Chapter Sixteen

INTEGRATING FUNCTIONS OF SEVERAL VARIABLES

Contents

16.1 THE DEFINITE INTEGRAL OF A FUNCTION OF TWO VARIABLES

The definite integral of a continuous one-variable function, f, is a limit of Riemann sums:

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_i f(x_i)\,\Delta x,$$

where x_i is a point in the i^{th} subdivision of the interval $[a, b]$. In this section we extend this definition to functions of two variables. We start by considering how to estimate total population from a two-variable population density.

Population Density of Foxes in England

The fox population in parts of England can be important to public health officials because animals can spread diseases, such as rabies. Biologists use a contour diagram to display the fox population density, D; see Figure 16.1, where D is in foxes per square kilometer.[1] The bold contour is the coastline, which may be thought of as the $D = 0$ contour; clearly the density is zero outside it. We can think of D as a function of position, $D = f(x, y)$ where x and y are in kilometers from the southwest corner of the map.

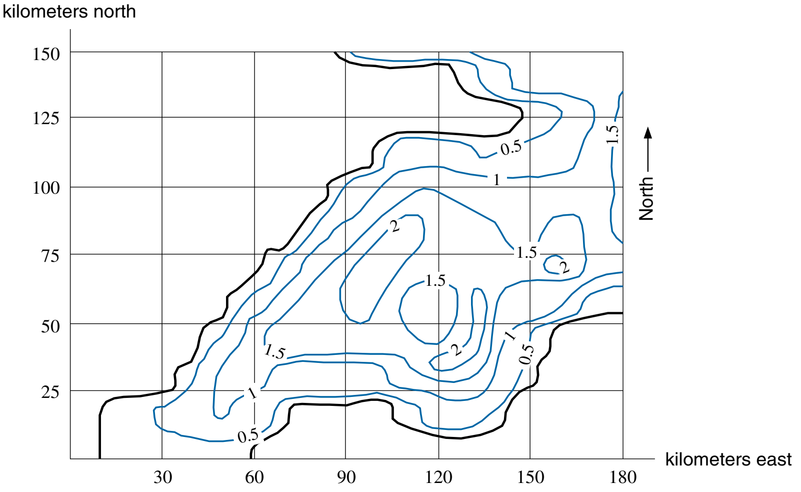


Figure 16.1: Population density of foxes in southwestern England

Example 1 Estimate the total fox population in the region represented by the map in Figure 16.1.

Solution We subdivide the map into the rectangles shown in Figure 16.1 and estimate the population in each rectangle. For simplicity, we use the population density at the northeast corner of each rectangle. For example, in the bottom left rectangle, the density is 0 at the northeast corner; in the next rectangle to the east (right), the density in the northeast corner is 1. Continuing in this way, we get the estimates in Table 16.1. To estimate the population in a rectangle, we multiply the density by the area of the rectangle, $30 \cdot 25 = 750$ km^2. Adding the results, we obtain

$$\begin{aligned}\text{Estimate of population } &= (0.2 + 0.7 + 1.2 + 1.2 + 0.1 + 1.6 + 0.5 + 1.4 \\ &+ 1.1 + 1.6 + 1.5 + 1.8 + 1.5 + 1.3 + 1.1 + 2.0 \\ &+ 1.4 + 1.0 + 1.0 + 0.6 + 1.2)750 = 18{,}000 \text{ foxes.}\end{aligned}$$

[1] From "On the spatial spread of rabies among foxes", Murray, J. D. et al, *Proc. R. Soc. Lond. B,* 229: 111–150, 1986.

Taking the upper and lower bounds for the population density on each rectangle enables us to find upper and lower estimates for the population. Using the same rectangles, the upper estimate is approximately 35,000 and the lower estimate is 4,000. There is a wide discrepancy between the upper and lower estimates; we could make them closer by taking finer subdivisions.

Table 16.1 *Estimates of population density (northeast corner)*

0.0	0.0	0.2	0.7	1.2	1.2
0.0	0.0	0.0	0.0	0.1	1.6
0.0	0.0	0.5	1.4	1.1	1.6
0.0	0.0	1.5	1.8	1.5	1.3
0.0	1.1	2.0	1.4	1.0	0.0
0.0	1.0	0.6	1.2	0.0	0.0

Definition of the Definite Integral

The sums used to approximate the fox population are Riemann sums. We now define the definite integral for a function f of two variables on a rectangular region. Given a continuous function $f(x, y)$ defined on a region $a \leq x \leq b$ and $c \leq y \leq d$, we subdivide each of the intervals $a \leq x \leq b$ and $c \leq y \leq d$ into n and m equal subintervals respectively, giving nm subrectangles. (See Figure 16.2.)

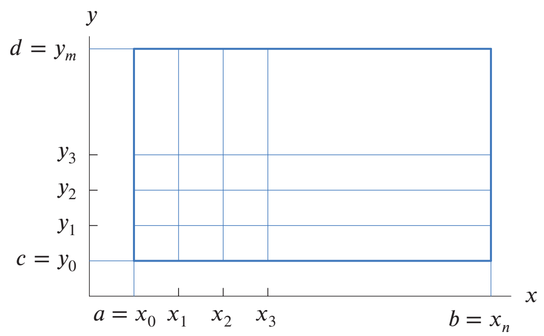


Figure 16.2: Subdivision of a rectangle into nm subrectangles

The area of each subrectangle is $\Delta A = \Delta x\, \Delta y$, where $\Delta x = (b - a)/n$ is the width of each subdivision on the x-axis, and $\Delta y = (d - c)/m$ is the width of each subdivision on the y-axis. To compute the Riemann sum, we multiply the area of each subrectangle by the value of the function at a point in the rectangle and add the resulting numbers. Choosing the maximum value, M_{ij}, of the function on each rectangle and adding for all i, j gives the *upper sum*, $\sum_{i,j} M_{ij}\Delta x\Delta y$.

The *lower sum*, $\sum_{i,j} L_{ij}\Delta x\Delta y$, is obtained by taking the minimum value on each rectangle. If (u_{ij}, v_{ij}) is any point in the ij-th subrectangle, any other Riemann sum satisfies

$$\sum_{i,j} L_{ij}\Delta x\Delta y \leq \sum_{i,j} f(u_{ij}, v_{ij})\,\Delta x\,\Delta y \leq \sum_{i,j} M_{ij}\Delta x\Delta y.$$

We define the definite integral by taking the limit as the numbers of subdivisions, n and m, tend to infinity. By comparing upper and lower sums, as we did for the fox population, it can be shown that the limit exists when the function, f, is continuous. We get the same limit by letting Δx and Δy tend to 0. Thus, we have the following definition:

Suppose the function f is continuous on R, the rectangle $a \le x \le b$, $c \le y \le d$. If (u_{ij}, v_{ij}) is any point in the ij-th subrectangle, we define the **definite integral** of f over R

$$\int_R f\,dA = \lim_{\Delta x, \Delta y \to 0} \sum_{i,j} f(u_{ij}, v_{ij}) \Delta x \Delta y.$$

Such an integral is called a **double integral**.

The case when R is not rectangular is considered on page 844. Sometimes we think of dA as being the area of an infinitesimal rectangle of length dx and height dy, so that $dA = dx\,dy$. Then we use the notation[2]

$$\int_R f\,dA = \int_R f(x, y)\,dx\,dy.$$

For this definition, we used a particular type of Riemann sum with equal-sized rectangular subdivisions. In a general Riemann sum, the subdivisions do not all have to be the same size.

Interpretation of the Double Integral as Volume

Just as the definite integral of a positive one-variable function can be interpreted as an area, so the double integral of a positive two-variable function can be interpreted as a volume. In the one-variable case we visualize the Riemann sums as the total area of rectangles above the subdivisions. In the two-variable case we get solid bars instead of rectangles. As the number of subdivisions grows, the tops of the bars approximate the surface better, and the volume of the bars gets closer to the volume under the graph of the function. (See Figure 16.3.)

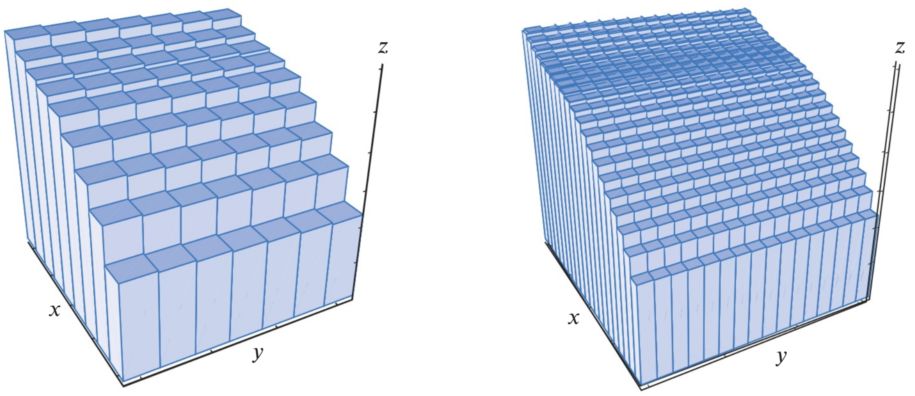


Figure 16.3: Approximating volume under a graph with finer and finer Riemann sums

Thus, we have the following result:

If x, y, z represent length and f is positive, then

$$\begin{array}{l}\text{Volume under graph} \\ \text{of } f \text{ above region } R\end{array} = \int_R f\,dA.$$

Example 2 Let R be the rectangle $0 \le x \le 1$ and $0 \le y \le 1$. Use Riemann sums to make upper and lower estimates of the volume of the region above R and under the graph of $z = e^{-(x^2+y^2)}$.

[2]Another common notation for the double integral is $\int\int_R f\,dA$.

Solution If R is the rectangle $0 \le x \le 1$, $0 \le y \le 1$, the volume we want is given by

$$\text{Volume} = \int_R e^{-(x^2+y^2)}\, dA.$$

We divide R into 16 subrectangles by dividing each edge into four parts. Figure 16.4 shows that $f(x,y) = e^{-(x^2+y^2)}$ decreases as we move away from the origin. Thus, to get an upper sum we evaluate f on each subrectangle at the corner nearest the origin. For example, in the rectangle $0 \le x \le 0.25$, $0 \le y \le 0.25$, we evaluate f at $(0,0)$. Using Table 16.2, we find that

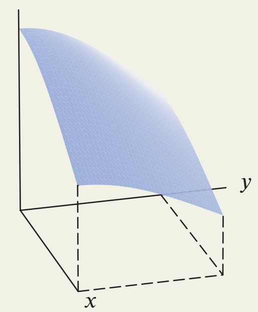


Figure 16.4: Graph of $e^{-(x^2+y^2)}$ above the rectangle R

$$\begin{aligned}\text{Upper sum} = (1 &+ 0.9394 + 0.7788 + 0.5698\\ &+ 0.9394 + 0.8825 + 0.7316 + 0.5353\\ &+ 0.7788 + 0.7316 + 0.6065 + 0.4437\\ &+ 0.5698 + 0.5353 + 0.4437 + 0.3247)(0.0625) = 0.68.\end{aligned}$$

To get a lower sum, we evaluate f at the opposite corner of each rectangle because the surface slopes down in both the x and y directions. This yields a lower sum of 0.44. Thus,

$$0.44 \le \int_R e^{-(x^2+y^2)}\, dA \le 0.68.$$

To get a better approximation of the volume under the graph, we use more subdivisions. See Table 16.3.

Table 16.2 *Values of $f(x,y) = e^{-(x^2+y^2)}$ on the rectangle R*

		y				
		0.0	0.25	0.50	0.75	1.00
	0.0	1	0.9394	0.7788	0.5698	0.3679
	0.25	0.9394	0.8825	0.7316	0.5353	0.3456
x	0.50	0.7788	0.7316	0.6065	0.4437	0.2865
	0.75	0.5698	0.5353	0.4437	0.3247	0.2096
	1.00	0.3679	0.3456	0.2865	0.2096	0.1353

Table 16.3 *Riemann sum approximations to $\int_R e^{-(x^2+y^2)}\, dA$*

	Number of subdivisions in x and y directions			
	8	16	32	64
Upper	0.6168	0.5873	0.5725	0.5651
Lower	0.4989	0.5283	0.5430	0.5504

The exact value of the double integral, 0.5577 …, is trapped between the lower and upper sums. Notice that the lower sum increases and the upper sum decreases as the number of subdivisions increases. However, even with 64 subdivisions, the lower and upper sums agree with the exact value of the integral only in the first decimal place.

Interpretation of the Double Integral as Area

In the special case that $f(x, y) = 1$ for all points (x, y) in the region R, each term in the Riemann sum is of the form $1 \cdot \Delta A = \Delta A$ and the double integral gives the area of the region R:

$$\text{Area}(R) = \int_R 1\, dA = \int_R dA$$

Interpretation of the Double Integral as Average Value

As in the one-variable case, the definite integral can be used to define the average value of a function:

$$\begin{matrix}\text{Average value of } f \\ \text{on the region } R\end{matrix} = \frac{1}{\text{Area of } R}\int_R f\, dA$$

We can rewrite this as

$$\text{Average value} \times \text{Area of } R = \int_R f\, dA.$$

If we interpret the integral as the volume under the graph of f, then we can think of the average value of f as the height of the box with the same volume that is on the same base. (See Figure 16.5.) Imagine that the volume under the graph is made out of wax. If the wax melted within the perimeter of R, then it would end up box-shaped with height equal to the average value of f.

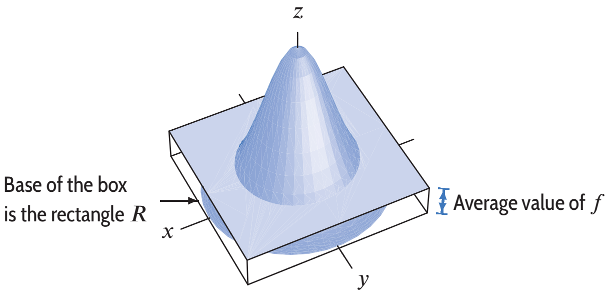


Figure 16.5: Volume and average value

Integral over Regions that Are Not Rectangles

We defined the definite integral $\int_R f(x, y)\, dA$, for a rectangular region R. Now we extend the definition to regions of other shapes, including triangles, circles, and regions bounded by the graphs of piecewise continuous functions.

To approximate the definite integral over a region, R, which is not rectangular, we use a grid of rectangles approximating the region. We obtain this grid by enclosing R in a large rectangle and subdividing that rectangle; we consider just the subrectangles which are inside R.

As before, we pick a point (u_{ij}, v_{ij}) in each subrectangle and form a Riemann sum

$$\sum_{i,j} f(u_{ij}, v_{ij})\Delta x \Delta y.$$

This time, however, the sum is over only those subrectangles within R. For example, in the case of the fox population we can use the rectangles which are entirely on land. As the subdivisions become finer, the grid approximates the region R more closely. For a function, f, which is continuous on R, we define the definite integral as follows:

$$\int_R f\, dA = \lim_{\Delta x, \Delta y \to 0} \sum_{i,j} f(u_{ij}, v_{ij}) \Delta x \Delta y$$

where the Riemann sum is taken over the subrectangles inside R.

You may wonder why we can leave out the rectangles which cover the edge of R—if we included them, might we get a different value for the integral? The answer is that for any region that we are likely to meet, the area of the subrectangles covering the edge tends to 0 as the grid becomes finer. Therefore, omitting these rectangles does not affect the limit.

Convergence of Upper and Lower Sums to Same Limit

We have said that if f is continuous on the rectangle R, then the difference between upper and lower sums for f converges to 0 as Δx and Δy approach 0. In the following example, we show this in a particular case. The ideas in this example can be used in a general proof.

Example 3 Let $f(x, y) = x^2 y$ and let R be the rectangle $0 \le x \le 1, 0 \le y \le 1$. Show that the difference between upper and lower Riemann sums for f on R converges to 0, as Δx and Δy approach 0.

Solution The difference between the sums is

$$\sum M_{ij} \Delta x \Delta y - \sum L_{ij}\, \Delta x \Delta y = \sum (M_{ij} - L_{ij})\, \Delta x \Delta y,$$

where M_{ij} and L_{ij} are the maximum and minimum of f on the ij-th subrectangle. Since f increases in both the x and y directions, M_{ij} occurs at the corner of the subrectangle farthest from the origin and L_{ij} at the closest. Moreover, since the slopes in the x and y directions don't decrease as x and y increase, the difference $M_{ij} - L_{ij}$ is largest in the subrectangle R_{nm} which is farthest from the origin. Thus,

$$\sum (M_{ij} - L_{ij})\, \Delta x \Delta y \le (M_{nm} - L_{nm}) \sum \Delta x \Delta y = (M_{nm} - L_{nm}) \text{Area}(R).$$

Thus, the difference converges to 0 as long as $(M_{nm} - L_{nm})$ does. The maximum M_{nm} of f on the nm-th subrectangle occurs at $(1, 1)$, the subrectangle's top right corner, and the minimum L_{nm} occurs at the opposite corner, $(1 - 1/n, 1 - 1/m)$. Substituting into $f(x, y) = x^2 y$ gives

$$M_{nm} - L_{nm} = (1)^2(1) - \left(1 - \frac{1}{n}\right)^2 \left(1 - \frac{1}{m}\right) = \frac{2}{n} - \frac{1}{n^2} + \frac{1}{m} - \frac{2}{nm} + \frac{1}{n^2 m}.$$

The right-hand side converges to 0 as $n, m \to \infty$, that is, as $\Delta x, \Delta y \to 0$.

Exercises and Problems for Section 16.1

Online Resource: Additional Problems for Section 16.1

EXERCISES

1. Table 16.4 gives values of the function $f(x, y)$, which is increasing in x and decreasing in y on the region $R : 0 \le x \le 6, 0 \le y \le 1$. Make the best possible upper and lower estimates of $\int_R f(x, y)\, dA$.

Table 16.4

y \ x	0	3	6
0	5	7	10
0.5	4	5	7
1	3	4	6

2. Values of $f(x, y)$ are in Table 16.5. Let R be the rectangle $1 \le x \le 1.2, 2 \le y \le 2.4$. Find Riemann sums which are reasonable over and underestimates for $\int_R f(x, y)\, dA$ with $\Delta x = 0.1$ and $\Delta y = 0.2$.

Table 16.5

y \ x	1.0	1.1	1.2
2.0	5	7	10
2.2	4	6	8
2.4	3	5	4

3. Figure 16.6 shows contours of $g(x, y)$ on the region R, with $5 \leq x \leq 11$ and $4 \leq y \leq 10$. Using $\Delta x = \Delta y = 2$, find an overestimate and an underestimate for $\int_R g(x, y)\,dA$.

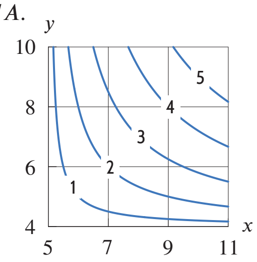


Figure 16.6

4. Figure 16.7 shows contours of $f(x, y)$ on the rectangle R with $0 \leq x \leq 30$ and $0 \leq y \leq 15$. Using $\Delta x = 10$ and $\Delta y = 5$, find an overestimate and an underestimate for $\int_R f(x, y)\,dA$.

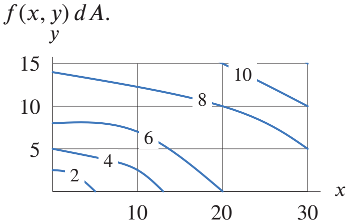


Figure 16.7

5. Figure 16.8 shows a contour plot of population density, people per square kilometer, in a rectangle of land 3 km by 2 km. Estimate the population in the region represented by Figure 16.8.

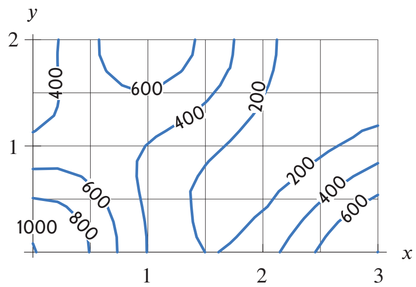


Figure 16.8

In Exercises 6–7, for x and y in meters and R a region on the xy-plane, what does the integral represent? Give units.

6. $\displaystyle\int_R \sigma(x, y)\,dA$, where $\sigma(x, y)$ is bacteria population, in thousands per m^2.

7. $\displaystyle\frac{1}{\text{Area of } R}\int_R h(x, y)\,dA$, where $h(x, y)$ is the height of a tent, in meters.

PROBLEMS

In Problems 8–14, decide (without calculation) whether the integrals are positive, negative, or zero. Let D be the region inside the unit circle centered at the origin, let R be the right half of D, and let B be the bottom half of D.

8. $\int_D 1\,dA$

9. $\int_R 5x\,dA$

10. $\int_B 5x\,dA$

11. $\int_D (y^3 + y^5)\,dA$

12. $\int_B (y^3 + y^5)\,dA$

13. $\int_D (y - y^3)\,dA$

14. $\int_B (y - y^3)\,dA$

15. Figure 16.9 shows contours of $f(x, y)$. Let R be the square $-0.5 \leq x \leq 1$, $-0.5 \leq y \leq 1$. Is the integral $\int_R f\,dA$ positive or negative? Explain your reasoning.

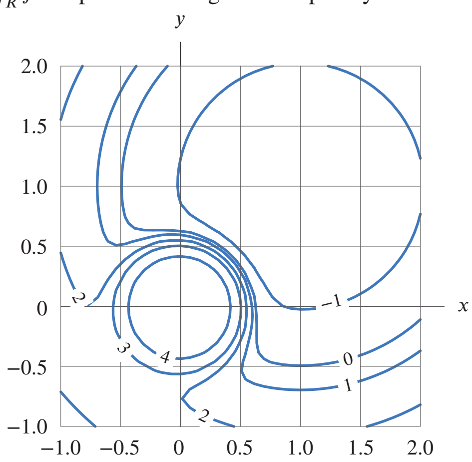


Figure 16.9

16. Table 16.6 gives values of $f(x, y)$, the number of milligrams of mosquito larvae per square meter in a swamp. If x and y are in meters and R is the rectangle $0 \leq x \leq 8$, $0 \leq y \leq 6$, estimate $\int_R f(x, y)\,dA$. Give units and interpret your answer.

Table 16.6

$y \backslash x$	0	4	8
0	1	3	6
3	2	5	9
6	4	9	15

17. Table 16.7 gives values of $f(x, y)$, the depth of volcanic ash, in meters, after an eruption. If x and y are in kilometers and R is the rectangle $0 \leq x \leq 100$, $0 \leq y \leq 100$, estimate the volume of volcanic ash in R in km^3.

Table 16.7

$y \backslash x$	0	50	100
0	0.82	0.56	0.43
50	0.63	0.45	0.3
100	0.55	0.44	0.26

18. Table 16.8 gives the density of cacti, $f(x, y)$, in a desert region, in thousands of cacti per km^2. If x and y are in kilometers and R is the square $0 \leq x \leq 30$, $0 \leq y \leq 30$, estimate the number of cacti in the region R.

Table 16.8

$y \backslash x$	0	10	20	30
0	8.5	8.2	7.9	8.1
10	9.5	10.6	10.5	10.1
20	9.3	10.5	10.4	9.5
30	8.3	8.6	9.3	9.1

19. Use four subrectangles to approximate the volume of the object whose base is the region $0 \le x \le 4$ and $0 \le y \le 6$, and whose height is given by $f(x, y) = x+y$. Find an overestimate and an underestimate and average the two.

20. Figure 16.10 shows the rainfall, in inches, in Tennessee on May 1–2, 2010.[3] Using three contours (red, yellow, and green), make a rough estimate of how many cubic miles of rain fell on the state during this time.

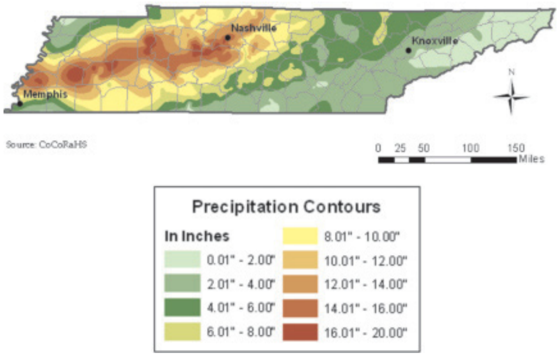


Figure 16.10

Strengthen Your Understanding

In Problems 21–22, explain what is wrong with the statement.

21. For all f, the integral $\int_R f(x, y)\, dA$ gives the volume of the solid under the graph of f over the region R.

22. If R is a region in the third quadrant where $x < 0$, $y < 0$, then $\int_R f(x, y)\, dA$ is negative.

In Problems 23–24, give an example of:

23. A function $f(x, y)$ and rectangle R such that the Riemann sums obtained using the lower left-hand corner of each subrectangle are an overestimate.

24. A function $f(x, y)$ whose average value over the square $0 \le x \le 1$, $0 \le y \le 1$ is negative.

Are the statements in Problems 25–34 true or false? Give reasons for your answer.

25. The double integral $\int_R f\, dA$ is always positive.

26. If $f(x, y) = k$ for all points (x, y) in a region R then $\int_R f\, dA = k \cdot \text{Area}(R)$.

27. If R is the rectangle $0 \le x \le 1, 0 \le y \le 1$ then $\int_R e^{xy}\, dA > 3$.

28. If R is the rectangle $0 \le x \le 2, 0 \le y \le 3$ and S is the rectangle $-2 \le x \le 0, -3 \le y \le 0$, then $\int_R f\, dA = -\int_S f\, dA$.

29. Let $\rho(x, y)$ be the population density of a city, in people per km^2. If R is a region in the city, then $\int_R \rho\, dA$ gives the total number of people in the region R.

30. If $\int_R f\, dA = 0$, then $f(x, y) = 0$ at all points of R.

31. If $g(x, y) = kf(x, y)$, where k is constant, then $\int_R g\, dA = k \int_R f\, dA$.

32. If f and g are two functions continuous on a region R, then $\int_R f \cdot g\, dA = \int_R f\, dA \cdot \int_R g\, dA$.

33. If R is the rectangle $0 \le x \le 1, 0 \le y \le 2$ and S is the square $0 \le x \le 1, 0 \le y \le 1$, then $\int_R f\, dA = 2\int_S f\, dA$.

34. If R is the rectangle $2 \le x \le 4, 5 \le y \le 9$, $f(x, y) = 2x$ and $g(x, y) = x + y$, then the average value of f on R is less than the average value of g on R.

16.2 ITERATED INTEGRALS

In Section 16.1 we approximated double integrals using Riemann sums. In this section we see how to compute double integrals exactly using one-variable integrals.

The Fox Population Again: Expressing a Double Integral as an Iterated Integral

To estimate the fox population, we computed a sum of the form

$$\text{Total population} \approx \sum_{i,j} f(u_{ij}, v_{ij})\Delta x\, \Delta y,$$

where $1 \le i \le n$ and $1 \le j \le m$ and the values $f(u_{ij}, v_{ij})$ can be arranged as in Table 16.9.

[3] www.srh.noaa.gov/images/ohx/rainfall/TN_May2010_rainfall_map.png, accessed June 13, 2016.

Table 16.9 *Estimates for fox population densities for $n = m = 6$*

0.0	0.0	0.2	0.7	1.2	1.2
0.0	0.0	0.0	0.0	0.1	1.6
0.0	0.0	0.5	1.4	1.1	1.6
0.0	0.0	1.5	1.8	1.5	1.3
0.0	1.1	2.0	1.4	1.0	0.0
0.0	1.0	0.6	1.2	0.0	0.0

For any values of n and m, we can either add across the rows first or add down the columns first. If we add rows first, we can write the sum in the form

$$\text{Total population} \approx \sum_{j=1}^{m} \left(\sum_{i=1}^{n} f(u_{ij}, v_{ij}) \Delta x \right) \Delta y.$$

The inner sum, $\sum_{i=1}^{n} f(u_{ij}, v_{ij})\, \Delta x$, approximates the integral $\int_0^{180} f(x, v_{ij})\, dx$. Thus, we have

$$\text{Total population} \approx \sum_{j=1}^{m} \left(\int_0^{180} f(x, v_{ij})\, dx \right) \Delta y.$$

The outer Riemann sum approximates another integral, this time with integrand $\int_0^{180} f(x, y)\, dx$, which is a function of y. Thus, we can write the total population in terms of nested, or *iterated*, one-variable integrals:

$$\text{Total population} = \int_0^{150} \left(\int_0^{180} f(x, y)\, dx \right) dy.$$

Since the total population is represented by $\int_R f\, dA$, this suggests the method of computing double integrals in the following theorem:[4]

Theorem 16.1: Writing a Double Integral as an Iterated Integral

If R is the rectangle $a \le x \le b$, $c \le y \le d$ and f is a continuous function on R, then the integral of f over R exists and is equal to the **iterated integral**

$$\int_R f\, dA = \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x, y)\, dx \right) dy.$$

The expression $\int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x, y)\, dx \right) dy$ can be written $\int_c^d \int_a^b f(x, y)\, dx\, dy$.

To evaluate the iterated integral, first perform the inside integral with respect to x, holding y constant; then integrate the result with respect to y.

Example 1 A building is 8 meters wide and 16 meters long. It has a flat roof that is 12 meters high at one corner and 10 meters high at each of the adjacent corners. What is the volume of the building?

Solution If we put the high corner on the z-axis, the long side along the y-axis, and the short side along the x-axis, as in Figure 16.11, then the roof is a plane with z-intercept 12, and x slope $(-2)/8 = -1/4$, and y slope $(-2)/16 = -1/8$. Hence, the equation of the roof is

$$z = 12 - \tfrac{1}{4}x - \tfrac{1}{8}y.$$

[4] For a proof, see M. Spivak, *Calculus on Manifolds*, pp. 53 and 58 (New York: Benjamin, 1965).

The volume is given by the double integral

$$\text{Volume} = \int_R (12 - \tfrac{1}{4}x - \tfrac{1}{8}y)\,dA,$$

where R is the rectangle $0 \le x \le 8$, $0 \le y \le 16$. Setting up an iterated integral, we get

$$\text{Volume} = \int_0^{16}\int_0^{8} (12 - \tfrac{1}{4}x - \tfrac{1}{8}y)\,dx\,dy.$$

The inside integral is

$$\int_0^{8} (12 - \tfrac{1}{4}x - \tfrac{1}{8}y)\,dx = \left(12x - \tfrac{1}{8}x^2 - \tfrac{1}{8}xy\right)\Big|_{x=0}^{x=8} = 88 - y.$$

Then the outside integral gives

$$\text{Volume} = \int_0^{16} (88 - y)\,dy = (88y - \tfrac{1}{2}y^2)\Big|_0^{16} = 1280.$$

The volume of the building is 1280 cubic meters.

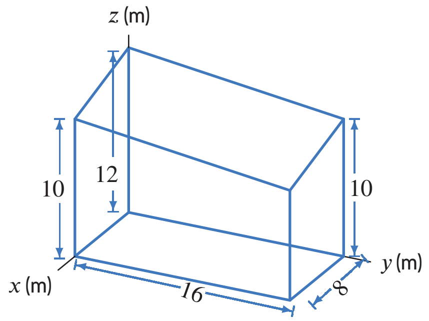


Figure 16.11: A slant-roofed building

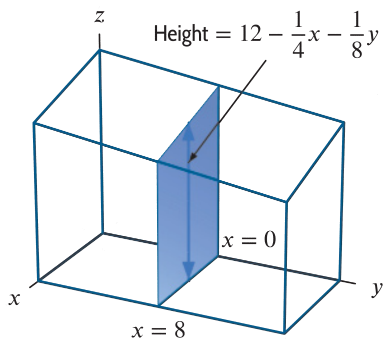


Figure 16.12: Cross-section of a building

Notice that the inner integral $\int_0^8 (12 - \frac{1}{4}x - \frac{1}{8}y)\,dx$ in Example 1 gives the area of the cross section of the building perpendicular to the y-axis in Figure 16.12.

The iterated integral $\int_0^{16}\int_0^8 (12 - \frac{1}{4}x - \frac{1}{8}y)\,dx\,dy$ thus calculates the volume by adding the volumes of thin cross-sectional slabs.

The Order of Integration

In computing the fox population, we could have chosen to add columns (fixed x) first, instead of the rows. This leads to an iterated integral where x is constant in the inner integral instead of y. Thus,

$$\int_R f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$$

where R is the rectangle $a \le x \le b$ and $c \le y \le d$.

For any function we are likely to meet, it does not matter in which order we integrate over a rectangular region R; we get the same value for the double integral either way.

$$\int_R f\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$$

Example 2 Compute the volume of Example 1 as an iterated integral by integrating with respect to y first.

Solution Rewriting the integral, we have

$$\text{Volume} = \int_0^8 \left(\int_0^{16} (12 - \tfrac{1}{4}x - \tfrac{1}{8}y)\, dy \right) dx = \int_0^8 \left((12y - \tfrac{1}{4}xy - \tfrac{1}{16}y^2) \Big|_{y=0}^{y=16} \right) dx$$

$$= \int_0^8 (176 - 4x)\, dx = (176x - 2x^2) \Big|_0^8 = 1280 \text{ meter}^3.$$

Iterated Integrals Over Non-Rectangular Regions

Example 3 The density at the point (x, y) of a triangular metal plate, as shown in Figure 16.13, is $\delta(x, y)$. Express its mass as an iterated integral.

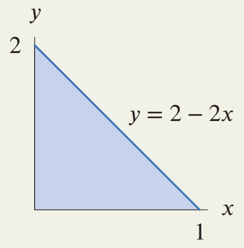


Figure 16.13: A triangular metal plate with density $\delta(x, y)$ at the point (x, y)

Solution Approximate the triangular region using a grid of small rectangles of sides Δx and Δy. The mass of one rectangle is given by

$$\text{Mass of rectangle} \approx \text{Density} \cdot \text{Area} \approx \delta(x, y)\Delta x \Delta y.$$

Summing over all rectangles gives a Riemann sum which approximates the double integral:

$$\text{Mass} = \int_R \delta(x, y)\, dA,$$

where R is the triangle. We want to compute this integral using an iterated integral.

Think about how the iterated integral over the rectangle $a \le x \le b$, $c \le y \le d$ works:

$$\int_a^b \int_c^d f(x, y)\, dy\, dx.$$

The inside integral with respect to y is along vertical strips which begin at the horizontal line $y = c$ and end at the line $y = d$. There is one such strip for each x between $x = a$ and $x = b$. (See Figure 16.14.)

For the triangular region in Figure 16.13, the idea is the same. The only difference is that the individual vertical strips no longer all go from $y = c$ to $y = d$. The vertical strip that starts at the point $(x, 0)$ ends at the point $(x, 2 - 2x)$, because the top edge of the triangle is the line $y = 2 - 2x$. See Figure 16.15. On this vertical strip, y goes from 0 to $2 - 2x$. Hence, the inside integral is

$$\int_0^{2-2x} \delta(x, y)\, dy.$$

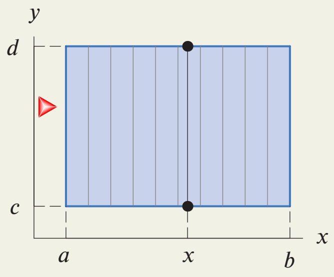


Figure 16.14: Integrating over a rectangle using vertical strips

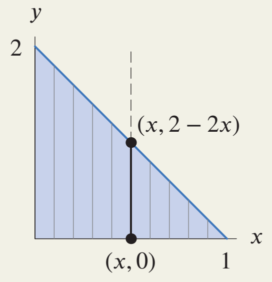


Figure 16.15: Integrating over a triangle using vertical strips

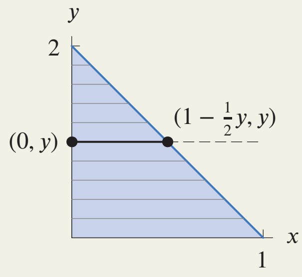


Figure 16.16: Integrating over a triangle using horizontal strips

Finally, since there is a vertical strip for each x between 0 and 1, the outside integral goes from $x = 0$ to $x = 1$. Thus, the iterated integral we want is

$$\text{Mass} = \int_0^1 \int_0^{2-2x} \delta(x, y)\, dy\, dx.$$

We could have chosen to integrate in the opposite order, keeping y fixed in the inner integral instead of x. The limits are formed by looking at horizontal strips instead of vertical ones, and expressing the x-values at the end points in terms of y. See Figure 16.16. To find the right endpoint of the strip, we use the equation of the top edge of the triangle in the form $x = 1 - \frac{1}{2}y$. Thus, a horizontal strip goes from $x = 0$ to $x = 1 - \frac{1}{2}y$. Since there is a strip for every y from 0 to 2, the iterated integral is

$$\text{Mass} = \int_0^2 \int_0^{1-\frac{1}{2}y} \delta(x, y)\, dx\, dy.$$

Limits on Iterated Integrals

- The limits on the outer integral must be constants.
- The limits on the inner integral can involve only the variable in the outer integral. For example, if the inner integral is with respect to x, its limits can be functions of y.

Example 4 Find the mass M of a metal plate R bounded by $y = x$ and $y = x^2$, with density given by $\delta(x, y) = 1 + xy$ kg/meter^2. (See Figure 16.17.)

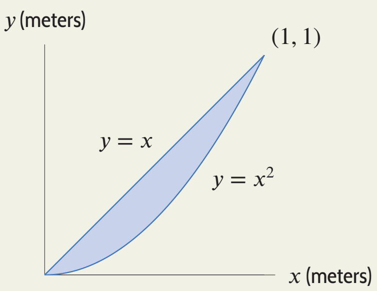


Figure 16.17: A metal plate with density $\delta(x, y)$

Solution The mass is given by

$$M = \int_R \delta(x, y)\, dA.$$

We integrate along vertical strips first; this means we do the y integral first, which goes from the bottom boundary $y = x^2$ to the top boundary $y = x$. The left edge of the region is at $x = 0$ and the right edge is at the intersection point of $y = x$ and $y = x^2$, which is $(1, 1)$. Thus, the x-coordinate of the vertical strips can vary from $x = 0$ to $x = 1$, and so the mass is given by

$$M = \int_0^1 \int_{x^2}^{x} \delta(x, y)\, dy\, dx = \int_0^1 \int_{x^2}^{x} (1 + xy)\, dy\, dx.$$

Calculating the inner integral first gives

$$\begin{aligned} M &= \int_0^1 \int_{x^2}^{x} (1 + xy)\, dy\, dx = \int_0^1 \left(y + x\frac{y^2}{2} \right)\Bigg|_{y=x^2}^{y=x} dx \\ &= \int_0^1 \left(x - x^2 + \frac{x^3}{2} - \frac{x^5}{2} \right) dx = \left(\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{8} - \frac{x^6}{12} \right)\Bigg|_0^1 = \frac{5}{24} = 0.208 \text{ kg}. \end{aligned}$$

Example 5 A semicircular city of radius 3 km borders the ocean on the straight side. Find the average distance from points in the city to the ocean.

Solution Think of the ocean as everything below the x-axis in the xy-plane and think of the city as the upper half of the circular disk of radius 3 bounded by $x^2 + y^2 = 9$. (See Figure 16.18.)

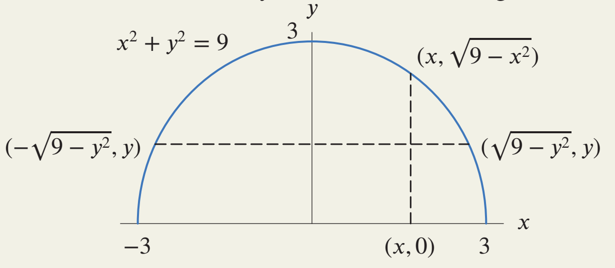


Figure 16.18: The city by the ocean showing a typical vertical strip and a typical horizontal strip

The distance from any point (x, y) in the city to the ocean is the vertical distance to the x-axis, namely y. Thus, we want to compute

$$\text{Average distance} = \frac{1}{\text{Area}(R)} \int_R y\, dA,$$

where R is the region between the upper half of the circle $x^2 + y^2 = 9$ and the x-axis. The area of R is $\pi 3^2/2 = 9\pi/2$.

To compute the integral, let's take the inner integral with respect to y. A vertical strip goes from the x-axis, namely $y = 0$, to the semicircle. The upper limit must be expressed in terms of x, so we solve $x^2 + y^2 = 9$ to get $y = \sqrt{9 - x^2}$. Since there is a strip for every x from -3 to 3, the integral is:

$$\begin{aligned} \int_R y\, dA &= \int_{-3}^{3} \left(\int_0^{\sqrt{9-x^2}} y\, dy \right) dx = \int_{-3}^{3} \left(\frac{y^2}{2} \Bigg|_{y=0}^{y=\sqrt{9-x^2}} \right) dx \\ &= \int_{-3}^{3} \frac{1}{2}(9 - x^2)\, dx = \frac{1}{2} \left(9x - \frac{x^3}{3} \right)\Bigg|_{-3}^{3} = \frac{1}{2}(18 - (-18)) = 18. \end{aligned}$$

Therefore, the average distance is $18/(9\pi/2) = 4/\pi = 1.273$ km.

What if we choose the inner integral with respect to x? Then we get the limits by looking at horizontal strips, not vertical, and we solve $x^2 + y^2 = 9$ for x in terms of y. We get $x = -\sqrt{9-y^2}$ at the left end of the strip and $x = \sqrt{9-y^2}$ at the right. There is a strip for every y from 0 to 3, so

$$\int_R y\,dA = \int_0^3 \left(\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} y\,dx\right) dy = \int_0^3 \left(yx\Big|_{x=-\sqrt{9-y^2}}^{x=\sqrt{9-y^2}} \right) dy = \int_0^3 2y\sqrt{9-y^2}\,dy$$

$$= -\frac{2}{3}(9-y^2)^{3/2}\Big|_0^3 = -\frac{2}{3}(0-27) = 18.$$

We get the same result as before. The average distance to the ocean is $(2/(9\pi))18 = 4/\pi = 1.273$ km.

In the examples so far, a region was given and the problem was to determine the limits for an iterated integral. Sometimes the limits are known and we want to determine the region.

Example 6 Sketch the region of integration for the iterated integral $\displaystyle\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\,dy\,dx$.

Solution The inner integral is with respect to y, so we imagine the region built of vertical strips. The bottom of each strip is on the line $y = x/3$, and the top is on the horizontal line $y = 2$. Since the limits of the outer integral are 0 and 6, the whole region is contained between the vertical lines $x = 0$ and $x = 6$. Notice that the lines $y = 2$ and $y = x/3$ meet where $x = 6$. See Figure 16.19.

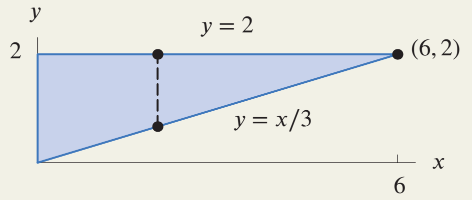


Figure 16.19: The region of integration for Example 6, showing the vertical strip

Reversing the Order of Integration

It is sometimes helpful to reverse the order of integration in an iterated integral. An integral which is difficult or impossible with the integration in one order can be quite straightforward in the other. The next example is such a case.

Example 7 Evaluate $\displaystyle\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\,dy\,dx$ using the region sketched in Figure 16.19.

Solution Since $\sqrt{y^3+1}$ has no elementary antiderivative, we cannot calculate the inner integral symbolically. We try reversing the order of integration. From Figure 16.19, we see that horizontal strips go from $x = 0$ to $x = 3y$ and that there is a strip for every y from 0 to 2. Thus, when we change the order of integration we get

$$\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\,dy\,dx = \int_0^2 \int_0^{3y} x\sqrt{y^3+1}\,dx\,dy.$$

Now we can at least do the inner integral because we know the antiderivative of x. What about the outer integral?

$$\int_0^2 \int_0^{3y} x\sqrt{y^3+1}\,dx\,dy = \int_0^2 \left(\frac{x^2}{2}\sqrt{y^3+1}\right)\Bigg|_{x=0}^{x=3y} dy = \int_0^2 \frac{9y^2}{2}(y^3+1)^{1/2}\,dy$$
$$= (y^3+1)^{3/2}\Big|_0^2 = 27 - 1 = 26.$$

Thus, reversing the order of integration made the integral in the previous problem much easier. Notice that to reverse the order it is essential first to sketch the region over which the integration is being performed.

Exercises and Problems for Section 16.2 Online Resource: Additional Problems for Section 16.2

EXERCISES

In Exercises 1–4, sketch the region of integration.

1. $\int_0^{\pi}\int_0^{x} y\sin x\,dy\,dx$

2. $\int_0^{1}\int_{y^2}^{y} xy\,dx\,dy$

3. $\int_0^{2}\int_0^{y^2} y^2x\,dx\,dy$

4. $\int_0^{1}\int_{x-2}^{\cos \pi x} y\,dy\,dx$

For Exercises 5–12, evaluate the integral.

5. $\int_0^{3}\int_0^{4} (4x+3y)\,dx\,dy$

6. $\int_0^{2}\int_0^{3} (x^2+y^2)\,dy\,dx$

7. $\int_0^{3}\int_0^{2} 6xy\,dy\,dx$

8. $\int_0^{1}\int_0^{2} x^2y\,dy\,dx$

9. $\int_0^{1}\int_0^{1} ye^{xy}\,dx\,dy$

10. $\int_0^{2}\int_0^{y} y\,dx\,dy$

11. $\int_0^{3}\int_0^{y} \sin x\,dx\,dy$

12. $\int_0^{\pi/2}\int_0^{\sin x} x\,dy\,dx$

For Exercises 13–20, sketch the region of integration and evaluate the integral.

13. $\int_1^{3}\int_0^{4} e^{x+y}\,dy\,dx$

14. $\int_0^{2}\int_0^{x} e^{x^2}\,dy\,dx$

15. $\int_1^{5}\int_x^{2x} \sin x\,dy\,dx$

16. $\int_1^{4}\int_{\sqrt{y}}^{y} x^2y^3\,dx\,dy$

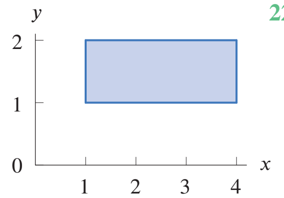
17. $\int_1^{2}\int_y^{3y} xy\,dx\,dy$

18. $\int_0^{1}\int_x^{\sqrt{x}} 30x\,dy\,dx$

19. $\int_0^{2}\int_0^{2x} xe^{x^3}\,dy\,dx$

20. $\int_0^{1}\int_1^{1+x^2} \frac{x}{\sqrt{y}}\,dy\,dx$

In Exercises 21–26, write $\int_R f\,dA$ as an iterated integral for the shaded region R.

21.


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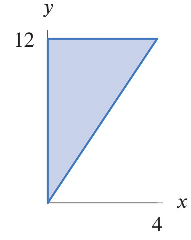

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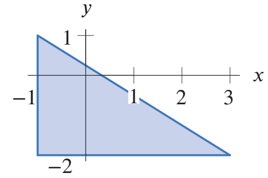

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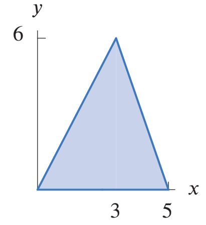

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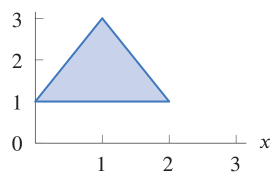

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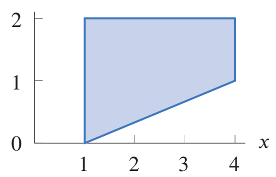
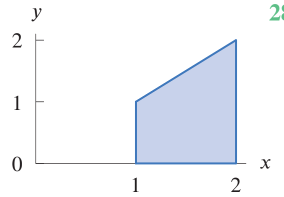


For Exercises 27–28, write $\int_R f\,dA$ as an iterated integral in two different ways for the shaded region R.

27.


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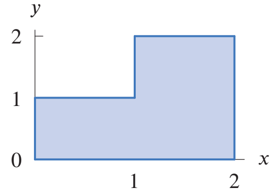


For Exercises 29–33, evaluate the integral.

29. $\int_R \sqrt{x+y}\, dA$, where R is the rectangle $0 \le x \le 1$, $0 \le y \le 2$.

30. The integral in Exercise 29 using the other order of integration.

31. $\int_R (5x^2+1)\sin 3y\, dA$, where R is the rectangle $-1 \le x \le 1$, $0 \le y \le \pi/3$.

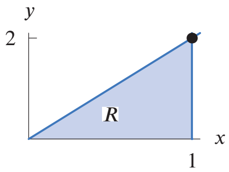
32. $\int_R xy\, dA$, where R is the triangle $x+y \le 1$, $x \ge 0$, $y \ge 0$.

33. $\int_R (2x+3y)^2\, dA$, where R is the triangle with vertices at $(-1,0)$, $(0,1)$, and $(1,0)$.

PROBLEMS

In Problems 34–37, integrate $f(x,y) = xy$ over the region R.

34.



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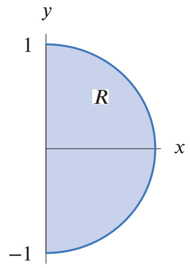

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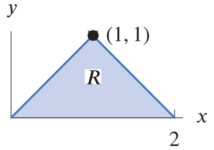

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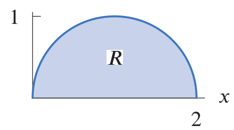


38. **(a)** Use four subrectangles to approximate the volume of the object whose base is the region $0 \le x \le 4$ and $0 \le y \le 6$, and whose height is given by $f(x,y) = xy$. Find an overestimate and an underestimate and average the two.

(b) Integrate to find the exact volume of the three-dimensional object described in part (a).

For Problems 39–42, sketch the region of integration then rewrite the integral with the order of integration reversed.

39. $\displaystyle\int_0^3 \int_{2y}^6 f(x,y)\, dx\, dy$

40. $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} f(x,y)\, dy\, dx$

41. $\displaystyle\int_{-3}^3 \int_0^{9-x^2} f(x,y)\, dy\, dx$

42. $\displaystyle\int_0^2 \int_{y-2}^{2-y} f(x,y)\, dx\, dy$

In Problems 43–50, evaluate the integral by reversing the order of integration.

43. $\displaystyle\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$

44. $\displaystyle\int_0^1 \int_y^1 \sin(x^2)\;\, dx\, dy$

45. $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \sqrt{2+x^3}\, dx\, dy$

46. $\displaystyle\int_0^3 \int_{y^2}^9 y\sin(x^2)\, dx\, dy$

47. $\displaystyle\int_0^1 \int_{e^y}^e \frac{x}{\ln x}\, dx\, dy$

48. $\displaystyle\int_0^1 \int_x^1 \cos(y^2)\, dy\, dx$

49. $\displaystyle\int_0^8 \int_{\sqrt[3]{y}}^2 \frac{1}{1+x^4}\, dx\, dy$

50. $\displaystyle\int_0^1 \int_0^x e^{2y-y^2}\, dy\, dx$

51. Each of the integrals (I)–(VI) takes one of two distinct values. Without evaluating, group them by value.

I. $\displaystyle\int_0^5 \int_0^{10} xy^2\, dx\, dy$ II. $\displaystyle\int_0^5 \int_0^{10} xy^2\, dy\, dx$

III. $\displaystyle\int_0^{10} \int_0^5 xy^2\, dx\, dy$ IV. $\displaystyle\int_0^{10} \int_0^5 xy^2\, dy\, dx$

V. $\displaystyle\int_0^5 \int_0^{10} uv^2\, du\, dv$ VI. $\displaystyle\int_0^5 \int_0^{10} uv^2\, dv\, du$

52. Find the volume under the graph of the function $f(x,y) = 6x^2 y$ over the region shown in Figure 16.20.

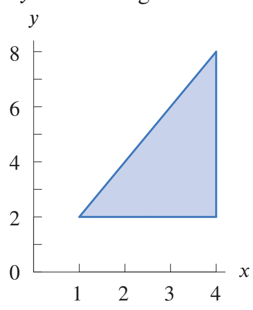


Figure 16.20

53. **(a)** Find the volume below the surface $z = x^2 + y^2$ and above the xy-plane for $-1 \le x \le 1, -1 \le y \le 1$.

(b) Find the volume above the surface $z = x^2 + y^2$ and below the plane $z = 2$ for $-1 \le x \le 1$, $-1 \le y \le 1$.

54. Compute the integral

$$\int\int_R (2x^2 + y)\, dA,$$

where R is the triangular region with vertices at $(0,1)$, $(-2,3)$ and $(2,3)$.

55. **(a)** Sketch the region in the xy-plane bounded by the x-axis, $y = x$, and $x + y = 1$.

(b) Express the integral of $f(x,y)$ over this region in terms of iterated integrals in two ways. (In one, use $dx\, dy$; in the other, use $dy\, dx$.)

(c) Using one of your answers to part (b), evaluate the integral exactly with $f(x,y) = x$.

56. Let $f(x, y) = x^2 e^{x^2}$ and let R be the triangle bounded by the lines $x = 3$, $x = y/2$, and $y = x$ in the xy-plane.

(a) Express $\int_R f\, dA$ as a double integral in two different ways.
(b) Evaluate one of them.

57. Find the average value of $f(x, y) = x^2 + 4y$ on the rectangle $0 \le x \le 3$ and $0 \le y \le 6$.

58. Find the average value of $f(x, y) = xy^2$ on the rectangle $0 \le x \le 4$, $0 \le y \le 3$.

59. Figure 16.21 shows two metal plates carrying electrical charges. The charge density (in coulombs per square meter) of each at the point (x, y) is $\sigma(x, y) = 6x + 6$ for x, y in meters.

(a) Without calculation, decide which plate carries a greater total charge, and explain your reasoning.
(b) Find the total charge on both plates, and compare to your answer from part (a).

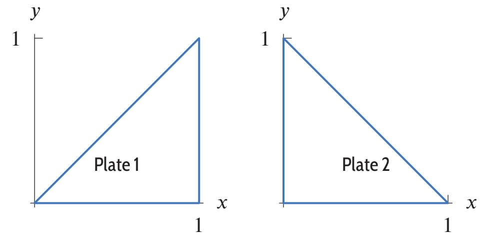


Figure 16.21

60. The population density in people per km^2 for the trapezoid-shaped town in Figure 16.22 for x, y in kilometers is $\delta(x, y) = 100x + 200y$. Find the town's population.

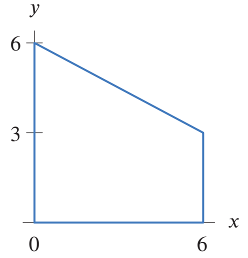


Figure 16.22

61. The quarter-disk-shaped metal plate in Figure 16.23 has radius 3 and density $\sigma(x, y) = 2y$ gm/cm^2, with x, y in cm. Find the mass of the plate.

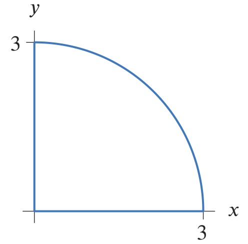


Figure 16.23

In Problems 62–63 set up, but do not evaluate, an iterated integral for the volume of the solid.

62. Under the graph of $f(x, y) = 25 - x^2 - y^2$ and above the xy-plane.

63. Below the graph of $f(x, y) = 25 - x^2 - y^2$ and above the plane $z = 16$.

64. A solid with flat base in the xy-plane is bounded by the vertical planes $y = 0$ and $y - x = 4$, and the slanted plane $2x + y + z = 4$.

(a) Draw the base of the solid.
(b) Set up, but do not evaluate, an iterated integral for the volume of the solid.

In Problems 65–69, find the volume of the solid region.

65. Under the graph of $f(x, y) = xy$ and above the square $0 \le x \le 2$, $0 \le y \le 2$ in the xy-plane.

66. Under the graph of $f(x, y) = x^2 + y^2$ and above the triangle $0 \le y \le x$, $0 \le x \le 1$.

67. Under the graph of $f(x, y) = x + y$ and above the region $y^2 \le x$, $0 \le x \le 9$, $y \ge 0$.

68. Under the graph of $2x + y + z = 4$ in the first octant.

69. The solid region R bounded by the coordinate planes and the graph of $ax + by + cz = 1$. Assume a, b, and $c > 0$.

70. If R is the region $x + y \ge a$, $x^2 + y^2 \le a^2$, with $a > 0$, evaluate the integral

$$\int_R xy\, dA.$$

71. The region W lies below the surface $f(x, y) = 2e^{-(x-1)^2 - y^2}$ and above the disk $x^2 + y^2 \le 4$ in the xy-plane.

(a) Describe in words the contours of f, using $f(x, y) = 1$ as an example.
(b) Write an integral giving the area of the cross-section of W in the plane $x = 1$.
(c) Write an iterated double integral giving the volume of W.

72. Find the average distance to the x-axis for points in the region in the first quadrant bounded by the x-axis and the graph of $y = x - x^2$.

73. Give the contour diagram of a function f whose average value on the square $0 \le x \le 1$, $0 \le y \le 1$ is

(a) Greater than the average of the values of f at the four corners of the square.
(b) Less than the average of the values of f at the four corners of the square.

74. The function $f(x, y) = ax + by$ has an average value of 20 on the rectangle $0 \le x \le 2$, $0 \le y \le 3$.

(a) What can you say about the constants a and b?

(b) Find two different choices for f that have average value 20 on the rectangle, and give their contour diagrams on the rectangle.

75. The function $f(x, y) = ax^2 + bxy + cy^2$ has an average value of 20 on the square $0 \le x \le 2$, $0 \le y \le 2$.

(a) What can you say about the constants a, b, and c?
(b) Find two different choices for f that have average value 20 on the square, and give their contour diagrams on the square.

Strengthen Your Understanding

In Problems 76–77, explain what is wrong with the statement.

76. $\int_0^1 \int_0^x f(x, y)\, dy\, dx = \int_0^1 \int_0^y f(x, y)\, dx\, dy$

77. $\int_0^1 \int_0^y xy\, dx\, dy = \int_0^y \int_0^1 xy\, dy\, dx$

In Problems 78–80, give an example of:

78. An iterated double integral, with limits of integration, giving the volume of a cylinder standing vertically with a circular base in the xy-plane.

79. A nonconstant function, f, whose integral is 4 over the triangular region with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$.

80. A double integral representing the volume of a triangular prism of base area 6.

Are the statements in Problems 81–88 true or false? Give reasons for your answer.

81. The iterated integral $\int_0^1 \int_5^{12} f\, dx\, dy$ is computed over the rectangle $0 \le x \le 1$, $5 \le y \le 12$.

82. If R is the region inside the triangle with vertices $(0, 0)$, $(1, 1)$ and $(0, 2)$, then the double integral $\int_R f\, dA$ can be evaluated by an iterated integral of the form $\int_0^2 \int_0^1 f\, dx\, dy$.

83. The region of integration of the iterated integral $\int_1^2 \int_{x^2}^{x^3} f\, dy\, dx$ lies completely in the first quadrant (that is, $x \ge 0$, $y \ge 0$).

84. If the limits a, b, c and d in the iterated integral $\int_a^b \int_c^d f\, dy\, dx$ are all positive, then the value of $\int_a^b \int_c^d f\, dy\, dx$ is also positive.

85. If $f(x, y)$ is a function of y only, then $\int_a^b \int_0^1 f\, dx\, dy = \int_a^b f\, dy$.

86. If R is the region inside a circle of radius a, centered at the origin, then $\int_R f\, dA = \int_{-a}^{a} \int_0^{\sqrt{a^2-x^2}} f\, dy\, dx$.

87. If $f(x, y) = g(x) \cdot h(y)$, where g and h are single-variable functions, then

$$\int_a^b \int_c^d f\, dy\, dx = \left(\int_a^b g(x)\, dx\right) \cdot \left(\int_c^d h(y)\, dy\right).$$

88. If $f(x, y) = g(x) + h(y)$, where g and h are single-variable functions, then

$$\int_a^b \int_c^d f\, dx\, dy = \left(\int_a^b g(x)\, dx\right) + \left(\int_c^d h(y)\, dy\right).$$

16.3 TRIPLE INTEGRALS

A continuous function of three variables can be integrated over a solid region W in 3-space in the same way as a function of two variables is integrated over a flat region in 2-space. Again, we start with a Riemann sum. First we subdivide W into smaller regions, then we multiply the volume of each region by a value of the function in that region, and then we add the results. For example, if W is the box $a \le x \le b$, $c \le y \le d$, $p \le z \le q$, then we subdivide each side into n, m, and l pieces, thereby chopping W into nml smaller boxes, as shown in Figure 16.24.

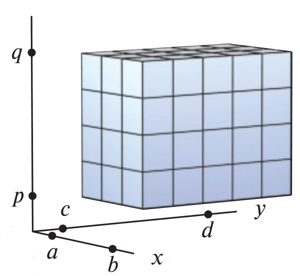


Figure 16.24: Subdividing a three-dimensional box

The volume of each smaller box is

$$\Delta V = \Delta x \Delta y \Delta z,$$

where $\Delta x = (b-a)/n$, and $\Delta y = (d-c)/m$, and $\Delta z = (q-p)/l$. Using this subdivision, we pick a point $(u_{ijk}, v_{ijk}, w_{ijk})$ in the ijk-th small box and construct a Riemann sum

$$\sum_{i,j,k} f(u_{ijk}, v_{ijk}, w_{ijk})\,\Delta V.$$

If f is continuous, as Δx, Δy, and Δz approach 0, this Riemann sum approaches the definite integral, $\int_W f\,dV$, called a *triple integral*, which is defined as

$$\int_W f\,dV = \lim_{\Delta x,\Delta y,\Delta z\to 0} \sum_{i,j,k} f(u_{ijk}, v_{ijk}, w_{ijk})\,\Delta x\,\Delta y\,\Delta z.$$

As in the case of a double integral, we can evaluate this integral as an iterated integral:

Triple integral as an iterated integral

$$\int_W f\,dV = \int_p^q \left(\int_c^d \left(\int_a^b f(x,y,z)\,dx \right) dy \right) dz,$$

where y and z are treated as constants in the innermost (dx) integral, and z is treated as a constant in the middle (dy) integral. Other orders of integration are possible.

Example 1 A cube C has sides of length 4 cm and is made of a material of variable density. If one corner is at the origin and the adjacent corners are on the positive x, y, and z axes, then the density at the point (x, y, z) is $\delta(x, y, z) = 1 + xyz$ gm/cm^3. Find the mass of the cube.

Solution Consider a small piece ΔV of the cube, small enough so that the density remains close to constant over the piece. Then

$$\text{Mass of small piece} = \text{Density} \cdot \text{Volume} \approx \delta(x,y,z)\,\Delta V.$$

To get the total mass, we add the masses of the small pieces and take the limit as $\Delta V \to 0$. Thus, the mass is the triple integral

$$M = \int_C \delta\,dV = \int_0^4 \int_0^4 \int_0^4 (1 + xyz)\,dx\,dy\,dz = \int_0^4 \int_0^4 \left(x + \frac{1}{2}x^2 yz \right)\Bigg|_{x=0}^{x=4} dy\,dz$$
$$= \int_0^4 \int_0^4 (4 + 8yz)\,dy\,dz = \int_0^4 \left(4y + 4y^2 z\right)\Bigg|_{y=0}^{y=4} dz = \int_0^4 (16 + 64z)\,dz = 576 \text{ gm}.$$

Example 2 Express the volume of the building described in Example 1 on page 848 as a triple integral.

Solution The building is given by $0 \le x \le 8$, $0 \le y \le 16$, and $0 \le z \le 12 - x/4 - y/8$. (See Figure 16.25.) To find its volume, divide it into small cubes of volume $\Delta V = \Delta x\,\Delta y\,\Delta z$ and add. First, make a vertical stack of cubes above the point $(x, y, 0)$. This stack goes from $z = 0$ to $z = 12 - x/4 - y/8$, so

$$\text{Volume of vertical stack} \approx \sum_z \Delta V = \sum_z \Delta x\,\Delta y\,\Delta z = \left(\sum_z \Delta z \right) \Delta x\,\Delta y.$$

Next, line up these stacks parallel to the y-axis to form a slice from $y = 0$ to $y = 16$. So

$$\text{Volume of slice} \approx \left(\sum_y \sum_z \Delta z\, \Delta y\right) \Delta x.$$

Finally, line up the slices along the x-axis from $x = 0$ to $x = 8$ and add up their volumes, to get

$$\text{Volume of building} \approx \sum_x \sum_y \sum_z \Delta z\, \Delta y\, \Delta x.$$

Thus, in the limit,

$$\text{Volume of building} = \int_0^8 \int_0^{16} \int_0^{12-x/4-y/8} 1\, dz\, dy\, dx.$$

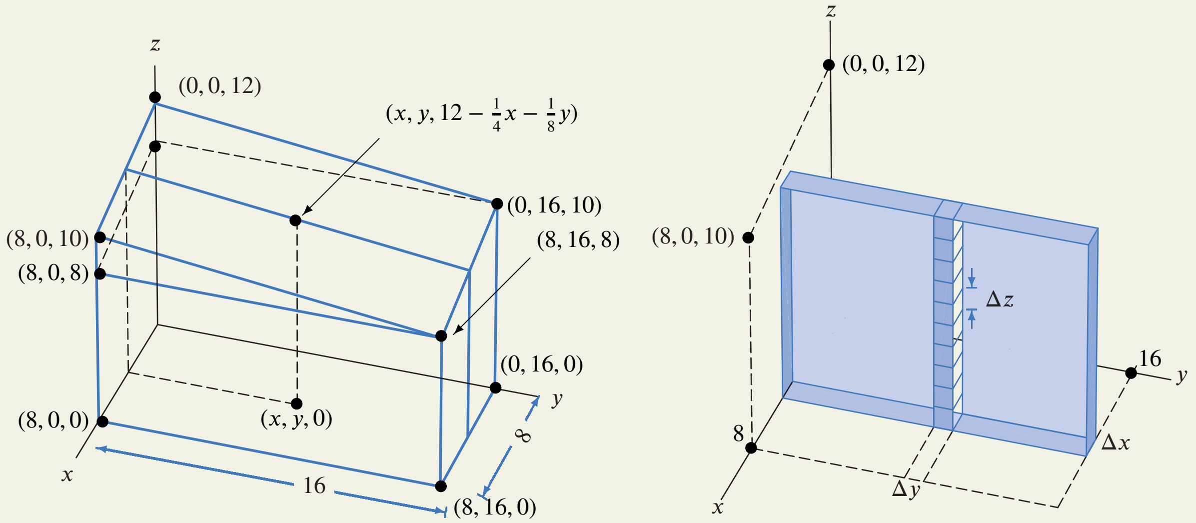


Figure 16.25: Volume of building (shown to left) divided into blocks and slabs for a triple integral

The preceding examples show that the triple integral has interpretations similar to the double integral:

- If $\rho(x, y, z)$ is density, then $\int_W \rho\, dV$ is the total quantity in the solid region W.
- $\int_W 1\, dV$ is the volume of the solid region W.

Example 3 Set up an iterated integral to compute the mass of the solid cone bounded by $z = \sqrt{x^2 + y^2}$ and $z = 3$, if the density is given by $\delta(x, y, z) = z$.

Solution We break the cone in Figure 16.26 into small cubes of volume $\Delta V = \Delta x\, \Delta y\, \Delta z$, on which the density is approximately constant, and approximate the mass of each cube by $\delta(x, y, z)\, \Delta x\, \Delta y\, \Delta z$. Stacking the cubes vertically above the point $(x, y, 0)$, starting on the cone at height $z = \sqrt{x^2 + y^2}$ and going up to $z = 3$, tells us that the inner integral is

$$\int_{\sqrt{x^2+y^2}}^{3} \delta(x, y, z)\, dz = \int_{\sqrt{x^2+y^2}}^{3} z\, dz.$$

There is a stack for every point in the xy-plane in the shadow of the cone. The cone $z = \sqrt{x^2+y^2}$ intersects the horizontal plane $z = 3$ in the circle $x^2 + y^2 = 9$, so there is a stack for all (x, y) in the region $x^2 + y^2 \leq 9$. Lining up the stacks parallel to the y-axis gives a slice from $y = -\sqrt{9-x^2}$ to $y = \sqrt{9-x^2}$, for each fixed value of x. Thus, the limits on the middle integral are

$$\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{3} z\, dz\, dy.$$

Finally, there is a slice for each x between -3 and 3, so the integral we want is

$$\text{Mass} = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{3} z\, dz\, dy\, dx.$$

Notice that setting up the limits on the two outer integrals was just like setting up the limits for a double integral over the region $x^2 + y^2 \leq 9$.

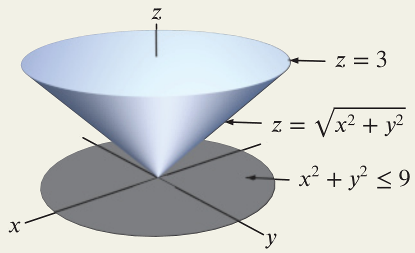


Figure 16.26: The cone $z = \sqrt{x^2+y^2}$ with its shadow on the xy-plane

As the previous example illustrates, for a region W contained between two surfaces, the innermost limits correspond to these surfaces. The middle and outer limits ensure that we integrate over the "shadow" of W in the xy-plane.

Limits on Triple Integrals

- The limits for the outer integral are constants.
- The limits for the middle integral can involve only one variable (that in the outer integral).
- The limits for the inner integral can involve two variables (those on the two outer integrals).

Exercises and Problems for Section 16.3 Online Resource: Additional Problems for Section 16.3

EXERCISES

In Exercises 1–4, find the triple integrals of the function over the region W.

1. $f(x, y, z) = x^2 + 5y^2 - z$, W is the rectangular box $0 \leq x \leq 2$, $-1 \leq y \leq 1$, $2 \leq z \leq 3$.

2. $h(x, y, z) = ax + by + cz$, W is the rectangular box $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 2$.

3. $f(x, y, z) = \sin x \cos(y + z)$, W is the cube $0 \leq x \leq \pi$, $0 \leq y \leq \pi$, $0 \leq z \leq \pi$.

4. $f(x, y, z) = e^{-x-y-z}$, W is the rectangular box with corners at $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$.

Sketch the region of integration in Exercises 5–13.

5. $\int_0^1 \int_{-1}^1 \int_0^{\sqrt{1-x^2}} f(x,y,z)\,dz\,dx\,dy$

6. $\int_0^1 \int_{-1}^1 \int_0^{\sqrt{1-z^2}} f(x,y,z)\,dy\,dz\,dx$

7. $\int_0^1 \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y,z)\,dz\,dx\,dy$

8. $\int_{-1}^1 \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} f(x,y,z)\,dy\,dz\,dx$

9. $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-z^2}} f(x,y,z)\,dy\,dz\,dx$

10. $\int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_0^{\sqrt{1-x^2-z^2}} f(x,y,z)\,dy\,dx\,dz$

11. $\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dx\,dy$

12. $\int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} f(x,y,z)\,dx\,dy\,dz$

13. $\int_0^1 \int_0^{\sqrt{1-z^2}} \int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} f(x,y,z)\,dy\,dx\,dz$

In Exercises 14–15, for x, y and z in meters, what does the integral over the solid region E represent? Give units.

14. $\int_E 1\,dV$

15. $\int_E \delta(x,y,z)\,dV$, where $\delta(x,y,z)$ is density, in kg/m^3.

PROBLEMS

In Problems 16–20, decide whether the integrals are positive, negative, or zero. Let S be the solid sphere $x^2+y^2+z^2 \le 1$, and T be the top half of this sphere (with $z \ge 0$), and B be the bottom half (with $z \le 0$), and R be the right half of the sphere (with $x \ge 0$), and L be the left half (with $x \le 0$).

16. $\int_T e^z\,dV$ 17. $\int_B e^z\,dV$ 18. $\int_S \sin z\,dV$

19. $\int_T \sin z\,dV$ 20. $\int_R \sin z\,dV$

Let W be the solid cone bounded by $z = \sqrt{x^2+y^2}$ and $z = 2$. For Problems 21–29, decide (without calculating its value) whether the integral is positive, negative, or zero.

21. $\int_W y\,dV$ 22. $\int_W x\,dV$

23. $\int_W z\,dV$ 24. $\int_W xy\,dV$

25. $\int_W xyz\,dV$ 26. $\int_W (z-2)\,dV$

27. $\int_W \sqrt{x^2+y^2}\,dV$ 28. $\int_W e^{-xyz}\,dV$

29. $\int_W (z - \sqrt{x^2+y^2})\,dV$

In Problems 30–34, let W be the solid cylinder bounded by $x^2+y^2=1$, $z=0$, and $z=2$. Decide (without calculating its value) whether the integral is positive, negative, or zero.

30. $\int_W x\,dV$ 31. $\int_W z\,dV$

32. $\int_W (x^2+y^2-2)\,dV$ 33. $\int_W (z-1)\,dV$

34. $\int_W e^{-y}\,dV$

35. Find the volume of the region bounded by the planes $z=3y$, $z=y$, $y=1$, $x=1$, and $x=2$.

36. Find the volume of the region bounded by $z = x^2$, $0 \le x \le 5$, and the planes $y=0$, $y=3$, and $z=0$.

37. Find the volume of the region in the first octant bounded by the coordinate planes and the surface $x+y+z=2$.

38. A trough with triangular cross-section lies along the x-axis for $0 \le x \le 10$. The slanted sides are given by $z=y$ and $z=-y$ for $0 \le z \le 1$ and the ends by $x=0$ and $x=10$, where x, y, z are in meters. The trough contains a sludge whose density at the point (x,y,z) is $\delta = e^{-3x}$ kg per m^3.

(a) Express the total mass of sludge in the trough in terms of triple integrals.
(b) Find the mass.

39. Find the volume of the region bounded by $z=x+y$, $z=10$, and the planes $x=0$, $y=0$.

In Problems 40–45, write a triple integral, including limits of integration, that gives the specified volume.

40. Between $z=x+y$ and $z=1+2x+2y$ and above $0 \le x \le 1, 0 \le y \le 2$.

41. Between the paraboloid $z=x^2+y^2$ and the sphere $x^2+y^2+z^2=4$ and above the disk $x^2+y^2 \le 1$.

42. Between $2x+2y+z=6$ and $3x+4y+z=6$ and above $x+y \le 1, x \ge 0, y \ge 0$.

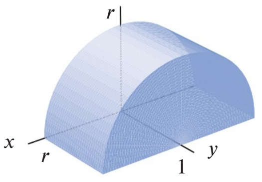
43. Under the sphere $x^2+y^2+z^2=9$ and above the region between $y=x$ and $y=2x-2$ in the xy-plane in the first quadrant.

44. Between the top portion of the sphere $x^2+y^2+z^2=9$ and the plane $z=2$.

45. Under the sphere $x^2+y^2+z^2=4$ and above the region $x^2+y^2 \le 4, 0 \le x \le 1, 0 \le y \le 2$ in the xy-plane.

In Problems 46–49, write limits of integration for the integral $\int_W f(x, y, z)\,dV$ where W is the quarter or half sphere or cylinder shown.

46.



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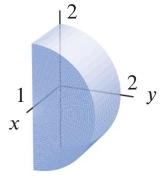

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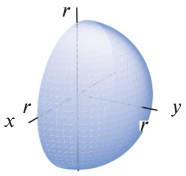

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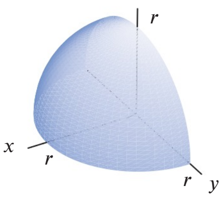


50. Find the volume of the region between the plane $z = x$ and the surface $z = x^2$, and the planes $y = 0$, and $y = 3$.

51. Find the volume of the region bounded by $z = x + y$, $0 \le x \le 5$, $0 \le y \le 5$, and the planes $x = 0$, $y = 0$, and $z = 0$.

52. Find the volume of the pyramid with base in the plane $z = -6$ and sides formed by the three planes $y = 0$ and $y - x = 4$ and $2x + y + z = 4$.

53. Find the volume between the planes $z = 1 + x + y$ and $x + y + z = 1$ and above the triangle $x + y \le 1, x \ge 0$, $y \ge 0$ in the xy-plane.

54. Find the mass of a triangular-shaped solid bounded by the planes $z = 1 + x$, $z = 1 - x$, $z = 0$, and with $0 \le y \le 3$. The density is $\delta = 10 - z$ gm/cm^3, and x, y, z are in cm.

55. Find the mass of the solid bounded by the xy-plane, yz-plane, xz-plane, and the plane $(x/3)+(y/2)+(z/6) = 1$, if the density of the solid is given by $\delta(x, y, z) = x + y$.

56. Find the mass of the pyramid with base in the plane $z = -6$ and sides formed by the three planes $y = 0$ and $y - x = 4$ and $2x + y + z = 4$, if the density of the solid is given by $\delta(x, y, z) = y$.

57. Let E be the solid pyramid bounded by the planes $x + z = 6$, $x - z = 0$, $y + z = 6$, $y - z = 0$, and above the plane $z = 0$ (see Figure 16.27). The density at any point in the pyramid is given by $\delta(x, y, z) = z$ grams per cm^3, where x, y, and z are measured in cm.

(a) Explain in practical terms what the triple integral $\int_E z\,dV$ represents.
(b) In evaluating the integral from part (a), how many separate triple integrals would be required if we chose to integrate in the z-direction first?
(c) Evaluate the triple integral from part (a) by integrating in a well-chosen order.

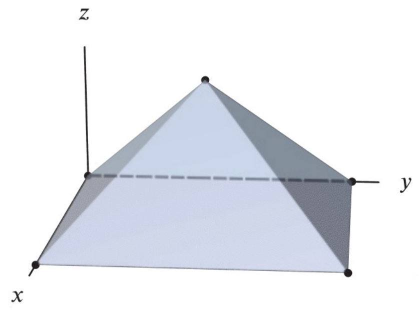


Figure 16.27

58. (a) What is the equation of the plane passing through the points (1, 0, 0), (0, 1, 0), and (0, 0, 1)?
(b) Find the volume of the region bounded by this plane and the planes $x = 0$, $y = 0$, and $z = 0$.

Problems 59–61 refer to Figure 16.28, which shows triangular portions of the planes $2x+4y+z = 4$, $3x-2y = 0$, $z = 2$, and the three coordinate planes $x = 0$, $y = 0$, and $z = 0$. For each solid region E, write down an iterated integral for the triple integral $\int_E f(x, y, z)\,dV$.

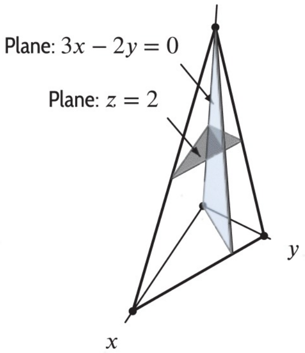


Figure 16.28

59. E is the region bounded by $y = 0$, $z = 0$, $3x - 2y = 0$, and $2x + 4y + z = 4$.

60. E is the region bounded by $x = 0$, $y = 0$, $z = 0$, $z = 2$, and $2x + 4y + z = 4$.

61. E is the region bounded by $x = 0$, $z = 0$, $3x - 2y = 0$, and $2x + 4y + z = 4$.

62. Figure 16.29 shows part of a spherical ball of radius 5 cm. Write an iterated triple integral which represents the volume of this region.

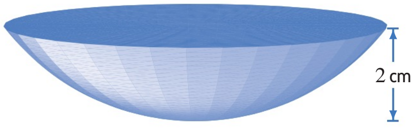


Figure 16.29

63. A solid region D is a half cylinder of radius 1 lying horizontally with its rectangular base in the xy-plane and its axis along the y-axis from $y = 0$ to $y = 10$. (The region is above the xy-plane.)

(a) What is the equation of the curved surface of this half cylinder?

(b) Write the limits of integration of the integral $\int_D f(x, y, z)\,dV$ in Cartesian coordinates.

64. Set up, but do not evaluate, an iterated integral for the volume of the solid formed by the intersections of the cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$.

Problems 65–67 refer to Figure 16.30, which shows E, the region in the first octant bounded by the parabolic cylinder $z = 6y^2$ and the elliptical cylinder $x^2 + 3y^2 = 12$. For the given order of integration, write an iterated integral equivalent to the triple integral $\int_E f(x, y, z)\,dV$.

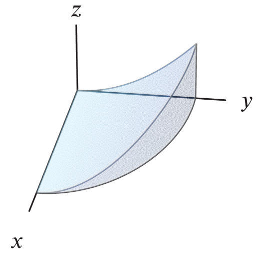


Figure 16.30

65. $dz\,dx\,dy$ **66.** $dx\,dz\,dy$ **67.** $dy\,dz\,dx$

Problems 68–71 refer to Figure 16.31, which shows E, the region in the first octant bounded by the planes $z = 5$ and $5x + 3z = 15$ and the elliptical cylinder $4x^2 + 9y^2 = 36$. For the given order of integration, write an iterated integral equivalent to the triple integral $\int_E f(x, y, z)\,dV$.

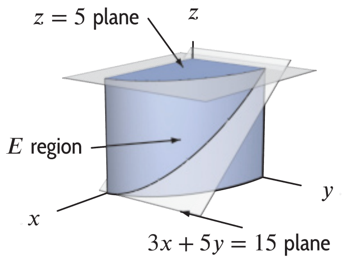


Figure 16.31

68. $dz\,dy\,dx$ **69.** $dz\,dx\,dy$

70. $dy\,dz\,dx$ **71.** $dy\,dx\,dz$

Problems 72–74 refer to Figure 16.32, which shows E, the region in the first octant bounded by the plane $x + y = 2$ and the parabolic cylinder $z = 4 - x^2$. For the given order of integration, write an iterated integral, or sum of integrals, equivalent to the triple integral $\int_E f(x, y, z)\,dV$.

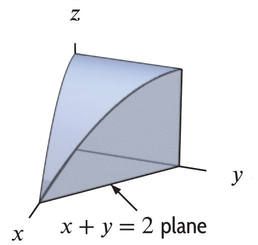


Figure 16.32

72. $dz\,dy\,dx$ **73.** $dy\,dz\,dx$ **74.** $dy\,dx\,dz$

Problems 75–76 concern the *center of mass*, the point at which the mass of a solid body in motion can be considered to be concentrated. If the object has density $\rho(x, y, z)$ at the point (x, y, z) and occupies a region W, then the coordinates $(\bar{x}, \bar{y}, \bar{z})$ of the center of mass are given by

$$\bar{x} = \frac{1}{m}\int_W x\rho\,dV \quad \bar{y} = \frac{1}{m}\int_W y\rho\,dV \quad \bar{z} = \frac{1}{m}\int_W z\rho\,dV$$

where $m = \int_W \rho\,dV$ is the total mass of the body.

75. A solid is bounded below by the square $z = 0, 0 \le x \le 1, 0 \le y \le 1$ and above by the surface $z = x + y + 1$. Find the total mass and the coordinates of the center of mass if the density is 1 gm/cm^3 and x, y, z are measured in centimeters.

76. Find the center of mass of the tetrahedron that is bounded by the xy, yz, xz planes and the plane $x + 2y + 3z = 1$. Assume the density is 1 gm/cm^3 and x, y, z are in centimeters.

Strengthen Your Understanding

In Problems 77–78, explain what is wrong with the statement.

77. Let S be the solid sphere $x^2 + y^2 + z^2 \le 1$ and let U be the upper half of S where $z \ge 0$. Then
$\int_S f(x, y, z)\,dV = 2\int_U f(x, y, z)\,dV$.

78. $\int_0^1 \int_0^x \int_0^y f(x, y, z)\,dz\,dy\,dx = \int_0^1 \int_y^1 \int_0^x f(x, y, z)\,dz\,dx\,dy$

In Problems 79–80, give an example of:

79. A function f such that $\int_R f\,dV = 7$, where R is the cylinder $x^2 + y^2 \le 4, 0 \le z \le 3$.

80. A nonconstant function $f(x, y, z)$ such that if B is the region enclosed by the sphere of radius 1 centered at the origin, the integral $\int_B f(x, y, z)\,dx\,dy\,dz$ is zero.

Are the statements in Problems 81–90 true or false? Give reasons for your answer.

81. If $\rho(x, y, z)$ is mass density of a material in 3-space, then $\int_W \rho(x, y, z)\, dV$ gives the volume of the solid region W.

82. The region of integration of the triple iterated integral $\int_0^1 \int_0^1 \int_0^x f\, dz\, dy\, dx$ lies above a square in the xy-plane and below a plane.

83. If W is the unit ball $x^2 + y^2 + z^2 \le 1$ then an iterated integral over W is $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} f\, dz\, dy\, dx$.

84. The iterated integrals $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} f\, dz\, dy\, dx$ and $\int_0^1 \int_0^{1-z} \int_0^{1-y-z} f\, dx\, dy\, dz$ are equal.

85. The iterated integrals $\int_{-1}^1 \int_0^1 \int_0^{1-x^2} f\, dz\, dy\, dx$ and $\int_0^1 \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} f\, dx\, dy\, dz$ are equal.

86. If W is a rectangular solid in 3-space, then $\int_W f\, dV = \int_a^b \int_c^d \int_e^k f\, dz\, dy\, dx$, where a, b, c, d, e, and k are constants.

87. If W is the unit cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ and $\int_W f\, dV = 0$, then $f = 0$ everywhere in the unit cube.

88. If $f > g$ at all points in the solid region W, then $\int_W f\, dV > \int_W g\, dV$.

89. If W_1 and W_2 are solid regions with volume$(W_1) >$ volume(W_2) then $\int_{W_1} f\, dV > \int_{W_2} f\, dV$.

90. Both double and triple integrals can be used to compute volume.

16.4 DOUBLE INTEGRALS IN POLAR COORDINATES

Integration in Polar Coordinates

We started this chapter by putting a rectangular grid on the fox population density map, to estimate the total population using a Riemann sum. However, sometimes a polar grid is more appropriate.

Example 1 A biologist studying insect populations around a circular lake divides the area into the polar sectors of radii 2, 3, and 4 km in Figure 16.33. The approximate population density in each sector is shown in millions per square km. Estimate the total insect population around the lake.

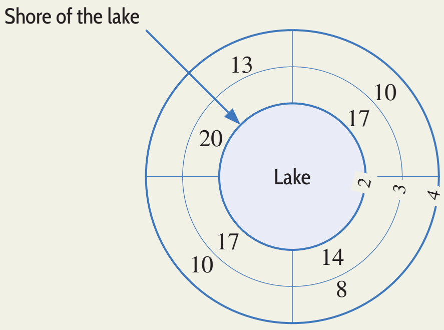


Figure 16.33: An insect-infested lake showing the insect population density by sector

Solution To get the estimate, we multiply the population density in each sector by the area of that sector. Unlike the rectangles in a rectangular grid, the sectors in this grid do not all have the same area. The inner sectors have area

$$\frac{1}{4}(\pi 3^2 - \pi 2^2) = \frac{5\pi}{4} \approx 3.93 \text{ km}^2,$$

and the outer sectors have area

$$\frac{1}{4}(\pi 4^2 - \pi 3^2) = \frac{7\pi}{4} \approx 5.50 \text{ km}^2,$$

so we estimate

$$\begin{aligned} \text{Population} &\approx (20)(3.93) + (17)(3.93) + (14)(3.93) + (17)(3.93) \\ &\quad + (13)(5.50) + (10)(5.50) + (8)(5.50) + (10)(5.50) \\ &= 492.74 \text{ million insects.} \end{aligned}$$

What Is dA in Polar Coordinates?

The previous example used a polar grid rather than a rectangular grid. A rectangular grid is constructed from vertical and horizontal lines of the form $x = k$ (a constant) and $y = l$ (another constant). In polar coordinates, $r = k$ gives a circle of radius k centered at the origin and $\theta = l$ gives a ray emanating from the origin (at angle l with the x-axis). A polar grid is built out of these circles and rays. Suppose we want to integrate $f(r, \theta)$ over the region R in Figure 16.34.

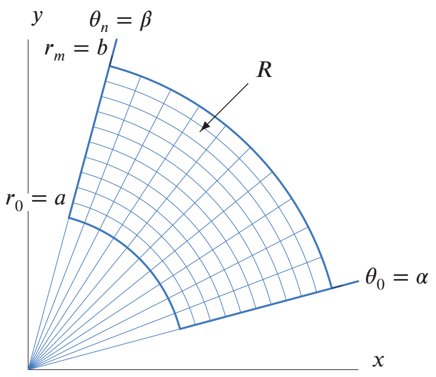


Figure 16.34: Dividing up a region using a polar grid

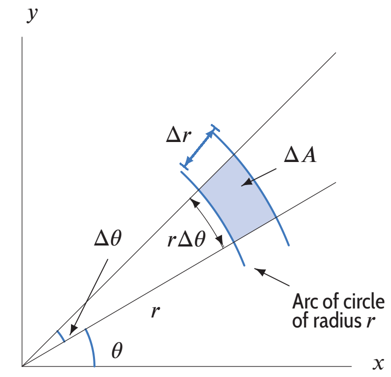


Figure 16.35: Calculating area ΔA in polar coordinates

Choosing (r_{ij}, θ_{ij}) in the ij-th bent rectangle in Figure 16.34 gives a Riemann sum:

$$\sum_{i,j} f(r_{ij}, \theta_{ij})\, \Delta A.$$

To calculate the area ΔA, look at Figure 16.35. If Δr and $\Delta\theta$ are small, the shaded region is approximately a rectangle with sides $r\,\Delta\theta$ and Δr, so

$$\Delta\, A \approx r\Delta\theta\Delta r.$$

Thus, the Riemann sum is approximately

$$\sum_{i,j} f(r_{ij}, \theta_{ij})\, r_{ij}\, \Delta\theta\, \Delta r.$$

If we take the limit as Δr and $\Delta\theta$ approach 0, we obtain

$$\int_R f\, dA = \int_\alpha^\beta \int_a^b f(r, \theta)\, r\, dr\, d\theta.$$

> When computing integrals in polar coordinates, use $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$. Put $dA = r\,dr\,d\theta$ or $dA = r\,d\theta\,dr$.

Example 2 Compute the integral of $f(x, y) = 1/(x^2 + y^2)^{3/2}$ over the region R shown in Figure 16.36.

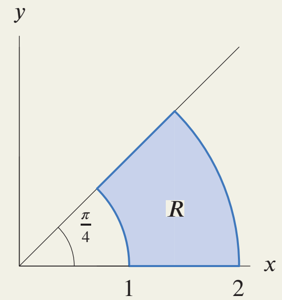


Figure 16.36: Integrate f over the polar region

Solution The region R is described by the inequalities $1 \leq r \leq 2$, $0 \leq \theta \leq \pi/4$. In polar coordinates, $r = \sqrt{x^2 + y^2}$, so we can write f as

$$f(x, y) = \frac{1}{(x^2 + y^2)^{3/2}} = \frac{1}{(r^2)^{3/2}} = \frac{1}{r^3}.$$

Then

$$\int_R f\, dA = \int_0^{\pi/4} \int_1^2 \frac{1}{r^3} r\, dr\, d\theta = \int_0^{\pi/4} \left(\int_1^2 r^{-2}\, dr \right) d\theta$$
$$= \int_0^{\pi/4} -\frac{1}{r} \bigg|_{r=1}^{r=2} d\theta = \int_0^{\pi/4} \frac{1}{2}\, d\theta = \frac{\pi}{8}.$$

Example 3 For each region in Figure 16.37, decide whether to integrate using polar or Cartesian coordinates. On the basis of its shape, write an iterated integral of an arbitrary function $f(x, y)$ over the region.

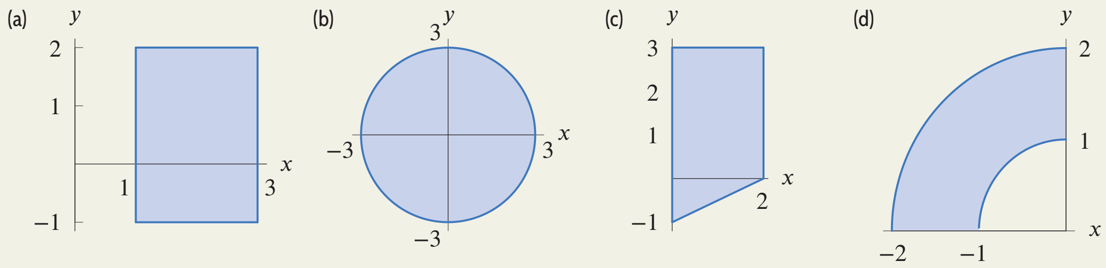


Figure 16.37

Solution (a) Since this is a rectangular region, Cartesian coordinates are likely to be a better choice. The rectangle is described by the inequalities $1 \leq x \leq 3$ and $-1 \leq y \leq 2$, so the integral is

$$\int_{-1}^{2} \int_1^3 f(x, y)\, dx\, dy.$$

(b) A circle is best described in polar coordinates. The radius is 3, so r goes from 0 to 3, and to describe the whole circle, θ goes from 0 to 2π. The integral is

$$\int_0^{2\pi} \int_0^3 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

(c) The bottom boundary of this trapezoid is the line $y = (x/2) - 1$ and the top is the line $y = 3$, so we use Cartesian coordinates. If we integrate with respect to y first, the lower limit of the integral is $(x/2) - 1$ and the upper limit is 3. The x limits are $x = 0$ to $x = 2$. So the integral is

$$\int_0^2 \int_{(x/2)-1}^{3} f(x, y)\, dy\, dx.$$

(d) This is another polar region: it is a piece of a ring in which r goes from 1 to 2. Since it is in the second quadrant, θ goes from $\pi/2$ to π. The integral is

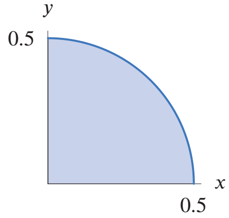
$$\int_{\pi/2}^{\pi} \int_1^2 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Exercises and Problems for Section 16.4 Online Resource: Additional Problems for Section 16.4

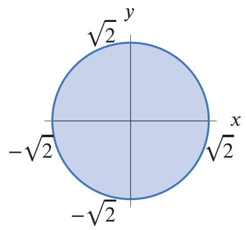
EXERCISES

For the regions R in Exercises 1–4, write $\int_R f\,dA$ as an iterated integral in polar coordinates.

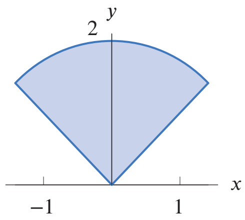
1.



2.



3.



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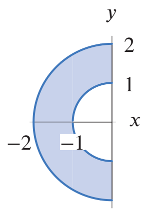
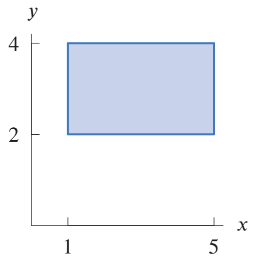
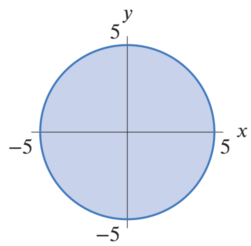


In Exercises 5–8, choose rectangular or polar coordinates to set up an iterated integral of an arbitrary function $f(x,y)$ over the region.

5.



6.



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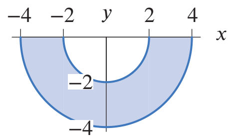

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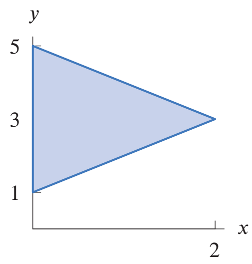


Sketch the region of integration in Exercises 9–15.

9. $\displaystyle\int_0^4 \int_{-\pi/2}^{\pi/2} f(r,\theta)\,r\,d\theta\,dr$

10. $\displaystyle\int_{\pi/2}^{\pi} \int_0^1 f(r,\theta)\,r\,dr\,d\theta$

11. $\displaystyle\int_0^{2\pi} \int_1^2 f(r,\theta)\,r\,dr\,d\theta$

12. $\displaystyle\int_{\pi/6}^{\pi/3} \int_0^1 f(r,\theta)\,r\,dr\,d\theta$

13. $\displaystyle\int_0^{\pi/4} \int_0^{1/\cos\theta} f(r,\theta)\,r\,dr\,d\theta$

14. $\displaystyle\int_3^4 \int_{3\pi/4}^{3\pi/2} f(r,\theta)\,r\,d\theta\,dr$

15. $\displaystyle\int_{\pi/4}^{\pi/2} \int_0^{2/\sin\theta} f(r,\theta)\,r\,dr\,d\theta$

PROBLEMS

In Problems 16–18, evaluate the integral.

16. $\int_R \sqrt{x^2+y^2}\,dx\,dy$ where R is $4 \le x^2+y^2 \le 9$.

17. $\int_R \sin(x^2+y^2)\,dA$, where R is the disk of radius 2 centered at the origin.

18. $\int_R (x^2-y^2)\,dA$, where R is the first quadrant region between the circles of radius 1 and radius 2.

Convert the integrals in Problems 19–21 to polar coordinates and evaluate.

19. $\displaystyle\int_{-1}^{0} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\,dy\,dx$

20. $\displaystyle\int_0^{\sqrt{6}} \int_{-x}^{x} dy\,dx$

21. $\displaystyle\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} xy\,dx\,dy$

Problems 22–26 concern Figure 16.38, which shows regions R_1, R_2, and R_3 contained in the semicircle $x^2+y^2=4$ with $y \ge 0$.

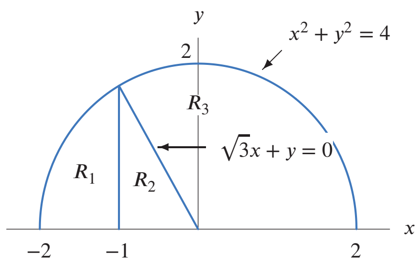


Figure 16.38

22. In Cartesian coordinates, write $\int_{R_1} 2y\,dA$ as an iterated integral in two different ways and then evaluate it.

23. In Cartesian coordinates, write $\int_{R_2} 2y\,dA$ as an iterated integral in two different ways.

24. Evaluate $\int_{R_3} (x^2+y^2)\,dA$.

25. Evaluate $\int_R 12y\,dA$, where R is the region formed by combining the regions R_1 and R_2.

26. Evaluate $\int_S x\,dA$, where S is the region formed by combining the regions R_2 and R_3.

27. Consider the integral $\int_0^3 \int_{x/3}^1 f(x, y)\,dy\,dx$.

(a) Sketch the region R over which the integration is being performed.
(b) Rewrite the integral with the order of integration reversed.
(c) Rewrite the integral in polar coordinates.

28. Describe the region of integration for $\int_{\pi/4}^{\pi/2} \int_{1/\sin\theta}^{4/\sin\theta} f(r, \theta) r\,dr\,d\theta$.

29. Evaluate the integral by converting it into Cartesian coordinates:

$$\int_0^{\pi/6} \int_0^{2/\cos\theta} r\,dr\,d\theta.$$

30. **(a)** Sketch the region of integration of

$$\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} x\,dy\,dx + \int_1^2 \int_0^{\sqrt{4-x^2}} x\,dy\,dx$$

(b) Evaluate the quantity in part (a).

31. Find the volume of the region between the graph of $f(x, y) = 25 - x^2 - y^2$ and the xy plane.

32. Find the volume of an ice cream cone bounded by the hemisphere $z = \sqrt{8 - x^2 - y^2}$ and the cone $z = \sqrt{x^2 + y^2}$.

33. **(a)** For $a > 0$, find the volume under the graph of $z = e^{-(x^2+y^2)}$ above the disk $x^2 + y^2 \le a^2$.
(b) What happens to the volume as $a \to \infty$?

34. A circular metal disk of radius 3 lies in the xy-plane with its center at the origin. At a distance r from the origin, the density of the metal per unit area is $\delta = \dfrac{1}{r^2 + 1}$.

(a) Write a double integral giving the total mass of the disk. Include limits of integration.
(b) Evaluate the integral.

35. A city surrounds a bay as shown in Figure 16.39. The population density of the city (in thousands of people per square km) is $\delta(r, \theta)$, where r and θ are polar coordinates and distances are in km.

(a) Set up an iterated integral in polar coordinates giving the total population of the city.

(b) The population density decreases the farther you live from the shoreline of the bay; it also decreases the farther you live from the ocean. Which of the following functions best describes this situation?

(i) $\delta(r, \theta) = (4 - r)(2 + \cos\theta)$
(ii) $\delta(r, \theta) = (4 - r)(2 + \sin\theta)$
(iii) $\delta(r, \theta) = (r + 4)(2 + \cos\theta)$

(c) Estimate the population using your answers to parts (a) and (b).

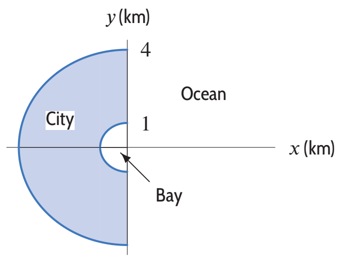


Figure 16.39

36. A disk of radius 5 cm has density 10 gm/cm^2 at its center and density 0 at its edge, and its density is a linear function of the distance from the center. Find the mass of the disk.

37. Electric charge is distributed over the xy-plane, with density inversely proportional to the distance from the origin. Show that the total charge inside a circle of radius R centered at the origin is proportional to R. What is the constant of proportionality?

38. **(a)** Graph $r = 1/(2\cos\theta)$ for $-\pi/2 \le \theta \le \pi/2$ and $r = 1$.
(b) Write an iterated integral representing the area inside the curve $r = 1$ and to the right of $r = 1/(2\cos\theta)$. Evaluate the integral.

39. **(a)** Sketch the circles $r = 2\cos\theta$ for $-\pi/2 \le \theta \le \pi/2$ and $r = 1$.
(b) Write an iterated integral representing the area inside the circle $r = 2\cos\theta$ and outside the circle $r = 1$. Evaluate the integral.

Strengthen Your Understanding

In Problems 40–44, explain what is wrong with the statement.

40. If R is the region bounded by $x = 1, y = 0, y = x$, then in polar coordinates $\int_R x\,dA = \int_0^{\pi/4} \int_0^1 r^2 \cos\theta\,dr\,d\theta$.

41. If R is the region $x^2 + y^2 \le 4$, then $\int_R (x^2 + y^2)\,dA = \int_0^{2\pi} \int_0^2 r^2\,dr\,d\theta$.

42. $\int_0^1 \int_0^1 \sqrt{x^2 + y^2}\,dy\,dx = \int_0^{\pi/2} \int_0^1 r^2\,dr\,d\theta$

43. $\int_1^2 \int_0^{\sqrt{4-x^2}} 1\,dy\,dx = \int_0^{\pi/2} \int_1^2 r\,dr\,d\theta$

44. $\int_0^1 \int_0^{\pi} r\,dr\,d\theta = \int_0^{\pi} \int_0^1 r\,dr\,d\theta$

In Problems 45–48, give an example of:

45. A region R of integration in the first quadrant which suggests the use of polar coordinates.

46. An integrand $f(x, y)$ that suggests the use of polar coordinates.

47. A function $f(x, y)$ such that $\int_R f(x, y)\, dy\, dx$ in polar coordinates has an integrand without a factor of r.

48. A region R such that $\int_R f(x, y)\, dA$ must be broken into two integrals in Cartesian coordinates, but only needs one integral in polar coordinates.

49. Which of the following integrals give the area of the unit circle?

(a) $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\, dx$ (b) $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\, dy\, dx$

(c) $\displaystyle\int_{0}^{2\pi}\int_{0}^{1} r\, dr\, d\theta$ (d) $\displaystyle\int_{0}^{2\pi}\int_{0}^{1} dr\, d\theta$

(e) $\displaystyle\int_{0}^{1}\int_{0}^{2\pi} r\, d\theta\, dr$ (f) $\displaystyle\int_{0}^{1}\int_{0}^{2\pi} d\theta\, dr$

Are the statements in Problems 50–55 true or false? Give reasons for your answer.

50. The integral $\int_0^{2\pi}\int_0^1 dr\, d\theta$ gives the area of the unit circle.

51. The quantity $8\int_5^7\int_0^{\pi/4} r\, d\theta\, dr$ gives the area of a ring with radius between 5 and 7.

52. Let R be the region inside the semicircle $x^2 + y^2 = 9$ with $y \geq 0$. Then $\int_R (x + y)\, dA = \int_0^{\pi}\int_0^3 r\, dr\, d\theta$

53. The integrals $\int_0^{\pi}\int_0^1 r^2 \cos\theta\, dr\, d\theta$ and $2\int_0^{\pi/2}\int_0^1 r^2 \cos\theta\, dr\, d\theta$ are equal.

54. The integral $\int_0^{\pi/4}\int_0^{1/\cos\theta} r\, dr\, d\theta$ gives the area of the region $0 \leq x \leq 1, 0 \leq y \leq x$.

55. The integral $\int_0^{2\pi}\int_0^1 r^3\, dr\, d\theta$ gives the area of the unit circle.

16.5 INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

Some double integrals are easier to evaluate in polar, rather than Cartesian, coordinates. Similarly, some triple integrals are easier in non-Cartesian coordinates.

Cylindrical Coordinates

The cylindrical coordinates of a point (x, y, z) in 3-space are obtained by representing the x and y coordinates in polar coordinates and letting the z-coordinate be the z-coordinate of the Cartesian coordinate system. (See Figure 16.40.)

Relation Between Cartesian and Cylindrical Coordinates

Each point in 3-space is represented using $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$, $-\infty < z < \infty$.

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$
$$z = z.$$

As with polar coordinates in the plane, note that $x^2 + y^2 = r^2$.

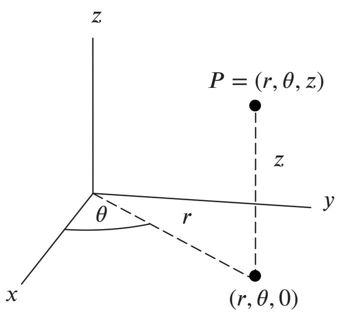


Figure 16.40: Cylindrical coordinates: (r, θ, z)

A useful way to visualize cylindrical coordinates is to sketch the surfaces obtained by setting one of the coordinates equal to a constant. See Figures 16.41–16.43.

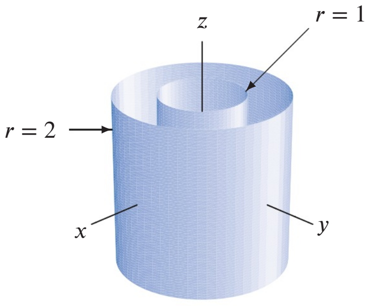


Figure 16.41: The surfaces $r = 1$ and $r = 2$

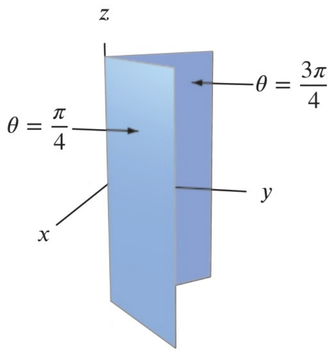


Figure 16.42: The surfaces $\theta = \pi/4$ and $\theta = 3\pi/4$

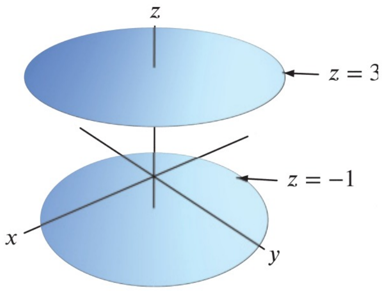


Figure 16.43: The surfaces $z = -1$ and $z = 3$

Setting $r = c$ (where c is constant) gives a cylinder around the z-axis whose radius is c. Setting $\theta = c$ gives a half-plane perpendicular to the xy plane, with one edge along the z-axis, making an angle c with the x-axis. Setting $z = c$ gives a horizontal plane $|c|$ units from the xy-plane. We call these *fundamental surfaces*.

The regions that can most easily be described in cylindrical coordinates are those regions whose boundaries are such fundamental surfaces. (For example, vertical cylinders, or wedge-shaped parts of vertical cylinders.)

Example 1 Describe in cylindrical coordinates a wedge of cheese cut from a cylinder 4 cm high and 6 cm in radius; this wedge subtends an angle of $\pi/6$ at the center. (See Figure 16.44.)

Solution The wedge is described by the inequalities $0 \le r \le 6$, and $0 \le z \le 4$, and $0 \le \theta \le \pi/6$.

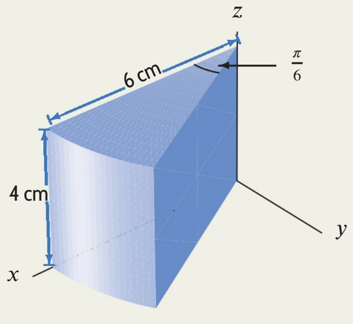


Figure 16.44: A wedge of cheese

Integration in Cylindrical Coordinates

To integrate a double integral $\int_R f\, dA$ in polar coordinates, we had to express the area element dA in terms of polar coordinates: $dA = r\,dr\,d\theta$. To evaluate a triple integral $\int_W f\, dV$ in cylindrical coordinates, we need to express the volume element dV in cylindrical coordinates.

In Figure 16.45, consider the volume element ΔV bounded by fundamental surfaces. The area of the base is $\Delta A \approx r\,\Delta r\,\Delta\theta$. Since the height is Δz, the volume element is given approximately by $\Delta V \approx r\,\Delta r\,\Delta\theta\,\Delta z$.

When computing integrals in cylindrical coordinates, put $dV = r\,dr\,d\theta\,dz$. Other orders of integration are also possible.

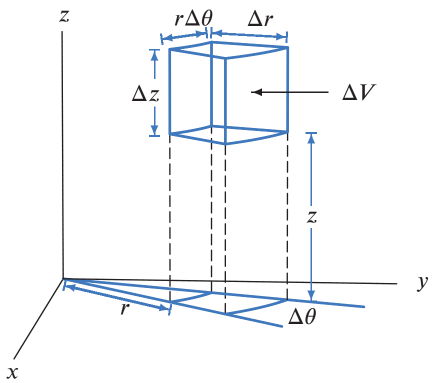


Figure 16.45: Volume element in cylindrical coordinates

Example 2 Find the mass of the wedge of cheese in Example 1, if its density is 1.2 grams/cm^3.

Solution If the wedge is W, its mass is

$$\int_W 1.2\,dV.$$

In cylindrical coordinates this integral is

$$\int_0^4 \int_0^{\pi/6} \int_0^6 1.2\,r\,dr\,d\theta\,dz = \int_0^4 \int_0^{\pi/6} 0.6r^2\Big|_0^6 d\theta\,dz = 21.6 \int_0^4 \int_0^{\pi/6} d\theta\,dz$$
$$= 21.6\left(\frac{\pi}{6}\right)4 = 45.239 \text{ grams.}$$

Example 3 A water tank in the shape of a hemisphere has radius a; its base is its plane face. Find the volume, V, of water in the tank as a function of h, the depth of the water.

Solution In Cartesian coordinates, a sphere of radius a has the equation $x^2+y^2+z^2 = a^2$. (See Figure 16.46.) In cylindrical coordinates, $r^2 = x^2 + y^2$, so this becomes

$$r^2 + z^2 = a^2.$$

Thus, if we want to describe the amount of water in the tank in cylindrical coordinates, we let r go from 0 to $\sqrt{a^2 - z^2}$, we let θ go from 0 to 2π, and we let z go from 0 to h, giving

$$\text{Volume of water} = \int_W 1\,dV = \int_0^{2\pi}\int_0^h\int_0^{\sqrt{a^2-z^2}} r\,dr\,dz\,d\theta = \int_0^{2\pi}\int_0^h \frac{r^2}{2}\Big|_{r=0}^{r=\sqrt{a^2-z^2}} dz\,d\theta$$
$$= \int_0^{2\pi}\int_0^h \frac{1}{2}(a^2 - z^2)\,dz\,d\theta = \int_0^{2\pi} \frac{1}{2}\left(a^2z - \frac{z^3}{3}\right)\Big|_{z=0}^{z=h} d\theta$$
$$= \int_0^{2\pi} \frac{1}{2}\left(a^2h - \frac{h^3}{3}\right) d\theta = \pi\left(a^2h - \frac{h^3}{3}\right).$$

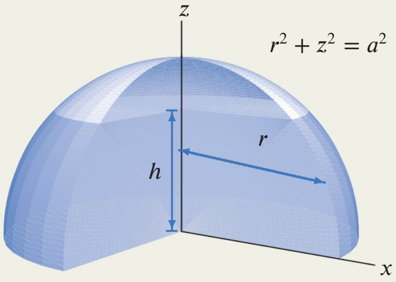


Figure 16.46: Hemispherical water tank with radius a and water of depth h

Spherical Coordinates

In Figure 16.47, the point P has coordinates (x, y, z) in the Cartesian coordinate system. We define spherical coordinates ρ, ϕ, and θ for P as follows: $\rho = \sqrt{x^2 + y^2 + z^2}$ is the distance of P from the origin; ϕ is the angle between the positive z-axis and the line through the origin and the point P; and θ is the same as in cylindrical coordinates.

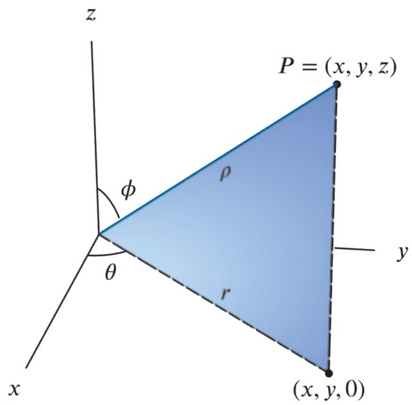


Figure 16.47: Spherical coordinates: (ρ, ϕ, θ)

In cylindrical coordinates,

$$x = r\cos\theta, \quad \text{and} \quad y = r\sin\theta, \quad \text{and} \quad z = z.$$

From Figure 16.47 we have $z = \rho\cos\phi$ and $r = \rho\sin\phi$, giving the following relationship:

Relation Between Cartesian and Spherical Coordinates

Each point in 3-space is represented using $0 \le \rho < \infty, 0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$.

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi.$$

Also, $\rho^2 = x^2 + y^2 + z^2$.

This system of coordinates is useful when there is spherical symmetry with respect to the origin, either in the region of integration or in the integrand. The fundamental surfaces in spherical coordinates are $\rho = k$ (a constant), which is a sphere of radius k centered at the origin, $\theta = k$ (a constant), which is the half-plane with its edge along the z-axis, and $\phi = k$ (a constant), which is a cone if $k \ne \pi/2$ and the xy-plane if $k = \pi/2$. (See Figures 16.48–16.50.)

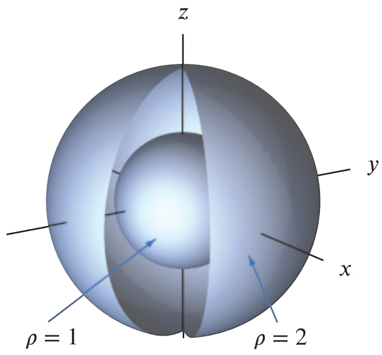


Figure 16.48: The surfaces $\rho = 1$ and $\rho = 2$

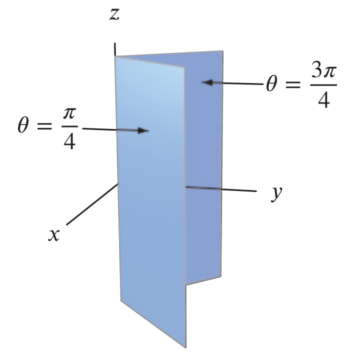


Figure 16.49: The surfaces $\theta = \pi/4$ and $\theta = 3\pi/4$

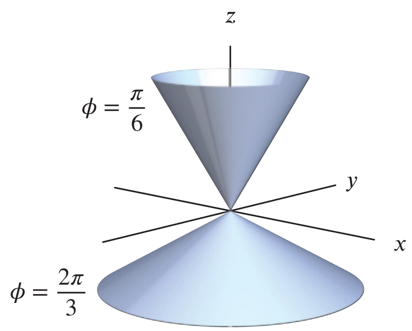


Figure 16.50: The surfaces $\phi = \pi/6$ and $\phi = 2\pi/3$

Integration in Spherical Coordinates

To use spherical coordinates in triple integrals we need to express the volume element, dV, in spherical coordinates. From Figure 16.51, we see that the volume element can be approximated by a box with curved edges. One edge has length $\Delta\rho$. The edge parallel to the xy-plane is an arc of a circle made from rotating the cylindrical radius r $(= \rho \sin\phi)$ through an angle $\Delta\theta$, and so has length $\rho \sin\phi\, \Delta\theta$. The remaining edge comes from rotating the radius ρ through an angle $\Delta\phi$, and so has length $\rho\, \Delta\phi$. Therefore, $\Delta V \approx \Delta\rho(\rho\, \Delta\phi)(\rho \sin\phi\, \Delta\theta) = \rho^2 \sin\phi\, \Delta\rho\, \Delta\phi\, \Delta\theta$.

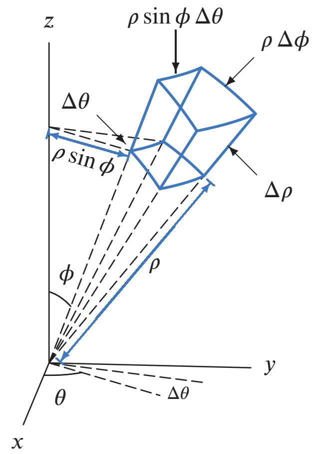


Figure 16.51: Volume element in spherical coordinates

Thus:

> When computing integrals in spherical coordinates, put $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$. Other orders of integration are also possible.

Example 4 Use spherical coordinates to derive the formula for the volume of a ball of radius a.

Solution In spherical coordinates, a ball of radius a is described by the inequalities $0 \le \rho \le a$, $0 \le \theta \le 2\pi$, and $0 \le \phi \le \pi$. Note that θ goes from 0 to 2π, whereas ϕ goes from 0 to π. We find the volume by integrating the constant density function 1 over the ball:

$$\text{Volume} = \int_R 1\, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi} \frac{1}{3} a^3 \sin\phi\, d\phi\, d\theta$$
$$= \frac{1}{3} a^3 \int_0^{2\pi} -\cos\phi \Big|_0^{\pi} d\theta = \frac{2}{3} a^3 \int_0^{2\pi} d\theta = \frac{4\pi a^3}{3}.$$

Example 5 Find the magnitude of the gravitational force exerted by a solid hemisphere of radius a and constant density δ on a unit mass located at the center of the base of the hemisphere.

Solution Assume the base of the hemisphere rests on the xy-plane with center at the origin. (See Figure 16.52.) Newton's law of gravitation says that the force between two masses m_1 and m_2 at a distance r apart is $F = Gm_1m_2/r^2$, where G is the gravitational constant.

In this example, symmetry shows that the net component of the force on the particle at the origin due to the hemisphere is in the z direction only. Any force in the x or y direction from some part of the hemisphere is canceled by the force from another part of the hemisphere directly opposite the first.

To compute the net z-component of the gravitational force, we imagine a small piece of the hemisphere with volume ΔV, located at spherical coordinates (ρ, θ, ϕ). This piece has mass $\delta\Delta V$ and exerts a force of magnitude F on the unit mass at the origin. The z-component of this force is given by its projection onto the z-axis, which can be seen from the figure to be $F\cos\phi$. The distance from the mass $\delta\Delta V$ to the unit mass at the origin is the spherical coordinate ρ. Therefore, the z-component of the force due to the small piece ΔV is

$$\begin{matrix} z\text{-component} \\ \text{of force} \end{matrix} = \frac{G(\delta\Delta V)(1)}{\rho^2}\cos\phi.$$

Adding the contributions of the small pieces, we get a vertical force with magnitude

$$\begin{aligned} F &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^a \left(\frac{G\delta}{\rho^2}\right)(\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/2} G\delta(\cos\phi\sin\phi)\rho\Big|_{\rho=0}^{\rho=a}\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/2} G\delta a\cos\phi\sin\phi\,d\phi\,d\theta = \int_0^{2\pi} G\delta a\left(-\frac{(\cos\phi)^2}{2}\right)\Big|_{\phi=0}^{\phi=\pi/2}\,d\theta \\ &= \int_0^{2\pi} G\delta a\left(\frac{1}{2}\right)\,d\theta = G\delta a\pi. \end{aligned}$$

The integral in this example is improper because the region of integration contains the origin, where the force is undefined. However, it can be shown that the result is nevertheless correct.

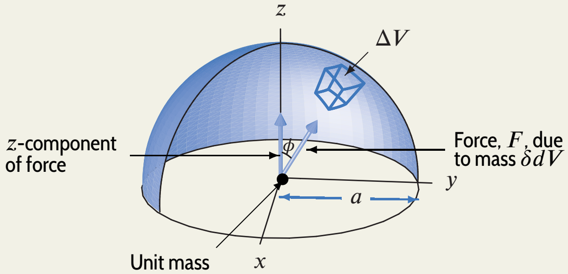


Figure 16.52: Gravitational force of hemisphere on mass at origin

Exercises and Problems for Section 16.5 Online Resource: Additional Problems for Section 16.5

EXERCISES

1. Match the equations in (a)–(f) with one of the surfaces in (I)–(VII).

(a) $x = 5$ **(b)** $x^2 + z^2 = 7$ **(c)** $\rho = 5$
(d) $z = 1$ **(e)** $r = 3$ **(f)** $\theta = 2\pi$

(I) Cylinder, centered on x-axis.
(II) Cylinder, centered on y-axis.
(III) Cylinder, centered on z-axis.
(IV) Plane, perpendicular to the x-axis.
(V) Plane, perpendicular to the y-axis.
(VI) Plane, perpendicular to the z-axis.
(VII) Sphere.

In Exercises 2–7, find an equation for the surface.

2. The vertical plane $y = x$ in cylindrical coordinates.
3. The top half of the sphere $x^2+y^2+z^2 = 1$ in cylindrical coordinates.
4. The cone $z = \sqrt{x^2 + y^2}$ in cylindrical coordinates.
5. The cone $z = \sqrt{x^2 + y^2}$ in spherical coordinates.
6. The plane $z = 10$ in spherical coordinates.
7. The plane $z = 4$ in spherical coordinates.

In Exercises 8–9, evaluate the triple integrals in cylindrical coordinates over the region W.

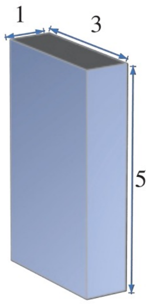
8. $f(x, y, z) = \sin(x^2 + y^2)$, W is the solid cylinder with height 4 and with base of radius 1 centered on the z axis at $z = -1$.
9. $f(x, y, z) = x^2 + y^2 + z^2$, W is the region $0 \le r \le 4$, $\pi/4 \le \theta \le 3\pi/4$, $-1 \le z \le 1$.

In Exercises 10–11, evaluate the triple integrals in spherical coordinates.

10. $f(\rho, \theta, \phi) = \sin\phi$, over the region $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/4$, $1 \le \rho \le 2$.
11. $f(x, y, z) = 1/(x^2 + y^2 + z^2)^{1/2}$ over the bottom half of the sphere of radius 5 centered at the origin.

For Exercises 12–18, choose coordinates and set up a triple integral, including limits of integration, for a density function f over the region.

12.



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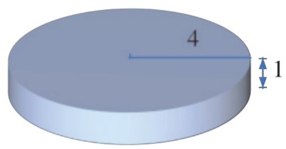

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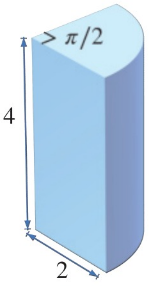

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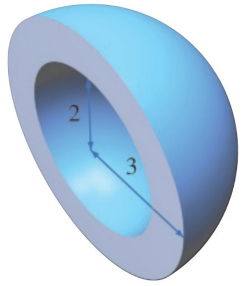
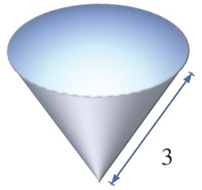


16. A piece of a sphere; angle at the center is $\pi/3$.



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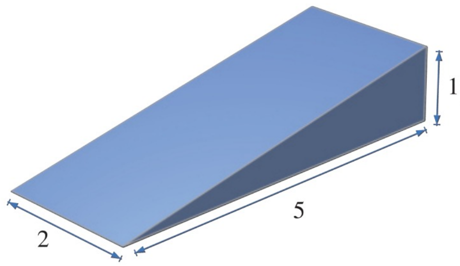

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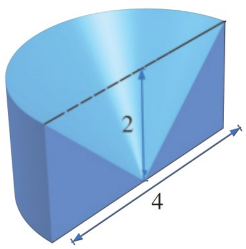


PROBLEMS

In Problems 19–21, if W is the region in Figure 16.53, what are the limits of integration?

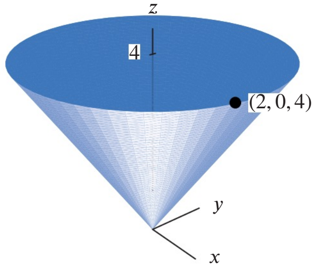


Figure 16.53: Cone with flat top, symmetric about z-axis

19. $$\int_?^? \int_?^? \int_?^? f(r, \theta, z) r\, dz\, dr\, d\theta$$

20. $$\int_?^? \int_?^? \int_?^? g(\rho, \phi, \theta)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

21. $$\int_?^? \int_?^? \int_?^? h(x, y, z)\, dz\, dy\, dx$$

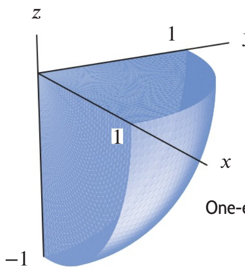
22. Write a triple integral in cylindrical coordinates giving the volume of a sphere of radius K centered at the origin. Use the order $dz\, dr\, d\theta$.

23. Write a triple integral in spherical coordinates giving the volume of a sphere of radius K centered at the origin. Use the order $d\theta\, d\rho\, d\phi$.

In Problems 24–26, for the regions W shown, write the limits of integration for $\int_W dV$ in the following coordinates:

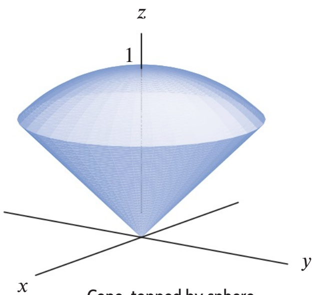
(a) Cartesian **(b)** Cylindrical **(c)** Spherical

24.



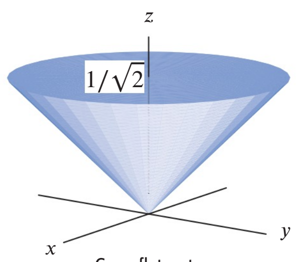
One-eighth sphere

25.



Cone, topped by sphere of radius 1 centered at origin, 90° at vertex

26.



Cone, flat on top, $\pi/2$ at vertex

27. Write a triple integral representing the volume above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere of radius 2 centered at the origin. Include limits of integration but do not evaluate. Use:

(a) Cylindrical coordinates
(b) Spherical coordinates

28. Write a triple integral representing the volume of the region between spheres of radius 1 and 2, both centered at the origin. Include limits of integration but do not evaluate. Use:

(a) Spherical coordinates.
(b) Cylindrical coordinates. Write your answer as the difference of two integrals.

In Problems 29–34, write a triple integral including limits of integration that gives the specified volume.

29. Under $\rho = 3$ and above $\phi = \pi/3$.

30. Under $\rho = 3$ and above $z = r$.

31. The region between $z = 5$ and $z = 10$, with $2 \le x^2 + y^2 \le 3$ and $0 \le \theta \le \pi$.

32. Between the cone $z = \sqrt{x^2 + y^2}$ and the first quadrant of the xy-plane, with $x^2 + y^2 \le 7$.

33. The cap of the solid sphere $x^2 + y^2 + z^2 \le 10$ cut off by the plane $z = 1$.

34. Below the cone $z = r$, above the xy-plane, and inside the sphere $x^2 + y^2 + z^2 = 8$.

35. **(a)** Write an integral (including limits of integration) representing the volume of the region inside the cone $z = \sqrt{3(x^2 + y^2)}$ and below the plane $z = 1$.
(b) Evaluate the integral.

36. Find the volume between the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 10 + x$ above the disk $x^2 + y^2 \le 1$.

37. Find the volume between the cone $x = \sqrt{y^2 + z^2}$ and the sphere $x^2 + y^2 + z^2 = 4$.

38. The sphere of radius 2 centered at the origin is sliced horizontally at $z = 1$. What is the volume of the cap above the plane $z = 1$?

39. Suppose W is the region outside the cylinder $x^2 + y^2 = 1$ and inside the sphere $x^2 + y^2 + z^2 = 2$. Calculate

$$\int_W (x^2 + y^2)\, dV.$$

40. Write and evaluate a triple integral representing the volume of a slice of the cylindrical cake of height 2 and radius 5 between the planes $\theta = \pi/6$ and $\theta = \pi/3$.

41. Write a triple integral representing the volume of the cone in Figure 16.54 and evaluate it.

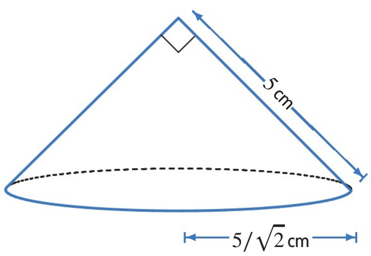


Figure 16.54

42. Find the average distance from the origin of

(a) The points in the interval $|x| \le 12$.
(b) The points in the plane in the disc $r \le 12$.
(c) The points in space in the ball $\rho \le 12$.

In Problems 43–44, without performing the integration, decide whether the integral is positive, negative, or zero.

43. W_1 is the unit ball, $x^2 + y^2 + z^2 \le 1$.

(a) $\int_{W_1} \sin\phi \, dV$ **(b)** $\int_{W_1} \cos\phi \, dV$

44. W_2 is $0 \le z \le \sqrt{1 - x^2 - y^2}$, the top half of the unit ball.

(a) $\int_{W_2} (z^2 - z) \, dV$ **(b)** $\int_{W_2} (-xz) \, dV$

45. The insulation surrounding a pipe of length l is the region between two cylinders with the same axis. The inner cylinder has radius a, the outer radius of the pipe, and the insulation has thickness h. Write a triple integral, including limits of integration, giving the volume of the insulation. Evaluate the integral.

46. Assume p, q, r are positive constants. Find the volume contained between the coordinate planes and the plane

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1.$$

47. A cone stands with its flat base on a table. The cone's circular base has radius a; the vertex (tip) is at a height of h above the center of the base. Write a triple integral, including limits of integration, representing the volume of the cone. Evaluate the integral.

48. A half-melon is approximated by the region between two concentric spheres, one of radius a and the other of radius b, with $0 < a < b$. Write a triple integral, including limits of integration, giving the volume of the half-melon. Evaluate the integral.

49. A bead is made by drilling a cylindrical hole of radius 1 mm through a sphere of radius 5 mm. See Figure 16.55.

(a) Set up a triple integral in cylindrical coordinates representing the volume of the bead.
(b) Evaluate the integral.

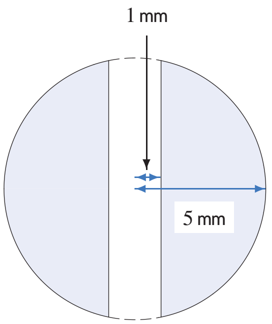


Figure 16.55

50. A pile of hay is in the region $0 \le z \le 2 - x^2 - y^2$, where x, y, z are in meters. At height z, the density of the hay is $\delta = (2 - z)$ kg/m^3.

(a) Write an integral representing the mass of hay in the pile.
(b) Evaluate the integral.

51. Find the mass M of the solid region W given in spherical coordinates by $0 \le \rho \le 3, 0 \le \theta < 2\pi, 0 \le \phi \le \pi/4$. The density, $\delta(P)$, at any point P is given by the distance of P from the origin.

52. Write an integral representing the mass of a sphere of radius 3 if the density of the sphere at any point is twice the distance of that point from the center of the sphere.

53. A sphere has density at each point proportional to the square of the distance of the point from the z-axis. The density is 2 gm/cm^3 at a distance of 2 cm from the axis. What is the mass of the sphere if it is centered at the origin and has radius 3 cm?

54. The density of a solid sphere at any point is proportional to the square of the distance of the point to the center of the sphere. What is the ratio of the mass of a sphere of radius 1 to a sphere of radius 2?

55. A spherical shell centered at the origin has an inner radius of 6 cm and an outer radius of 7 cm. The density, δ, of the material increases linearly with the distance from the center. At the inner surface, $\delta = 9$ gm/cm^3; at the outer surface, $\delta = 11$ gm/cm^3.

(a) Using spherical coordinates, write the density, δ, as a function of radius, ρ.
(b) Write an integral giving the mass of the shell.
(c) Find the mass of the shell.

56. **(a)** Write an iterated integral which represents the mass of a solid ball of radius a. The density at each point in the ball is k times the distance from that point to a fixed plane passing through the center of the ball.
(b) Evaluate the integral.

57. In the region under $z = 4 - x^2 - y^2$ and above the xy-plane the density of a gas is $\delta = e^{-x-y}$gm/cm^3, where x, y, z are in cm. Write an integral, with limits of integration, representing the mass of the gas.

58. The density, δ, of the cylinder $x^2 + y^2 \le 4, 0 \le z \le 3$ varies with the distance, r, from the z-axis:

$$\delta = 1 + r \text{ gm/cm}^3.$$

Find the mass of the cylinder if x, y, z are in cm.

59. The density of material at a point in a solid cylinder is proportional to the distance of the point from the z-axis. What is the ratio of the mass of the cylinder $x^2 + y^2 \le 1$, $0 \le z \le 2$ to the mass of the cylinder $x^2 + y^2 \le 9$, $0 \le z \le 2$?

60. Electric charge is distributed throughout 3-space, with density proportional to the distance from the xy-plane. Show that the total charge inside a cylinder of radius R and height h, sitting on the xy-plane and centered along the z-axis, is proportional to R^2h^2.

61. Electric charge is distributed throughout 3-space with density inversely proportional to the distance from the origin. Show that the total charge inside a sphere of radius R is proportional to R^2.

For Problems 62–65, use the definition of center of mass given on page 863. Assume x, y, z are in cm.

62. Let C be a solid cone with both height and radius 1 and contained between the surfaces $z = \sqrt{x^2 + y^2}$ and $z = 1$. If C has constant mass density of 1 gm/cm^3, find the z-coordinate of C's center of mass.

63. The density of the cone C in Problem 62 is given by $\delta(z) = z^2$ gm/cm^3. Find

(a) The mass of C.
(b) The z-coordinate of C's center of mass.

64. For $a > 0$, consider the family of solids bounded below by the paraboloid $z = a(x^2 + y^2)$ and above by the plane $z = 1$. If the solids all have constant mass density 1 gm/cm^3, show that the z-coordinate of the center of mass is $2/3$ and so independent of the parameter a.

65. Find the location of the center of mass of a hemisphere of radius a and density b gm/cm^3.

Strengthen Your Understanding

In Problems 66–68, explain what is wrong with the statement.

66. The integral $\int_0^{2\pi}\int_0^{\pi}\int_0^1 1\, d\rho\, d\phi\, d\theta$ gives the volume inside the sphere of radius 1.

67. Changing the order of integration gives

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2/\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2/\cos\phi}\int_0^{\pi/4}\int_0^{2\pi} \rho^2 \sin\phi\, d\theta\, d\phi\, d\rho.$$

68. The volume of a cylinder of height and radius 1 is $\int_0^{2\pi}\int_0^{1}\int_0^{1} 1\, dz\, dr\, d\theta$.

In Problems 69–70, give an example of:

69. An integral in spherical coordinates that gives the volume of a hemisphere.

70. An integral for which it is more convenient to use spherical coordinates than to use Cartesian coordinates.

71. Which of the following integrals give the volume of the unit sphere?

(a) $\int_0^{2\pi}\int_0^{2\pi}\int_0^1 1\, d\rho\, d\theta\, d\phi$

(b) $\int_0^{\pi}\int_0^{2\pi}\int_0^1 1\, d\rho\, d\theta\, d\phi$

(c) $\int_0^{\pi}\int_0^{2\pi}\int_0^1 \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$

(d) $\int_0^{\pi}\int_0^{2\pi}\int_0^1 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

(e) $\int_0^{\pi}\int_0^{2\pi}\int_0^1 \rho\, d\rho\, d\phi\, d\theta$

16.6 APPLICATIONS OF INTEGRATION TO PROBABILITY

To represent how a quantity such as height or weight is distributed throughout a population, we use a density function. To study two or more quantities at the same time and see how they are related, we use a multivariable density function.

Density Functions

Distribution of Weight and Height in Expectant Mothers

Table 16.10 shows the distribution of weight and height in a survey of expectant mothers. The histogram in Figure 16.56 is constructed so that the volume of each bar represents the percentage in the corresponding weight and height range. For example, the bar representing the mothers who weighed 60–70 kg and were 160–165 cm tall has base of area 10 kg $\cdot$ 5 cm $=$ 50 kg cm. The volume of this bar is 12%, so its height is $12\%/50$ kg cm $= 0.24\%/$ kg cm. Notice that the units on the vertical axis are % per kg cm, so the volume of a bar is a %. The total volume is $100\% = 1$.

Table 16.10 *Distribution of weight and height in a survey of expectant mothers, in %*

	45-50 kg	50-60 kg	60-70 kg	70-80 kg	80-105 kg	Totals by height
150-155 cm	2	4	4	2	1	13
155-160 cm	0	12	8	2	1	23
160-165 cm	1	7	12	4	3	27
165-170 cm	0	8	12	6	2	28
170-180 cm	0	1	3	4	1	9
Totals by weight	3	32	39	18	8	100

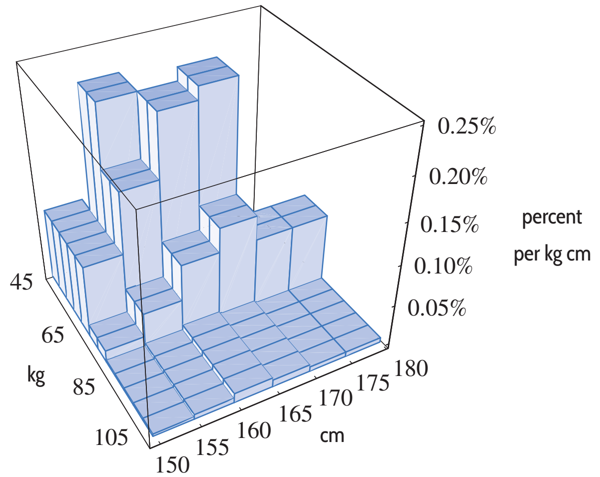


Figure 16.56: Histogram representing the data in Table 16.10

Example 1 Find the percentage of mothers in the survey with height between 170 and 180 cm.

Solution We add the percentages across the row corresponding to the 170–180 cm height range; this is equivalent to adding the volumes of the corresponding rectangular solids in the histogram.

$$\text{Percentage of mothers} = 0 + 1 + 3 + 4 + 1 = 9\%.$$

Smoothing the Histogram

If we group the data using narrower weight and height groups (and a larger sample), we can draw a smoother histogram and get finer estimates. In the limit, we replace the histogram with a smooth surface, in such a way that the volume under the surface above a rectangle is the percentage of mothers in that rectangle. We define a *density function*, $p(w, h)$, to be the function whose graph is the smooth surface. It has the property that

$$\begin{array}{c}\text{Fraction of sample with}\\ \text{weight between } a \text{ and } b \text{ and}\\ \text{height between } c \text{ and } d\end{array} = \begin{array}{c}\text{Volume under graph of } p\\ \text{over the rectangle}\\ a \le w \le b,\ c \le h \le d\end{array} = \int_a^b \int_c^d p(w,h)\, dh\, dw.$$

This density also gives the probability that a mother is in these height and weight groups.

Joint Probability Density Functions

We generalize this idea to represent any two characteristics, x and y, distributed throughout a population.

> A function $p(x, y)$ is called a **joint probability density function**, or **pdf**, for x and y if
>
> $$\begin{pmatrix}\text{Probability that member of}\\ \text{population has } x \text{ between } a \text{ and } b\\ \text{and } y \text{ between } c \text{ and } d\end{pmatrix} = \begin{pmatrix}\text{Volume under graph of } p\\ \text{above the rectangle}\\ a \le x \le b,\ c \le y \le d\end{pmatrix} = \int_a^b \int_c^d p(x, y)\, dy\, dx,$$
>
> where
>
> $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y)\, dy\, dx = 1 \quad \text{and} \quad p(x, y) \ge 0 \text{ for all } x \text{ and } y.$$
>
> The probability that x falls in an interval of width Δx around x_0 and y falls in an interval of width Δy around y_0 is approximately $p(x_0, y_0)\Delta x \Delta y$.

A joint density function need not be continuous, as in Example 2. In addition, as in Example 4, the integrals involved may be improper and must be computed by methods similar to those used for improper one-variable integrals.

Example 2 Let $p(x, y)$ be defined on the square $0 \le x \le 1, 0 \le y \le 1$ by $p(x, y) = x + y$; let $p(x, y) = 0$ if (x, y) is outside this square. Check that p is a joint density function. In terms of the distribution of x and y in the population, what does it mean that $p(x, y) = 0$ outside the square?

Solution First, we have $p(x, y) \ge 0$ for all x and y. To check that p is a joint density function, we show that the total volume under the graph is 1:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y)\, dy\, dx = \int_0^1\int_0^1 (x + y)\, dy\, dx$$
$$= \int_0^1 \left(xy + \frac{y^2}{2}\right)\Big|_0^1 dx = \int_0^1 \left(x + \frac{1}{2}\right) dx = \left(\frac{x^2}{2} + \frac{x}{2}\right)\Big|_0^1 = 1.$$

The fact that $p(x, y) = 0$ outside the square means that the variables x and y never take values outside the interval $[0, 1]$; that is, the value of x and y for any individual in the population is always between 0 and 1.

Example 3 Two variables x and y are distributed in a population according to the density function of Example 2. Find the fraction of the population with $x \le 1/2$, the fraction with $y \le 1/2$, and the fraction with both $x \le 1/2$ and $y \le 1/2$.

Solution The fraction with $x \le 1/2$ is the volume under the graph to the left of the line $x = 1/2$:

$$\int_0^{1/2}\int_0^1 (x + y)\, dy\, dx = \int_0^{1/2} \left(xy + \frac{y^2}{2}\right)\Big|_0^1 dx = \int_0^{1/2} \left(x + \frac{1}{2}\right) dx$$
$$= \left(\frac{x^2}{2} + \frac{x}{2}\right)\Big|_0^{1/2} = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}.$$

Since the function and the regions of integration are symmetric in x and y, the fraction with $y \le 1/2$

is also $3/8$. Finally, the fraction with both $x \le 1/2$ and $y \le 1/2$ is

$$\int_0^{1/2}\int_0^{1/2}(x+y)\,dy\,dx = \int_0^{1/2}\left(xy+\frac{y^2}{2}\right)\Bigg|_0^{1/2} dx = \int_0^{1/2}\left(\frac{1}{2}x+\frac{1}{8}\right)dx$$
$$= \left(\frac{1}{4}x^2+\frac{1}{8}x\right)\Bigg|_0^{1/2} = \frac{1}{16}+\frac{1}{16}=\frac{1}{8}.$$

Recall that a one-variable density function $p(x)$ is a function such that $p(x) \ge 0$ for all x, and $\int_{-\infty}^{\infty} p(x)\,dx = 1$.

Example 4 Let p_1 and p_2 be one-variable density functions for x and y, respectively. Check that $p(x,y) = p_1(x)p_2(y)$ is a joint density function.

Solution Since both p_1 and p_2 are density functions, they are nonnegative everywhere. Thus, their product $p_1(x)p_2(x) = p(x,y)$ is nonnegative everywhere. Now we must check that the volume under the graph of p is 1. Since $\int_{-\infty}^{\infty} p_2(y)\,dy = 1$ and $\int_{-\infty}^{\infty} p_1(x)\,dx = 1$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dy\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_1(x)p_2(y)\,dy\,dx = \int_{-\infty}^{\infty} p_1(x)\left(\int_{-\infty}^{\infty} p_2(y)\,dy\right)dx$$
$$= \int_{-\infty}^{\infty} p_1(x)(1)\,dx = \int_{-\infty}^{\infty} p_1(x)\,dx = 1.$$

Example 5 A machine in a factory is set to produce components 10 cm long and 5 cm in diameter. In fact, there is a slight variation from one component to the next. A component is usable if its length and diameter deviate from the correct values by less than 0.1 cm. With the length, x, in cm and the diameter, y, in cm, the probability density function is

$$p(x,y) = \frac{50\sqrt{2}}{\pi}e^{-100(x-10)^2}e^{-50(y-5)^2}.$$

What is the probability that a component is usable? (See Figure 16.57.)

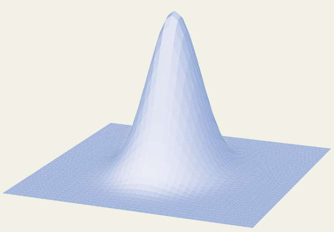


Figure 16.57: The density function $p(x,y) = \frac{50\sqrt{2}}{\pi}e^{-100(x-10)^2}e^{-50(y-5)^2}$

Solution We know that

$$\begin{array}{c}\text{Probability that } x \text{ and } y \text{ satisfy}\\ x_0-\Delta x \le x \le x_0+\Delta x\\ y_0-\Delta y \le y \le y_0+\Delta y\end{array} = \frac{50\sqrt{2}}{\pi}\int_{y_0-\Delta y}^{y_0+\Delta y}\int_{x_0-\Delta x}^{x_0+\Delta x} e^{-100(x-10)^2}e^{-50(y-5)^2}\,dx\,dy.$$

Thus,

$$\begin{array}{c}\text{Probability that}\\ \text{component is usable}\end{array} = \frac{50\sqrt{2}}{\pi}\int_{4.9}^{5.1}\int_{9.9}^{10.1} e^{-100(x-10)^2} e^{-50(y-5)^2}\,dx\,dy.$$

The double integral must be evaluated numerically. This yields

$$\begin{array}{c}\text{Probability that}\\ \text{component is usable}\end{array} = \frac{50\sqrt{2}}{\pi}(0.02556) = 0.57530.$$

Thus, there is a 57.530% chance that the component is usable.

Exercises and Problems for Section 16.6

EXERCISES

In Exercises 1–6, check whether p is a joint density function. Assume $p(x, y) = 0$ outside the region R.

1. $p(x, y) = 1/2$, where R is $4 \le x \le 5, -2 \le y \le 0$
2. $p(x, y) = 1$, where R is $0 \le x \le 1, 0 \le y \le 2$
3. $p(x, y) = x + y$, where R is $-1 \le x \le 1, 0 \le y \le 1$
4. $p(x, y) = 6(y - x)$, where R is $0 \le x \le y \le 2$
5. $p(x, y) = (2/\pi)(1 - x^2 - y^2)$, where R is $x^2 + y^2 \le 1$
6. $p(x, y) = xye^{-x-y}$, where R is $x \ge 0, y \ge 0$

In Exercises 7–10, a joint probability density function is given by $p(x, y) = xy/4$ in R, the rectangle $0 \le x \le 2$, $0 \le y \le 2$, and $p(x, y) = 0$ else. Find the probability that a point (x, y) satisfies the given conditions.

7. $x \le 1$ and $y \le 1$
8. $x \ge 1$ and $y \ge 1$
9. $x \ge 1$ and $y \le 1$
10. $1/3 \le x \le 1$

In Exercises 11–14, a joint probability density function is given by $p(x, y) = 0.005x + 0.025y$ in R, the rectangle $0 \le x \le 10$, $0 \le y \le 2$, and $p(x, y) = 0$ else. Find the probability that a point (x, y) satisfies the given conditions.

11. $x \le 4$
12. $y \ge 1$
13. $x \le 4$ and $y \ge 1$
14. $x \ge 5$ and $y \ge 1$

In Exercises 15–22, let p be the joint density function such that $p(x, y) = xy$ in R, the rectangle $0 \le x \le 2, 0 \le y \le 1$, and $p(x, y) = 0$ outside R. Find the fraction of the population satisfying the given constraints.

15. $x \ge 3$
16. $x = 1$
17. $x + y \le 3$
18. $-1 \le x \le 1$
19. $x \ge y$
20. $x + y \le 1$
21. $0 \le x \le 1, 0 \le y \le 1/2$
22. Within a distance 1 from the origin

PROBLEMS

23. Let x and y have joint density function

$$p(x, y) = \begin{cases} \frac{2}{3}(x + 2y) & \text{for } 0 \le x \le 1,\ 0 \le y \le 1,\\ 0 & \text{otherwise.}\end{cases}$$

Find the probability that

(a) $x > 1/3$. **(b)** $x < (1/3) + y$.

24. The joint density function for x, y is given by

$$f(x, y) = \begin{cases} kxy & \text{for } 0 \le x \le y \le 1,\\ 0 & \text{otherwise.}\end{cases}$$

(a) Determine the value of k.
(b) Find the probability that (x, y) lies in the shaded region in Figure 16.58.

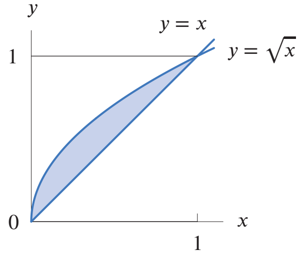


Figure 16.58

25. A joint density function is given by

$$f(x,y) = \begin{cases} kx^2 & \text{for } 0 \le x \le 2 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant k.
(b) Find the probability that (x, y) satisfies $x + y \le 2$.
(c) Find the probability that (x, y) satisfies $x \le 1$ and $y \le 1/2$.

26. A point is chosen at random from the region S in the xy-plane containing all points (x, y) such that $-1 \le x \le 1, -2 \le y \le 2$ and $x - y \ge 0$ ("at random" means that the density function is constant on S).

(a) Determine the joint density function for x and y.
(b) If T is a subset of S with area α, then find the probability that a point (x, y) is in T.

27. A probability density function on a square has constant values in different triangular regions as shown in Figure 16.59. Find the probability that

(a) $x \ge 2$
(b) $y \ge x$
(c) $y \ge x$ and $x \ge 2$

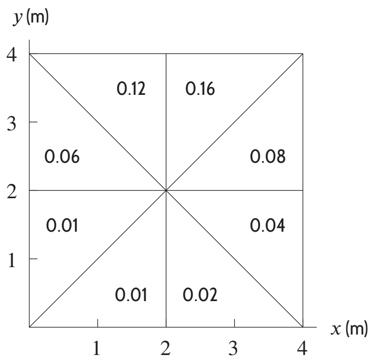


Figure 16.59: Probability density on a square (per m^2)

28. A health insurance company wants to know what proportion of its policies are going to cost the company a lot of money because the insured people are over 65 and sick. In order to compute this proportion, the company defines a *disability index*, x, with $0 \le x \le 1$, where $x = 0$ represents perfect health and $x = 1$ represents total disability. In addition, the company uses a density function, $f(x, y)$, defined in such a way that the quantity

$$f(x, y)\,\Delta x\,\Delta y$$

approximates the fraction of the population with disability index between x and $x+\Delta x$, and aged between y and $y+\Delta y$. The company knows from experience that a policy no longer covers its costs if the insured person is over 65 and has a disability index exceeding 0.8. Write an expression for the fraction of the company's policies held by people meeting these criteria.

29. The probability that a radioactive substance will decay at time t is modeled by the density function

$$p(t) = \lambda e^{-\lambda t}$$

for $t \ge 0$, and $p(t) = 0$ for $t < 0$. The positive constant λ depends on the material, and is called the decay rate.

(a) Check that p is a density function.
(b) Two materials with decay rates λ and μ decay independently of each other; their joint density function is the product of the individual density functions. Write the joint density function for the probability that the first material decays at time t and the second at time s.
(c) Find the probability that the first substance decays before the second.

30. Figure 16.60 represents a baseball field, with the bases at $(1, 0)$, $(1, 1)$, $(0, 1)$, and home plate at $(0, 0)$. The outer bound of the outfield is a piece of a circle about the origin with radius 4. When a ball is hit by a batter we record the spot on the field where the ball is caught. Let $p(r, \theta)$ be a function in the plane that gives the density of the distribution of such spots. Write an expression that represents the probability that a hit is caught in

(a) The right field (region R).
(b) The center field (region C).

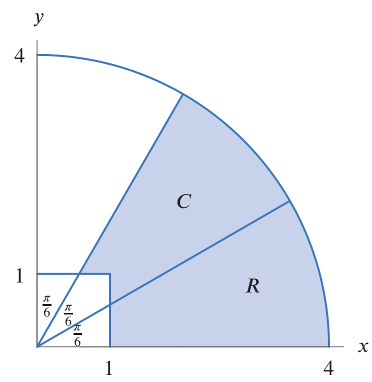


Figure 16.60

31. Two independent random numbers x and y between 0 and 1 have joint density function

$$p(x,y) = \begin{cases} 1 & \text{if } 0 \le x, y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

This problem concerns the average $z = (x+y)/2$, which has a one-variable probability density function of its own.

(a) Find $F(t)$, the probability that $z \le t$. Treat separately the cases $t \le 0$, $0 < t \le 1/2$, $1/2 < t \le 1$, $1 < t$. Note that $F(t)$ is the cumulative distribution function of z.
(b) Find and graph the probability density function of z.
(c) Are x and y more likely to be near 0, 1/2, or 1? What about z?

Strengthen Your Understanding

In Problems 32–33, explain what is wrong with the statement.

32. If $p_1(x, y)$ and $p_2(x, y)$ are joint density functions, then $p_1(x, y) + p_2(x, y)$ is a joint density function.

33. If $p(w, h)$ is the probability density function of the weight and height of mothers discussed in Section 16.6, then the probability that a mother weighs 60 kg and has a height of 170 cm is $p(60, 170)$.

In Problems 34–35, give an example of:

34. Values for a, b, c and d such that f is a joint density function:

$$f(x, y) = \begin{cases} 1 & \text{for } a \le x \le b \text{ and } c \le y \le d, \\ 0 & \text{otherwise} \end{cases}$$

35. A one-variable function $g(y)$ such that f is a joint density function:

$$f(x, y) = \begin{cases} g(y) & \text{for } 0 \le x \le 2 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$

For Problems 36–39, let $p(x, y)$ be a joint density function for x and y. Are the following statements true or false?

36. $\int_a^b \int_{-\infty}^{\infty} p(x, y)\, dy\, dx$ is the probability that $a \le x \le b$.

37. $0 \le p(x, y) \le 1$ for all x.

38. $\int_a^b p(x, y)\, dx$ is the probability that $a \le x \le b$.

39. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y)\, dy\, dx = 1$.

Online Resource: Review problems and Projects