in [0,K]$

$u(x,L,t) = 0, \quad x \in [0,K]$

$u(0,y,t) = 0, \quad y \in [0,L]$

$u(K,y,t) = 0, \quad y \in [0,L]$



I.C. $\quad u(x,y,0) = f(x,y)$

$$\frac{\partial u}{\partial t}(x,y,0) = g(x,y)$$

$$u(x,y,t) = S(x,y)\,T(t)$$

We separate space variables (x,y) from time (t)

$$\frac{\partial^2 u}{\partial t^2} = S\,T''$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\nabla^2 S\right) T$$

$$S\,T'' = c^2 \left(\nabla^2 S\right) T$$

$$\frac{T''(t)}{c^2\,T(t)} = \frac{\left(\nabla^2 S\right)(x,y)}{S(x,y)} = -\lambda$$

This is a successful separation of variables ∇

$$T''(t) = -c^2 \lambda T(t)$$

the Fundamental Set of sols (for $\lambda > 0$) is

$$\{\cos(c\sqrt{\lambda}\, t), \sin(c\sqrt{\lambda}\, t)\}$$

We truly expect $\lambda > 0$ since otherwise we would get unbounded solutions for $T(t)$ cosh, sinh

$$-(\nabla^2 S)(x,y) = \lambda S(x,y)$$

B.Cs for S are

$S(x,0)=0$, $S(x,L)=0$, $x \in [0,K]$

$S(0,y)=0$, $S(K,y)=0$, $y \in [0,L]$

To solve this eigenvalue problem we separate variables x and y. Look for the eigenfunctions of the form:

$$S(x,y) = A(x)\,B(y)$$

then $(\nabla^2 S)(x) = A''(x)\,B(y) + A(x)\,B''(y)$

The Eigenproblem DE is

$$-A''(x)\,B(y) - A(x)\,B''(y) = \lambda\,A(x)\,B(y)$$

$$-\frac{A''(x)}{A(x)} - \frac{B''(y)}{B(y)} = \lambda$$

$$\underbrace{-\frac{A''(x)}{A(x)}}_{\text{depends on } x \text{ only}} = \underbrace{\lambda + \frac{B''(y)}{B(y)}}_{\text{depends on } y \text{ only}} = \nu$$

$$A(0) = 0, \; A(K) = 0$$

$$B(0) = 0, \; B(L) = 0$$

Now I have two eigenvalue problems

$$-A''(x) = \nu A(x) \quad A(0) = 0, A(K) = 0$$

$$-B''(y) = \underbrace{(\lambda - \nu)}_{\mu} B(y)$$

$$B(0)=0, \; B(L)=0$$

We solved this eigenvalue problem before

$$\nu = \left(\frac{n\pi}{K}\right)^2, \text{ eigenfunction } \sin\left(\frac{n\pi}{K}x\right), \; n \in \mathbb{N}$$

$$\mu = \left(\frac{m\pi}{L}\right)^2, \text{ eigenfunctions } \sin\left(\frac{m\pi}{L}y\right) \quad m \in \mathbb{N}$$

$$\boxed{\text{Therefore } \lambda = \left(\frac{n\pi}{K}\right)^2 + \left(\frac{m\pi}{L}\right)^2 \quad m, n \in \mathbb{N}}$$

a corresponding eigenfunction is

$$\underbrace{S_{n,m}}_{\text{eigenfunction corresp. to eigenvalue } \left(\frac{n\pi}{K}\right)^2+\left(\frac{m\pi}{L}\right)^2}(x,y) = \sin\left(\frac{n\pi}{K}x\right)\sin\left(\frac{m\pi}{L}y\right) \qquad (x,y)\in[0,K]\times[0,L]$$

$$\underbrace{-\left(\nabla^2 S\right)(x,y)}_{-\frac{\partial^2 S}{\partial x^2}-\frac{\partial^2 S}{\partial y}} \overset{?\ \text{true}}{=} \left(\left(\frac{n\pi}{K}\right)^2+\left(\frac{m\pi}{L}\right)^2\right) S(x,y)$$

$$-\frac{\partial^2 S}{\partial x^2}-\frac{\partial^2 S}{\partial y} = +\left(\frac{n\pi}{K}\right)^2 \sin\left(\frac{n\pi}{K}x\right)\sin\left(\frac{m\pi}{2}y\right)$$

$$+\left(\frac{m\pi}{L}\right)^2 \sin\left(\frac{n\pi}{K}x\right)\sin\left(\frac{m\pi}{L}y\right)$$

$$= \left(\left(\frac{n\pi}{K}\right)^2+\left(\frac{m\pi}{L}\right)^2\right)\sin(\ )\sin(\ )$$

The eigenfunctions corresp. to distinct eigenvalues are orthogonal in the sense

$(n,m) \neq (k,j)$ then

$$\int_0^L \int_0^K \sin\left(\frac{n\pi}{K}x\right) \sin\left(\frac{m\pi}{L}y\right) \sin\left(\frac{k\pi}{K}x\right) \sin\left(\frac{j\pi}{L}y\right) dx\,dy$$

$$= \underbrace{\left(\int_0^K \sin\left(\frac{n\pi}{K}x\right) \sin\left(\frac{k\pi}{K}x\right) dx\right)}_{\substack{n \neq k \\ = 0}} \text{ OR } \underbrace{\left(\int_0^L \sin\left(\frac{m\pi}{L}y\right) \sin\left(\frac{j\pi}{L}y\right) dy\right)}_{\substack{m \neq j \\ = 0}}$$

OR

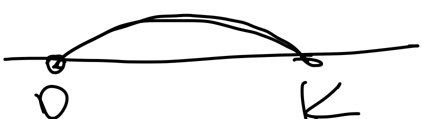
The natural modes of vibration of this rectangular membrane are:

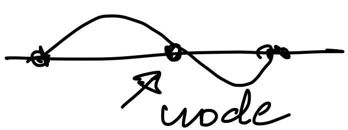
$$\sin\left(\frac{n\pi}{K}x\right)\sin\left(\frac{m\pi}{L}y\right)\cos\left(c\sqrt{\left(\frac{n\pi}{K}\right)^2+\left(\frac{m\pi}{L}\right)^2}\,t\right)$$

$n, m \in \mathbb{N}$

$n = m = 1$

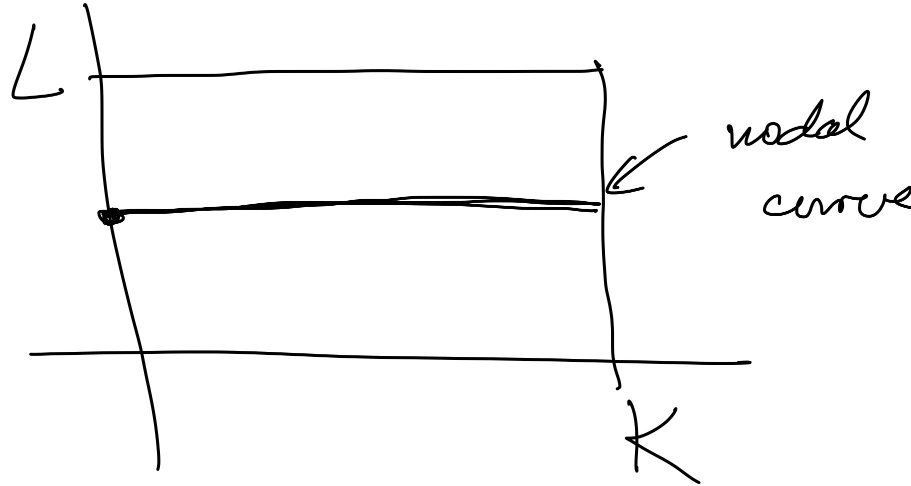
$$\sin\left(\frac{\pi}{K}x\right)\sin\left(\frac{\pi}{L}y\right)\underbrace{\cos\left(c\sqrt{\left(\frac{\pi}{K}\right)^2+\left(\frac{\pi}{L}\right)^2}\,t\right)}_{\substack{1 \\ \text{at } t=0}}$$





$n = 1 \quad , m = 2$

$\sin\left(\frac{\pi}{K}x\right)\sin\left(\frac{2\pi}{L}y\right)$



$n = 2, m = 2$