Limits and Infinite Series

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Contents

CHAPTER 1

Preliminaries

1.1. Real Numbers

All numbers in these notes are real numbers. The set of all real numbers is denoted by $\mathbb{R}$. All mathematical proofs are constructed from axioms using the mathematical logic that we reviewed in "A Brief Review of Mathematical Logic." In the next subsection we present Axioms of the set of the real numbers $\mathbb{R}$.

1.1.1. Axioms for the Set $\mathbb{R}$ of Real Numbers.

Axiom 1 (AE: Addition exists). If $a, b \in \mathbb{R}$, then the sum of a and b, denoted by $a + b$, is a uniquely defined number in $\mathbb{R}$.

Axiom 2 (AA: Addition is associative). For all $a, b, c \in \mathbb{R}$ we have $a+(b+c) = (a+b)+c$.

Axiom 3 (AC: Addition is commutative). For all $a, b \in \mathbb{R}$ we have $a + b = b + a$.

Axiom 4 (AZ: Addition has 0). There is an element 0 in $\mathbb{R}$ such that $0 + a = a + 0 = a$ for all $a \in \mathbb{R}$.

Axiom 5 (AO: Addition has opposites). If $a \in \mathbb{R}$, then the equation $a + x = 0$ has a solution $-a \in \mathbb{R}$. The number $-a$ is called the *opposite* of a.

Axiom 6 (ME: Multiplication exists). If $a, b \in \mathbb{R}$, then the product of a and b, denoted by ab, is a uniquely defined number in $\mathbb{R}$.

Axiom 7 (MA: Multiplication is associative). For all $a, b, c \in \mathbb{R}$ we have $a(bc) = (ab)c$.

Axiom 8 (MC: Multiplication is commutative). For all $a, b \in \mathbb{R}$ we have $ab = ba$.

Axiom 9 (MO: Multiplication has 1). There is an element $1 \neq 0$ in $\mathbb{R}$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathbb{R}$.

Axiom 10 (MR: Multiplication has reciprocals). If $a \in \mathbb{R}$ is such that $a \neq 0$, then the equation $a \cdot x = 1$ has a solution $a^{-1} = \dfrac{1}{a}$ in $\mathbb{R}$. The number $a^{-1} = \dfrac{1}{a}$ is called the *reciprocal* of a.

Axiom 11 (DL: Distributive law, the connection between addition and multiplication). For all $a, b, c \in \mathbb{R}$ we have $a(b + c) = ab + ac$.

Axiom 12 (OE: Order exists). Given any $a, b \in \mathbb{R}$, exactly one of these statements is true: $a < b$, $a = b$, or $b < a$. (The symbol $a \leq b$ stands for $a < b$ or $a = b$.)

Axiom 13 (OT: Order is transitive). Given any $a, b, c \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$.

Axiom 14 (OA: Order respects addition). Given any $a, b, c \in \mathbb{R}$, if $a < b$ then $a + c < b + c$.

Axiom 15 (OM: Order respects multiplication). Given any $a, b, c \in \mathbb{R}$, if $a < b$ and $0 < c$, then $ac < bc$.

Axiom 16 (CA: Completeness Axiom). If A and B are nonempty subsets of $\mathbb{R}$ such that for every $a \in A$ and for every $b \in B$ we have $a \leq b$, then there exists $c \in \mathbb{R}$ such that $a \leq c \leq b$ for all $a \in A$ and all $b \in B$.

1.1.2. Basic properties of the set of real numbers. In the next theorem we list several most important properties of the real numbers that follow from the Axioms.

Theorem 1.1. *Let a, b, c be real numbers. The following statements hold.*

(i) $a + c = b + c \quad \Leftrightarrow \quad a = b$
(ii) $-0 = 0$
(iii) $-a = (-1)a$
(iv) $ab = 0 \quad \Leftrightarrow \quad (a = 0) \vee (b = 0)$
(v) $a < b \quad \Leftrightarrow \quad -b < -a$
(vi) $(a < b) \wedge (c < 0) \quad \Rightarrow \quad bc < ac$
(vii) $a \neq 0 \quad \Leftrightarrow \quad aa > 0$
(viii) $0 < a \quad \Leftrightarrow \quad 0 < \frac{1}{a}$
(ix) *If a and b are positive, the following equivalence holds $a < b \;\Leftrightarrow\; \frac{1}{b} < \frac{1}{a}$.*

We can discuss proofs of these statements in Discussions on Canvas.

Now I will make a far-reaching statement: All properties about real numbers you learned in high school algebra and precalculus courses can be deduced from the Axioms using mathematical logic. Sometimes these deductions, more commonly known as proofs, are tedious. That is probably why these proofs are not done formally in high school and beginning college courses. However, we will assume that you have already learned a lot of algebra. You can use all the algebra that you learned in your proofs. We will refer to that knowledge as Background Knowledge. In each proof, it is a good idea to identify the Background Knowledge you need and ensure that all the statements you are using are true. The first step towards verifying the validity of Background Knowledge is a precise formulation. Then you can draw graphs, try a few well-chosen cases, make up a real-life illustration, or raise a question in Discussions on Canvas. If you find an algebraic property particularly interesting, you can always challenge me to prove it from axioms.

The next theorem I call the **Pizza-Party Inequality**. It is often used in proofs.

Theorem 1.2. *Let a, b, c and d are positive real numbers. The following implication holds:*

$$(a \leq b) \ \wedge \ (c \leq d) \quad \Rightarrow \quad \frac{c}{b} \leq \frac{d}{a}.$$

From a pizza-party perspective, the implication in the preceding theorem is clear: If your objective is getting more pizza, would you rather attend a smaller party that is sharing in a larger pizza or a larger party sharing in a smaller pizza? Should this be taught in a math class, or kids learn that in kindergarten? Ok, this is a math class, so we can prove it!

1.2. Sets

In this course we will use the standard set notation. We will be dealing with sets which consist of real numbers. A set can be described by a clear statement such as "Let S be the set of real solutions of the equation $x^2 - x = 0$." A set can also be described by a listing of all its elements; for example. In the preceding case: $S = \{0, 1\}$. To describe sets we often use the set builder notation:

$$S = \big\{x \in \mathbb{R} \,:\, x^2 = x\big\}.$$

The above expression is read as: "The set A is the set of all real numbers x such that $x^2 = x$." Here the colon ":" is used as an abbreviation for the phrase "such that". Instead of colon many books use the vertical bar symbol $|$.

Pay attention to the usage of the braces (or curly brackets) $\{$ and $\}$ in the set notation. The braces are used to delimit sets. Notice the difference between the symbols 0 and $\{0\}$. The symbol 0 stands for the real number 0 and $\{0\}$ denotes the set whose only element is 0.

Next we review some important subsets of the real numbers. The set of all integers is denoted by $\mathbb{Z}$. In the set notation it is written as

$$\mathbb{Z} = \big\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\big\}.$$

Since we cannot list all the integers, we use the ellipses to indicate that the pattern continues infinitely. The set of positive integers is denoted by $\mathbb{N}$. In the set notation we can write this set as follows

$$\mathbb{N} = \big\{1, 2, 3, \ldots\big\} = \big\{x \in \mathbb{Z} \,:\, x > 0\big\}.$$

The synonym for "positive integer" is "natural number". A rational number is any real number that can be expressed as a fraction p/q where p is an integer and q is a positive integer. The set of rational numbers is denoted by $\mathbb{Q}$. In the set notation we can write this set as follows

$$\mathbb{Q} = \left\{x \in \mathbb{R} \,:\, x = \frac{p}{q} \text{ where } p \in \mathbb{Z},\ q \in \mathbb{N}\right\}.$$

Further important subsets of $\mathbb{R}$ are intervals. Let a and b be real numbers such that $a < b$. Below we list all possible intervals with endpoints a and b. The symbol $\wedge$ denotes the logical conjunction between two mathematical statements. We read it as *and*.

$$\begin{aligned}
[a, b] &= \big\{x \in \mathbb{R} \,:\, (a \le x) \wedge (x \le b)\big\} && \text{is called } \textit{a closed interval},\\
(a, b) &= \big\{x \in \mathbb{R} \,:\, (a < x) \wedge (x < b)\big\} && \text{is called } \textit{an open interval},\\
[a, b) &= \big\{x \in \mathbb{R} \,:\, (a \le x) \wedge (x < b)\big\} && \text{is called } \textit{a half-open interval},\\
(a, b] &= \big\{x \in \mathbb{R} \,:\, (a < x) \wedge (x \le b)\big\} && \text{is called } \textit{a half-open interval}.
\end{aligned}$$

The intervals above are called *finite intervals*. We also define four types of *infinite intervals*:

$$\begin{aligned}
[a, +\infty) &= \big\{x \in \mathbb{R} \,:\, a \le x\big\} && \text{is called } \textit{a closed unbounded interval},\\
(a, +\infty) &= \big\{x \in \mathbb{R} \,:\, a < x\big\} && \text{is called } \textit{an open unbounded interval}\\
(-\infty, b] &= \big\{x \in \mathbb{R} \,:\, x \le b\big\} && \text{is called } \textit{an unbounded closed interval},\\
(-\infty, b) &= \big\{x \in \mathbb{R} \,:\, x < b\big\} && \text{is called } \textit{an unbounded open interval}.
\end{aligned}$$

Geometric illustrations of these intervals are given in Figures 1 through 8.

The infinity symbols $-\infty$ and $+\infty$ are used to indicate that the set is unbounded in the negative $(-\infty)$ or positive $(+\infty)$ direction of the real number line. The symbols $-\infty$ and $+\infty$ are just symbols; they are not real numbers. Therefore we always exclude them as endpoints by using parentheses.

The set $\mathbb{R}$ is also an infinite interval. Sometimes it is written as $(-\infty, +\infty)$.

FIG. 1. A closed interval

FIG. 2. An open interval

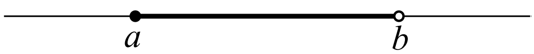

FIG. 3. A half-open interval

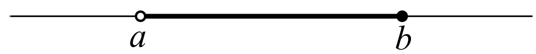

FIG. 4. A half-open interval

FIG. 5. A closed infinite interval

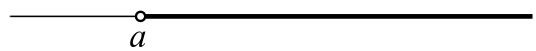

FIG. 6. An open infinite interval

FIG. 7. An infinite closed interval

FIG. 8. An infinite open interval

Let S be a subset of $\mathbb{R}$. If u is the smallest number in S, then u is called a *minimum* of S and we write $u = \min S$. If v is the greatest number in S, then v is called a *maximum* of S and we write $v = \max S$. More formally, we express these definitions as logical statements:

$$u = \min S \quad \text{if and only if} \quad u \in S \quad \text{and} \quad u \leq x \quad \text{for all} \quad x \in S,$$

$$v = \max S \quad \text{if and only if} \quad v \in S \quad \text{and} \quad v \geq x \quad \text{for all} \quad x \in S.$$

Notice that the set $\mathbb{Z}$ has neither a minimum nor a maximum. Also, the open interval (a, b) has neither a minimum nor a maximum. The set $\mathbb{N}$ has no maximum and $\min \mathbb{N} = 1$. Each finite subset of $\mathbb{R}$ has both a minimum and a maximum.

1.3. Functions

1.3.1. The definition. Next we review the definition of a function. Let A and B be nonempty sets. A *function* f from A to B is a rule that assigns **exactly one** element of B to **each** element in A. This relationship between the sets A and B and the rule f is indicated by the following notation:

$$f : A \to B.$$

This notation can be read as: "f maps the set A into the set B." For $x \in A$ the unique element of B which is assigned to x by the function f is called the value of f at x. This element is denoted by $f(x)$. Sometimes this relationship between x and $f(x)$ is emphasized by the following notation:

$$x \mapsto f(x) \quad \text{where} \quad x \in A.$$

This notation is particularly convenient when a function is given by a formula and it is not given a letter name. For example,

$$x \mapsto x^2 \quad \text{where} \quad x \in [0, +\infty).$$

Let $f : A \to B$ be a function. The set A is called the *domain of f*. The set B is called the *codomain of f*. The subset

$$\{f(x) \in B \ : \ x \in A\}$$

of B is called the *range of f*.

In this class we are interested in functions for which both sets A and B are subsets of the set of real numbers $\mathbb{R}$. Some examples of such functions are given next.

1.3.2. The sign and the unit step function. Let sign : $\mathbb{R} \to \mathbb{R}$ be given by the formula

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{for } \ x > 0, \\ 0 & \text{for } \ x = 0, \\ -1 & \text{for } \ x < 0. \end{cases}$$

This function is called the *sign* function. A graph of the sign function is given in Fig. 9. Notice the use of small circles and small disks on the graph of the sign function. The small circles are placed at the points $(0, -1)$ and $(0, 1)$. They emphasise the fact that the value of the sign function at 0 is neither -1, nor 1. The disk placed at the point $(0, 0)$ emphasises that the value of the sign function at 0 is 0. This is the standard notation used on the graphs of piecewise defined function whenever a confusion could arise.

The domain of the sign function is the set $\mathbb{R}$ of real numbers. The range of the sign function is the set $\{-1, 0, 1\}$.

Let us : $\mathbb{R} \to \mathbb{R}$ be given by the formula

$$\operatorname{us}(x) = \begin{cases} 1 & \text{for } \ x \geq 0, \\ 0 & \text{for } \ x < 0. \end{cases}$$

This function is called the *unit step* function. A graph of the unit step function is given in Fig. 10. Notice that a disk is placed at the point $(0, 1)$ and a circle is placed at $(0, 0)$. This notation emphasises that $\operatorname{us}(0) = 1$.

The domain of the unit step function is the set $\mathbb{R}$ of real numbers. The range of the unit step function is the set $\{0, 1\}$.

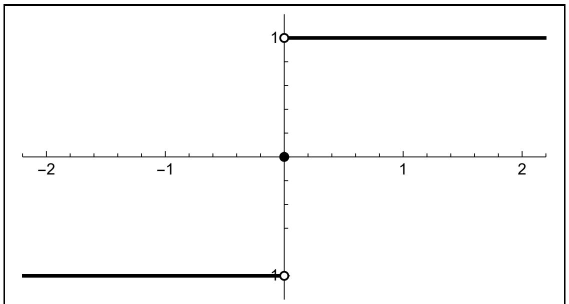

FIG. 9. The sign function

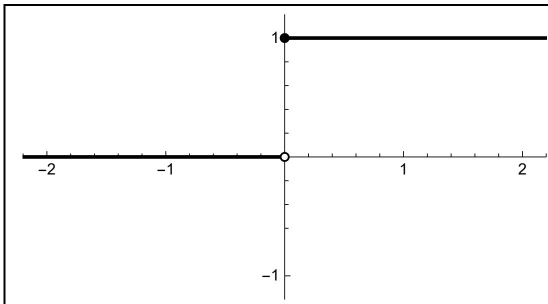

FIG. 10. The unit step function

EXERCISE 1.3. Prove that $\max\{u, v\} = v + (u - v)\operatorname{us}(u - v)$ for all $u, v \in \mathbb{R}$.

1.3.3. The floor and the ceiling function. The *floor* function,

$$\text{floor} : \mathbb{R} \to \mathbb{R},$$

is defined by the formula

$$\text{floor}(x) = \lfloor x \rfloor = \max\big\{k \in \mathbb{Z} : k \leq x\big\}.$$

A graph of the floor function is given in Fig. 11. Since the floor is piecewise defined function and without disks and circles on its graph there could be confusion as of the exact values at the integers, we placed disks at the following set of points:

$$\big\{(n, n) \,:\, n \in \mathbb{Z}\big\}$$

and we placed circles at the following set of points:

$$\big\{(n-1, n) \,:\, n \in \mathbb{Z}\big\}.$$

It follows from the properties of the maximum that for an arbitrary $x \in \mathbb{R}$ we have the following equivalence

$$m = \lfloor x \rfloor \qquad \text{if and only if} \qquad m \leq x < m+1 \quad \text{and} \quad m \in \mathbb{Z}.$$

It is important to notice the following equivalence: For all $x \in \mathbb{R}$ we have

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1 \quad \Leftrightarrow \quad x - 1 < \lfloor x \rfloor \leq x.$$

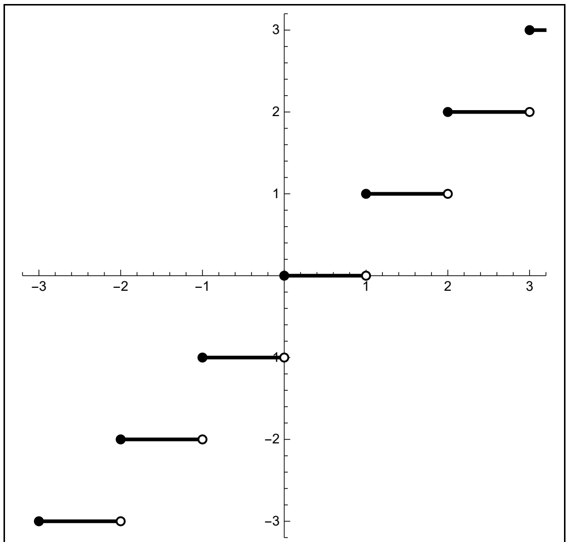

Fig. 11. The floor function

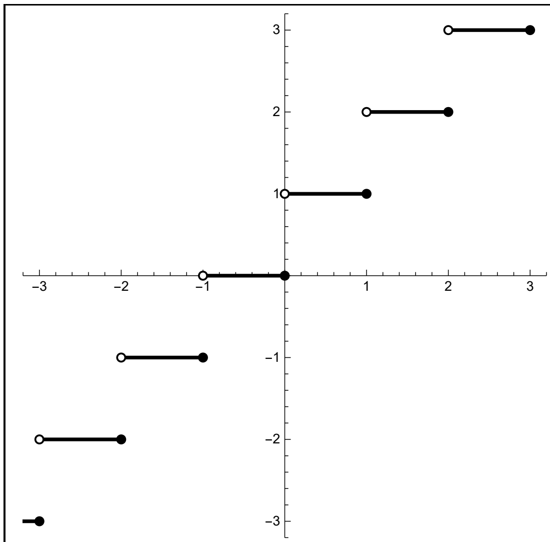

Fig. 12. The ceiling function

The *ceiling* function,

$$\text{ceiling} : \mathbb{R} \to \mathbb{R},$$

is defined by the formula

$$\text{ceiling}(x) = \lceil x \rceil = \min\{k \in \mathbb{Z} : k \geq x\}.$$

A graph of the ceiling function is given in Fig. 12. In Fig. 12 we placed disks at the following set of points:

$$\big\{(n, n) \,:\, n \in \mathbb{Z}\big\}$$

and we placed circles at the following set of points:

$$\{(n, n+1) \,:\, n \in \mathbb{Z}\}.$$

It follows from the properties of the minimum that for an arbitrary $x \in \mathbb{R}$ we have the following equivalence

$$n = \lceil x \rceil \qquad \text{if and only if} \qquad n-1 < x \leq n \quad \text{and} \quad n \in \mathbb{Z}.$$

Notice that the inequalities $\lceil x \rceil - 1 < x \leq \lceil x \rceil$ are equivalent to

$$x \leq \lceil x \rceil < x+1. \tag{1.1}$$

EXERCISE 1.4. State clearly the domain and the range of the floor and the ceiling function.

EXERCISE 1.5. Prove that for all $x \in \mathbb{R}$ we have

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \tfrac{1}{2} \rfloor.$$

Discover and prove the analogous identity for the ceiling function.

1.3.4. A rounding function. Since rounding function is probably the most used function in everyday life, I wanted to include it in these notes. For a real number x by $\lceil x \rfloor$ we denote the closest integer to x. The previous statement is ambiguous for the odd multiples of $1/2$. To be specific we define that $\lceil 1/2 + m \rfloor = m+1$ for all $m \in \mathbb{Z}$, see Fig. 13. This function can be expressed using the floor function:

$$\lceil x \rfloor = \lfloor x + 1/2 \rfloor = \left\lceil \frac{\lfloor 2x \rfloor}{2} \right\rceil \qquad \text{for all} \quad x \in \mathbb{R}.$$

Or, explicitly,

$$\forall x \in \mathbb{R} \quad \forall m \in \mathbb{Z} \qquad m = \lceil x \rfloor \quad \Leftrightarrow \quad m - \frac{1}{2} \leq x < m + \frac{1}{2}. \tag{1.2}$$

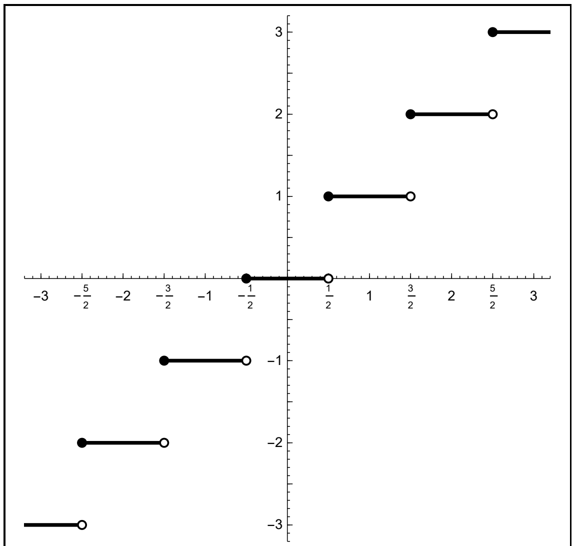

FIG. 13. The graph of our $\lceil x \rfloor$

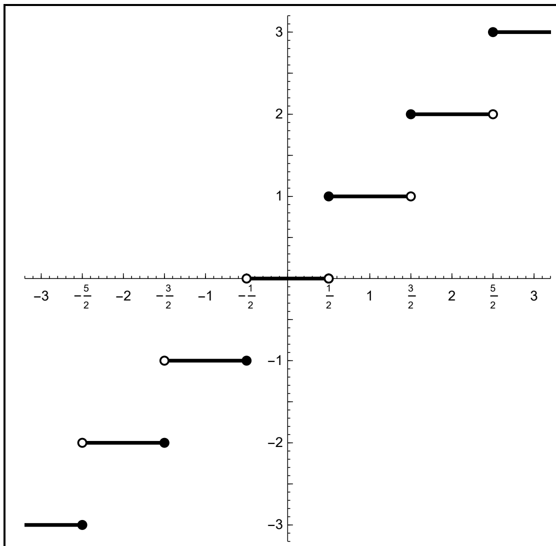

FIG. 14. An alternative rounding

The Wikipedia page on rounding lists six ways of rounding to the nearest integer. The advantage of the rounding shown in Fig. 14 is that the function is odd. That is, the rounding shown in Fig. 14 treats positive and negative values symmetrically, $-1/2$ is rounded -1, while $1/2$ is rounded to 1. This rounding is used in commercial transactions.

1.3.5. The absolute value function.

DEFINITION 1.6. Let $\text{abs} : \mathbb{R} \to \mathbb{R}$ be defined by the piecewise formula

$$\text{abs}(x) = |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

This function is called the *absolute value* function. For a given real number x the number $|x|$ is called the *absolute value* of x.

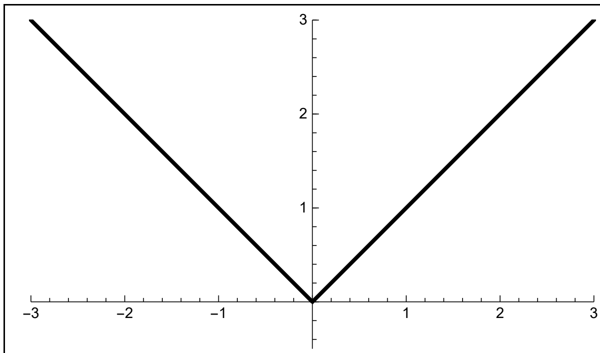

FIG. 15. The absolute value function

From calculus you are familiar with the geometric representation of real numbers as points on a straight line. This is done by choosing a point on the line to represent 0 and another point to represent 1. Then, every real number will correspond to a point on this line (called the number line), and every point on the number line will correspond to a real number. This geometric representation might be very helpful in doing problems.

Geometrically, the absolute value of a represents the distance between 0 and a, or, generally $|a - b|$ is the *distance* between the real numbers a and b on the number line.

EXERCISE 1.7. In the following problems write your solution as a set.
(a) Find all values of x such that $|5x - 3| = 4$.
(b) Find all values of x such that $|5x - 3| < 4$.
(c) Find all values of x such that $|5x - 3| > 4$.

EXERCISE 1.8. In the following problems write your solution as a set.
(a) Find all values of x such that $|7x + 3| = 5$.
(b) Find all values of x such that $|7x + 3| < 5$.
(c) Find all values of x such that $|7x + 3| > 5$.

The basic properties of the absolute value are given in the following theorem.

THEOREM 1.9. *The following statements hold.*
(i) *For all* $x \in \mathbb{R}$ *we have* $|x| = \max\{x, -x\}$.
(ii) *For all* $x \in \mathbb{R}$ *we have* $|x| \geq 0$.
(iii) *For all* $x \in \mathbb{R}$ *we have* $|-x| = |x|$.
(iv) *for all* $x \in \mathbb{R}$ *we have* $-x \leq |x|$ *and* $x \leq |x|$.

(v) *For all* $x, y \in \mathbb{R}$ *we have* $|xy| = |x||y|$.

(vi) *For all* $x, y \in \mathbb{R}$ *with* $y \neq 0$ *we have* $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$.

PROOF. To prove (i) we consider two cases. **Case I.** Assume $x \geq 0$. Then $-x \leq 0$. Since $-x \leq 0$ and $0 \leq x$, it follows that $-x \leq x$. Therefore $\max\{x, -x\} = x$. By Definition 1.6 for $x \geq 0$ we have that $\operatorname{abs}(x) = x$. Hence, we conclude that $\operatorname{abs}(x) = \max\{x, -x\}$ in this case. **Case II.** Assume $x < 0$. Then $-x > 0$. Since $-x > 0$ and $0 > x$, it follows that $-x > x$. Therefore $\max\{x, -x\} = -x$. By Definition 1.6 for $x < 0$ we have that $\operatorname{abs}(x) = -x$. Hence, we conclude that $\operatorname{abs}(x) = \max\{x, -x\}$ in this case.

Since Cases I and II cover all real numbers x, the equality $\operatorname{abs}(x) = \max\{x, -x\}$ is proved.

The statement (ii) can also be proved by considering two cases.

To prove (iii) note that by (i) $|x| = \max\{x, -x\}$ and also $|-x| = \max\{-x, -(-x)\} = \max\{-x, x\}$. Since $\max\{x, -x\} = \max\{-x, x\}$, we conclude that $|x| = |-x|$.

By the definition of max we have $\max\{a, b\} \geq a$ and $\max\{a, b\} \geq b$ for any real numbers a and b. Therefore $\max\{x, -x\} \geq x$ and $\max\{x, -x\} \geq -x$. Using (i) we conclude $|x| \geq x$ and $|x| \geq -x$. This proves (iv).

The proof of (v) is by considering four cases. The proof of (vi) first considers the case $x = 1$ by two cases and then applies (v). □

EXERCISE 1.10. Let x and y be real numbers. Prove that

$$\max\{x, y\} = \frac{1}{2}\big(x + y + |x - y|\big).$$

EXERCISE 1.11. Let $x \in \mathbb{R}$ and $a > 0$. Prove that $|x| < a$ if and only if $-a < x < a$.

THEOREM 1.12. (**Triangle Inequalities**)

(a) *For all* $a, b \in \mathbb{R}$ *we have* $|a + b| \leq |a| + |b|$.

(b) *For all* $x, y, z \in \mathbb{R}$ *we have* $|x - y| \leq |x - z| + |z - y|$.

(c) *For all* $x, y \in \mathbb{R}$ *we have* $\big||x| - |y|\big| \leq |x - y|$.

PROOF. Proof of (a). By Theorem 1.9 (iv) we know that $a \leq |a|$ and $b \leq |b|$. Therefore we conclude that

$$a + b \leq |a| + |b|. \tag{1.3}$$

By Theorem 1.9 (iv) we know that $-a \leq |a|$ and $-b \leq |b|$. Therefore we conclude

$$-(a + b) = -a + (-b) \leq |a| + |b|. \tag{1.4}$$

The inequalities (1.3) and (1.4) imply

$$\max\{a + b, -(a + b)\} \leq |a| + |b|. \tag{1.5}$$

By Theorem 1.9 (i) the inequality (1.5) yields $|a + b| \leq |a| + |b|$. This proves (a).

Prove (b) and (c) as an exercise. □

The inequalities in Theorem 1.12 are called the **Triangle Inequalities**.

EXERCISE 1.13. Let a, b, c be real numbers such that $a \neq 0$ and $c > 0$. Write your solution as a set.

(a) Find all values of x such that $|ax + b| = c$.

(b) Find all values of x such that $|ax + b| < c$.

(c) Find all values of x such that $|ax+b| > c$.

EXERCISE 1.14. Let a be a real number and let ϵ be a positive real number. Prove that

$$|x-a| < \epsilon \qquad \text{if and only if} \qquad x \in (a-\epsilon, a+\epsilon).$$

1.3.6. New functions from old.

DEFINITION 1.15. Given two functions $f : A \to B$ and $g : A \to B$, with $A, B \subseteq \mathbb{R}$, and two real numbers α and β we form a new function $\alpha f + \beta g : A \to B$ defined by

$$(\alpha f + \beta g)(x) = a\, f(x) + \beta\, g(x), \quad \text{for all} \quad x \in A.$$

Notice that $f(x)$ and $g(x)$ are real numbers so that $\alpha\, f(x)$ and $\beta\, g(x)$ in the above formula is just a multiplication of real numbers. The function $\alpha f + \beta g$ is called a *linear combination* of the functions f and g.

DEFINITION 1.16. Given two functions $f : A \to B$ and $g : A \to B$, with $A, B \subseteq \mathbb{R}$ we form a new function $fg : A \to B$ defined by

$$(fg)(x) = f(x)g(x), \quad \text{for all} \quad x \in A.$$

Notice that $f(x)$ and $g(x)$ are real numbers so that $f(x)g(x)$ in the above formula is just a multiplication of real numbers. The function fg is called the *product* of the functions f and g.

DEFINITION 1.17. Given two functions $f : A \to B$ and $g : B \to C$ a new function $g \circ f : A \to C$ is defined by

$$(g \circ f)(x) = g(f(x)), \qquad x \in A.$$

The function $g \circ f$ is called the *composition* of the functions f and g.

Applying these definitions to familiar functions gives rise to new, sometimes very interesting functions.

EXERCISE 1.18. For each of the functions given below answer the following questions: (a) What are the domain and the range of the function? (b) Plot the function using your graphing calculator. Plot the function by hand emphasizing the details missed by your graphing calculator.

(a) $x \mapsto x\,\mathrm{abs}(x)$	(b) $x \mapsto x(1-\mathrm{abs}(x))$
(c) $x \mapsto x\,\mathrm{sign}(x)$	(d) $x \mapsto \mathrm{ceiling}(x) - \mathrm{floor}(x)$
(e) $x \mapsto x - \mathrm{floor}(x)$	(f) $x \mapsto x\,\mathrm{floor}(1/x)$
(g) $x \mapsto (1+\mathrm{sign}(x))/2$	(h) $x \mapsto x\ \mathrm{us}(x)$
(i) $x \mapsto \mathrm{sign}(\mathrm{abs}(x))$	(j) $x \mapsto \mathrm{abs}(\mathrm{sign}(x))$
(k) $x \mapsto \mathrm{floor}(\mathrm{abs}(x))$	(l) $x \mapsto \mathrm{ceiling}(\mathrm{abs}(x))$

CHAPTER 2

Limits

2.1. Limit of a function as x approaches $+\infty$

2.1.1. The definition.

DEFINITION 2.1. Let D be a subset of $\mathbb{R}$ and $L \in \mathbb{R}$. A function $f : D \to \mathbb{R}$ has the limit L as x approaches $+\infty$ if the following two conditions are satisfied:

(I) There exists $X_0 \in D$ such that $[X_0, +\infty) \subseteq D$.

(II) For every real number $\epsilon > 0$ there exists a real number $X(\epsilon) \geq X_0$ such that

$$x > X(\epsilon) \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

If the conditions (I) and (II) in Definition 2.1 are satisfied we write

$$\lim_{x \to +\infty} f(x) = L.$$

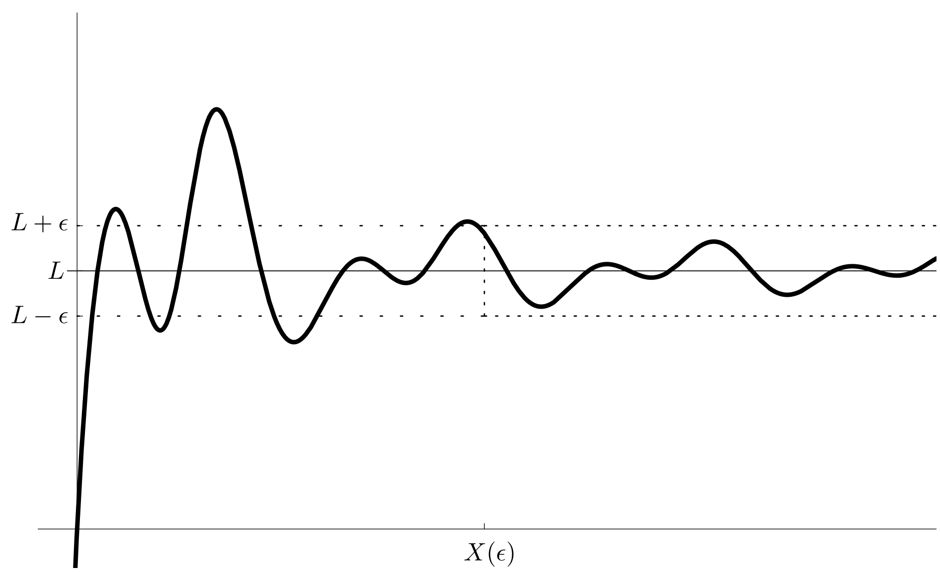

FIG. 1. An illustration for the condition (II) in Definition 2.1

2.1.2. Examples for Definition 2.1.

EXAMPLE 2.2. Prove that $\displaystyle\lim_{x \to +\infty} \frac{1}{\sqrt{x-1}} = 0$.

SOLUTION. We have to show that the conditions (I) and (II) in Definition 2.1 are satisfied. In this example $L = 0$ and we can take $D = (1, +\infty)$, since if $x > 1$, then $x - 1 > 0$

and $1/\sqrt{x-1}$ is defined. Next, we have to provide X_0. We can take $X_0 = 2$, since clearly $[2, +\infty) \subseteq (1, +\infty)$.

Next we show that the condition (II) is satisfied. Let $\epsilon > 0$ be given. We have to find a real number $X(\epsilon) \geq 2$ such that

$$x > X(\epsilon) \quad \Rightarrow \quad \left| \frac{1}{\sqrt{x-1}} - 0 \right| < \epsilon. \tag{2.1}$$

In some sense we have to solve the inequality

$$\left| \frac{1}{\sqrt{x-1}} - 0 \right| < \epsilon.$$

for x. The first step is to simplify it. Clearly

$$\left| \frac{1}{\sqrt{x-1}} - 0 \right| = \frac{1}{\sqrt{x-1}} \quad \text{for} \quad x \geq 2.$$

Thus we need to solve

$$\frac{1}{\sqrt{x-1}} < \epsilon.$$

This inequality is solved for x by using the following sequence of algebraic steps:

$$\frac{1}{\sqrt{x-1}} < \epsilon \quad \Leftrightarrow \quad \sqrt{x-1} > \frac{1}{\epsilon} \quad \Leftrightarrow \quad x - 1 > \frac{1}{\epsilon^2} \quad \Leftrightarrow \quad x > \frac{1}{\epsilon^2} + 1. \tag{2.2}$$

Since we need $X(\epsilon) \geq 2$, we choose $X(\epsilon) = \max\left\{ \frac{1}{\epsilon^2} + 1, 2 \right\}$.

It remains to prove that the implication (2.1) is satisfied. Assume that

$$x > X(\epsilon). \tag{2.3}$$

Since $X(\epsilon) \geq 2$, we conclude that $x > 2$. Therefore $x - 1 > 0$ and $1/\sqrt{x-1}$ is defined. Since $X(\epsilon) \geq 1/\epsilon^2 + 1$, we conclude that

$$x > \frac{1}{\epsilon^2} + 1.$$

Now the equivalences (2.2) imply that

$$\frac{1}{\sqrt{x-1}} < \epsilon. \tag{2.4}$$

Since $1/\sqrt{x-1}$ is positive we conclude that

$$\frac{1}{\sqrt{x-1}} = \left| \frac{1}{\sqrt{x-1}} \right| = \left| \frac{1}{\sqrt{x-1}} - 0 \right|. \tag{2.5}$$

Combining (2.4) and (2.5), yields

$$\left| \frac{1}{\sqrt{x-1}} - 0 \right| < \epsilon. \tag{2.6}$$

Thus, we have proved that the assumption (2.3) implies the inequality (2.6). This is exactly the implication (2.1). □

EXAMPLE 2.3. Determine the limit of the function $x \mapsto \dfrac{\lceil x \rceil}{x}$ as x approaches $+\infty$ and prove your claim.

Solution. In Subsection 1.3.3, see (1.1), we established that $x \leq \lceil x \rceil < x+1$ for every real number x. Therefore, for large x, the value of $\lceil x \rceil$ does not differ much from x. Therefore it is reasonable to make the following claim

$$\lim_{x\to+\infty} \frac{\lceil x \rceil}{x} = 1.$$

Next we will prove this claim using Definition 2.1. The function $x \mapsto \dfrac{\lceil x \rceil}{x}$ is defined for all $x \neq 0$. Thus, we can take $D = \mathbb{R}\backslash\{0\}$, and $X_0 = 1$. In this example $L = 1$.

Next we show that the condition (II) is satisfied. Let $\epsilon > 0$ be given. We have to find a real number $X(\epsilon) \geq 1$ such that

$$x > X(\epsilon) \quad \Rightarrow \quad \left| \frac{\lceil x \rceil}{x} - 1 \right| < \epsilon. \tag{2.7}$$

Solving for x the inequality

$$\left| \frac{\lceil x \rceil}{x} - 1 \right| < \epsilon \tag{2.8}$$

is not easy. To find solutions of this inequality we first need to simplify it. In this process of simplification we can replace the expression

$$\left| \frac{\lceil x \rceil}{x} - 1 \right|$$

with an expression which is greater or equal to it. By the definition of the ceiling function we know that

$$x \leq \lceil x \rceil < x+1. \tag{2.9}$$

Since we consider only $x \geq 1$, we can divide by x in (2.9) and subtract 1 from each term to get

$$0 \leq \frac{\lceil x \rceil}{x} - 1 < \frac{x+1}{x} - 1 = \frac{1}{x}.$$

Therefore

$$\left| \frac{\lceil x \rceil}{x} - 1 \right| \leq \frac{1}{x} \qquad \text{for all} \qquad x \geq 1. \tag{2.10}$$

This inequality is the key step in this proof. Therefore I call it the BIg INequality, or BIN. (Each of the proofs involving the definition of limit involves a BIN.) The importance of BIN lies in the fact that instead of solving (2.8), we can solve for x the simpler inequality

$$\frac{1}{x} < \epsilon.$$

The solution of this inequality (remember $x \geq 1$) is $x > \dfrac{1}{\epsilon}$.

Now we can define $X(\epsilon) = \max\left\{\dfrac{1}{\epsilon}, 1\right\}$. With this $X(\epsilon)$ the implication (2.7) is true. It is easy to prove this claim: Assume that

$$x > X(\epsilon) = \max\left\{\frac{1}{\epsilon}, 1\right\}.$$

Then $x \geq 1$ and $x > \dfrac{1}{\epsilon}$. Since $x \geq 1$ the BIN inequality (see (2.10))

$$\left| \frac{\lceil x \rceil}{x} - 1 \right| \leq \frac{1}{x}$$

is true. Since also $x > \dfrac{1}{\epsilon}$, we conclude that

$$\frac{1}{x} < \epsilon.$$

The last two displayed inequalities imply that

$$\left| \frac{\lceil x \rceil}{x} - 1 \right| < \epsilon.$$

This proves the implication (2.7). ☐

Exercise 2.4. Determine whether the following functions have limits as x approaches $+\infty$. Prove your statements using the definition.

(a) $x \mapsto \dfrac{x}{3x-2}$ (b) $x \mapsto \dfrac{2x}{x^2+x+1}$ (c) $x \mapsto \dfrac{x+\sin(x)}{x-1}$

(d) $x \mapsto \dfrac{x^2+x}{x^3+3}$ (e) $x \mapsto \dfrac{x^3-2x^2+1}{x^3+x+101}$ (f) $x \mapsto \sqrt{x+1} - \sqrt{x}$

(g) $x \mapsto \dfrac{x^2 + x\cos(x)}{x^2-x+5}$ (h) $x \mapsto \left(\dfrac{1}{x}\right)^{1/\ln x}$ (i) $x \mapsto \dfrac{x^2-1}{x^2+2x\sin(x)}$

(j) $x \mapsto x - \sqrt{x^2 - x}$

Exercise 2.5. Guess the limit of the function $x \mapsto \ln\left(1+\dfrac{1}{x}\right)^x$ and prove your guess. Hint: 1) Use the rules for logarithms to simplify the expression. 2) Use the representation of the logarithm function $u \mapsto \ln(u)$ as an integral (area under the graph of the function $u \mapsto 1/u$) to find an upper and lower bound for the given function $x \mapsto \ln\left(1+\frac{1}{x}\right)^x$ for large values of x. The bounds should be very simple functions of x.

2.1.3. Negative results. How can we prove that $\lim\limits_{x\to+\infty} f(x) = L$ is false? This means that the condition (I) or the condition (II) in Definition 2.1 is not satisfied. Since the condition (II) is the essence of the definition of limit we will focus on the negation of the condition (II).

The negation of (II): There exists $\epsilon > 0$ such that for every $X \in \mathbb{R}$ there exists $x > X$ such that $|f(x) - L| \geq \epsilon$.

Example 2.6. Prove that $\lim\limits_{x\to+\infty} \sin(x) = 0$ is false.

Solution. Let $\epsilon = 1/2$. For arbitrary $X \in \mathbb{R}$ we have

$$\pi\lceil X/\pi \rceil + \pi/2 > X$$

and, for $x = \pi\lceil X/\pi \rceil + \pi/2$, we have $|\sin(x)| = 1$. Therefore

$$|\sin(x) - 0| \geq 1/2.$$ ☐

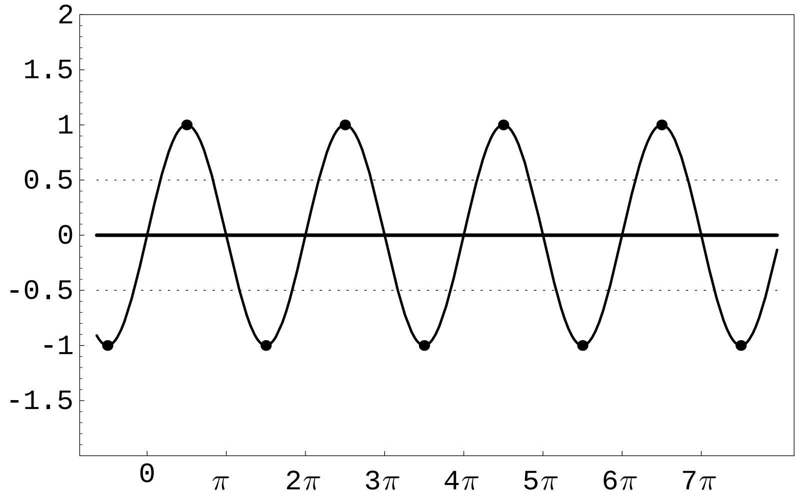

FIG. 2. Illustration for the solution of Example 2.6

Now we consider the statement "$\lim_{x\to+\infty} f(x)$ does not exist."

This means that for every $L \in \mathbb{R}$, $\lim_{x\to+\infty} f(x) = L$ is false.

EXAMPLE 2.7. Prove that $\lim_{x\to+\infty} \sin(x)$ does not exist.

SOLUTION. Let $L \in \mathbb{R}$ be arbitrary. We need to prove that $\lim_{x\to+\infty} \sin(x) = L$ is false. Consider three cases $L = 0$, $L < 0$ and $L > 0$. The case $L = 0$ is done in Example 2.6. Now assume $L < 0$. Let $\epsilon = 1/2$. For arbitrary $X \in \mathbb{R}$ we have

$$2\pi \left\lceil \frac{X}{2\pi} \right\rceil + \frac{\pi}{2} > X$$

and, for $x = 2\pi \left\lceil \frac{X}{2\pi} \right\rceil + \frac{\pi}{2}$, we have $\sin(x) = 1$. Therefore

$$|\sin(x) - L| = |1 - L| = 1 + |L| \geq 1/2.$$

Do the case $L > 0$ as an exercise. □

2.1.4. Infinite limits.

DEFINITION 2.8. Let D be a subset of $\mathbb{R}$. A function $f : D \to \mathbb{R}$ has the limit $+\infty$ as x approaches $+\infty$ if the following two conditions are satisfied:

(I) There exists a real number $X_0 \in D$ such that $[X_0, +\infty) \subseteq D$.
(II) For every real number M there exists a real number $X(M) \geq X_0$ such that

$$x > X(M) \quad \Rightarrow \quad f(x) > M.$$

The symbolic notation for this limit is

$$\lim_{x\to+\infty} f(x) = +\infty.$$

DEFINITION 2.9. Let D be a subset of $\mathbb{R}$. A function $f : D \to \mathbb{R}$ has the limit $-\infty$ as x approaches $+\infty$ if the following two conditions are satisfied:

(I) There exists a real number $X_0 \in D$ such that $[X_0, +\infty) \subseteq D$.
(II) For every real number M there exists a real number $X(M) \geq X_0$ such that
$$x > X(M) \quad \Rightarrow \quad f(x) < M.$$
The symbolic notation for this limit is
$$\lim_{x \to +\infty} f(x) = -\infty.$$

2.1.5. Examples of infinite limits.

EXAMPLE 2.10. Let $f(x) = \sqrt{x}$. Prove that $\lim\limits_{x \to +\infty} \sqrt{x} = +\infty$.

SOLUTION. The function $\sqrt{\cdot}$ is defined for all $x \geq 0$. Therefore we can take $X_0 = 0$ in the part (I) of the definition.

Now consider the part (II) of the definition. Let $M \in \mathbb{R}$ be arbitrary. we have to determine a real number $X(M)$ such that
$$x > X(M) \quad \Rightarrow \quad \sqrt{x} > M.$$
This will be accomplished if we solve the inequality $\sqrt{x} > M$. If $M < 0$, then all $x \geq 0$ satisfy this inequality. If $M \geq 0$ then the solution of the inequality is $x > M^2$. Thus, we can take
$$X(M) = \begin{cases} M^2 & \text{if} \quad M \geq 0, \\ 0 & \text{if} \quad M < 0\,. \end{cases}$$
Clearly, $X(M) \geq 0$ for all $M \in \mathbb{R}$ and
$$x > X(M) \quad \Rightarrow \quad \sqrt{x} > M. \qquad \square$$

EXAMPLE 2.11. Let $f(x) = \operatorname{floor}(x)$. Prove that $\lim\limits_{x \to +\infty} \operatorname{floor}(x) = +\infty$.

SOLUTION. The function floor is defined for all $x \in \mathbb{R}$. Therefore we can take $X_0 = 0$ in the part (I) of the definition.

Now consider the part (II) of the definition. Let $M \in \mathbb{R}$ be arbitrary. We have to determine a real number $X(M) \geq X_0$ such that
$$x > X(M) \quad \Rightarrow \quad \operatorname{floor}(x) > M. \tag{2.11}$$
This will be accomplished if we solve the inequality
$$\operatorname{floor}(x) > M. \tag{2.12}$$
Since we don't know much about floor it is not easy to solve (2.12). To achieve the implication (2.11), we can replace $\operatorname{floor}(x)$ in (2.12) with a smaller quantity $g(x)$ such that $g(x) > M$ is easy to solve. Thus we need $g(x)$ such that

(A) $\operatorname{floor}(x) \geq g(x)$ for all $x > X_0$.
(B) $g(x) > M$ is easy to solve.

By the definition of $\operatorname{floor}(x)$ we conclude that $0 \leq x - \operatorname{floor}(x) < 1$ for all $x \in \mathbb{R}$. Therefore
$$x - 1 < \operatorname{floor}(x) \quad \text{for all} \quad x \in \mathbb{R}. \tag{2.13}$$
Clearly $x - 1 > M$ is easy to solve: $x > M + 1$. Thus, we can take $X(M) = \max\{M+1, 0\}$ in the part (II) of the definition. Clearly $X(M) \geq X_0 = 0$. Let $x > X(M)$. Then $x > M+1$ and therefore $x - 1 > M$. By the inequality (2.13) we conclude that
$$\operatorname{floor}(x) > x - 1 > M.$$

Thus $x > X(M)$ implies $\operatorname{floor}(x) > M$. □

The key step in the solution of Example 2.11 was the discovery of the function $g(x)$ such that

(A) $f(x) \geq g(x)$ for all $x > X_0$.
(B) $g(x) > M$ is easy to solve.

Most proofs about limits follow this same pattern. Therefore I refer to the discovery of the function g as a *Big Inequality* or BIN for short.

EXERCISE 2.12. Determine whether the following functions have the limit $+\infty$ when x approaches $+\infty$.

(a) $x \mapsto \dfrac{x^2}{2x+1}$, (b) $x \mapsto \ln x$, (c) $x \mapsto x - \sqrt{x}$,

(d) $x \mapsto x - \ln(x)$, (e) $x \mapsto \dfrac{x^2 - x - 1}{x + 2\sqrt{x} + 1}$, (f) $x \mapsto \dfrac{1}{\sin\left(\frac{1}{x}\right)}$,

(g) $x \mapsto \sqrt{x - \sqrt{x - \sqrt{x}}}$, (h) $x \mapsto \dfrac{(\cos x)^2 x}{\sqrt{x} + \sin(x)}$, (j) $x \mapsto \dfrac{(2+\cos(x))x}{\sqrt{x} + \sin(x)}$.

2.2. Limit of a function at a real number a

2.2.1. The definition.

DEFINITION 2.13. Let D be a subset of $\mathbb{R}$ and let a and L be real numbers. A function $f : D \to \mathbb{R}$ has the limit L as x approaches a if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $\bigl(a - \delta_0, a\bigr) \cup \bigl(a, a + \delta_0\bigr) \subseteq D$.
(II) For every real number $\epsilon > 0$ there exists a real number $\delta(\epsilon)$ such that $0 < \delta(\epsilon) \leq \delta_0$ and

$$0 < |x - a| < \delta(\epsilon) \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

If the conditions (I) and (II) in Definition 2.13 are satisfied we write $\lim\limits_{x \to a} f(x) = L$.

Figure 3 illustrates Definition 2.13.

Next we restate Definition 2.13 using the terminology of a calculator screen. The figure below shows a fictional calculator screen with 35 pixels. We assume that $ymin$ and $ymax$ are chosen in such a way that the number L is in the middle of the y-range and that $xmin$ and $xmax$ are such that a is in the middle of the x-range.

In Definition 2.14 below we assume that the function f satisfies (I) in Definition 2.13. We rephrase (II) from Definition 2.13 in terms of a calculator screen.

For the specific fictional calculator screen shown in Figure 4, the connection between Definition 2.13 and Definition 2.14 is given by $\epsilon = (ymax - ymin)/8$, $xmin = a - \delta(\epsilon)$, $xmax = a + \delta(\epsilon)$ and $\delta(\epsilon) = \Delta$.

The fictional screen in Figure 4 is chosen for its simplicity. The screen of TI-92 (see the manual p. 321) is 239 pixels wide and 103 pixels tall; it has 24617 pixels. The screen of TI-83 (see the manual p. 8-16) and of TI-82 is 95 pixels wide and 63 pixels tall; it has 5985 pixels. The screen of TI-85 (see the manual p. 4-13) is 127 pixels wide and 63 pixels tall; it has 8001 pixels. The screen of TI-89 (see the manual p. 222) is 159 pixels wide and 77 pixels tall; it has 12243 pixels. Using these numbers you can calculate the connection between ϵ and $\delta(\epsilon)$ in Definition 2.13 and the screen of your calculator.

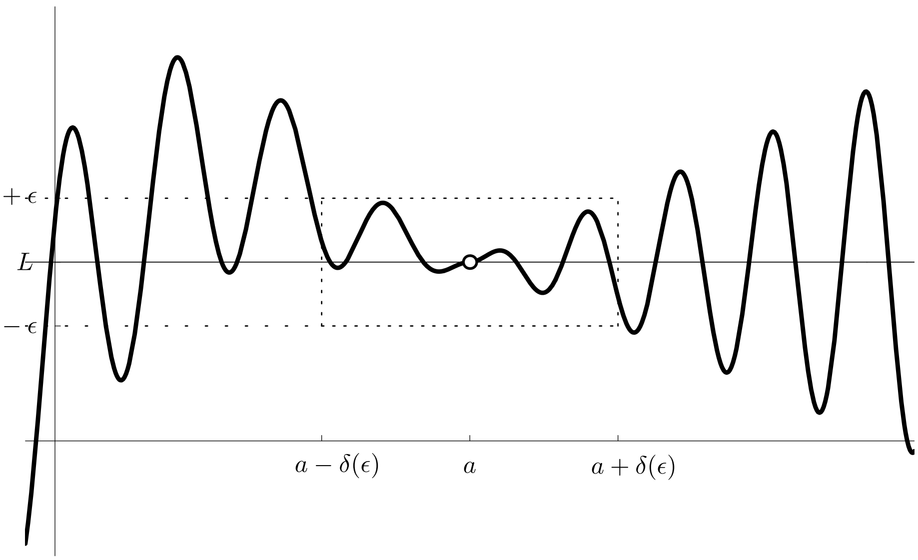

FIG. 3

DEFINITION 2.14 (Calculator Screen). A function f has a limit L as x approaches a for every choice of $ymin$ and $ymax$ there exists Δ (which depends on $ymin$ and $ymax$) such that whenever we choose $xmin$ and $xmax$ such that $xmax - xmin < 2\Delta$ the graph of the function f will appear to be a straight horizontal line on the calculator screen with the only possible exception at the pixel containing $x = a$.

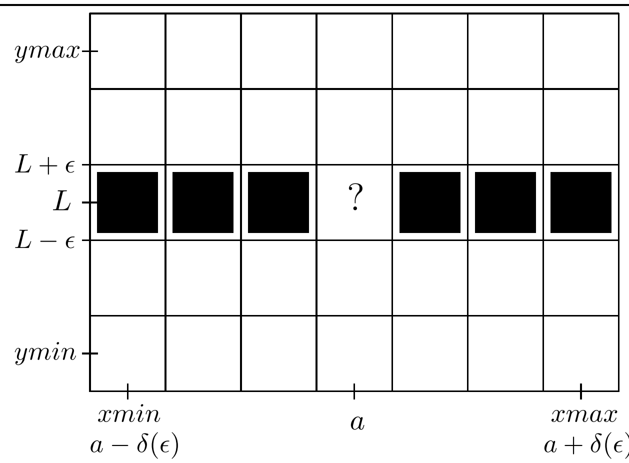

FIG. 4. A fictional calculator screen

2.2.2. Examples for Definition 2.13.

EXAMPLE 2.15. Prove $\lim_{x\to 2}(3x-1) = 5$.

SOLUTION. In this example $a = 2$, $L = 5$, $D = \mathbb{R}$ and $f(x) = 3x - 1$.
(I) We can take any positive number for δ_0. Since it might be useful to have a specific δ_0 to work with, we set $\delta_0 = 1$.
(II) Let $\epsilon > 0$ be given. Let $\delta(\epsilon) = \min\{\epsilon/3, 1\}$. Assume $0 < |x-2| < \delta(\epsilon)$. Since $\delta(\epsilon) \le \epsilon/3$, we conclude that $|x-2| < \epsilon/3$. Next, we calculate

$$|(3x-1)-5| = |3x-6| = 3\,|x-2|\,. \tag{2.14}$$

It follows from the assumption $0 < |x-2| < \delta(\epsilon)$ that $|x-2| < \epsilon/3$. Therefore we conclude

$$|(3x-1)-5| = 3\,|x-2| < 3\frac{\epsilon}{3} = \epsilon.$$

Thus we proved that

$$0 < |x-2| < \delta(\epsilon) \quad \Rightarrow \quad |(3x-1)-5| < \epsilon.$$

This is exactly the implication in (II) in Definition 2.13. Since $\epsilon > 0$ was arbitrary this completes the proof. □

REMARK 2.16. How did we guess the formula for $\delta(\epsilon)$ in the previous proof? We first studied the implication in the statement (II) in Definition 2.13. The goal in that implication is to prove

$$|(3x-1)-5| < \epsilon.$$

To prove this inequality we need to assume something about $|x-2|$. To find out what to assume, we simplified the expression $|(3x-1)-5|$ until $|x-2|$ appeared (see (2.14)). Then we solved for $|x-2|$. In this process of simplification we can afford to make the right-hand side larger. This will be illustrated in the next example.

EXAMPLE 2.17. Prove $\lim\limits_{x\to 2}\left(3x^2-2x-1\right) = 7$.

SOLUTION. We will use Definition 2.13 to prove that the statement in the example is correct. In this example $a=2$, $L=7$, $D=\mathbb{R}$, and $f(x) = 3x^2-2x-1$, .

Next we prove (I). Since the given function is defined on $\mathbb{R}$, we can take any positive number for δ_0. In this example it is essential to specify δ_0, so, we put $\delta_0 = 1$. (Please pay attention how this is used in an essential way in the proof below. Notice that this choice of $\delta_0 = 1$ in essence implies that, from now on, we consider only in the values of x which are in the set $(1,2)\cup(2,3)$.)

Next we will discover an inequality which will help us find a formula for $\delta(\epsilon)$:

$$\bigl|(3x^2-2x-1)-7\bigr| = \bigl|3x^2-2x-8\bigr| = \bigl|(3x+4)(x-2)\bigr| = |3x+4|\,|x-2|.$$

Now we use the fact that we are considering only the values of x which are in the set $(1,2)\cup(2,3)$. For $x\in(1,2)\cup(2,3)$ the value of $|3x+4|$ does not exceed 13. Therefore

$$|(3x^2-2x-1)-7| \leq 13\,|x-2| \quad \text{for all} \quad x\in(1,2)\cup(2,3).$$

Let $\epsilon > 0$ be given. The inequality $13\,|x-2| < \epsilon$ is easy to solve for $|x-2|$. The solution is $|x-2| < \epsilon/13$. Now we define $\delta(\epsilon)$:

$$\delta(\epsilon) = \min\left\{\frac{\epsilon}{13}, 1\right\}.$$

The remaining step of the proof is to prove the implication

$$|x-2| < \delta(\epsilon) \quad \Rightarrow \quad |(3x^2-2x-1)-7| < \epsilon.$$

We hope that at this point you can prove this implication on your own. □

EXAMPLE 2.18. Prove $\lim\limits_{x\to 2}\dfrac{x^3-x-4}{x-1} = 2$.

SOLUTION. We will use Definition 2.13 to prove that the statement in the example is correct. In this example $a=2$, $L=2$, $D=\mathbb{R}\backslash\{1\}$ and $f(x) = (x^3-x-4)/(x-1)$.

Next we prove (I). Notice that the function $f(x)$ is defined on $\mathbb{R}\setminus\{1\}$. In this proof we are interested in the values of x near $a=2$. Therefore, for δ_0 we can take any positive number which is smaller than 1. Since it is useful to have a specific number, we put $\delta_0 = 1/2$. This implies that from now on we consider only the values of x which are in the set $(3/2,2)\cup(2,5/2)$.

Next we will discover an inequality which will help us find a formula for $\delta(\epsilon)$:

$$\left|\frac{x^3-x-4}{x-1}-2\right| = \left|\frac{x^3-3x-2}{x-1}\right| = \left|\frac{(x^2+2x+1)(x-2)}{x-1}\right| = \left|\frac{x^2+2x+1}{x-1}\right| |x-2|. \quad (2.15)$$

Now remember that we are interested only in the values of x which are in the set $(3/2,2)\cup(2,5/2)$. For $x\in(3/2,2)\cup(2,5/2)$ we estimate

$$\left|\frac{x^2+2x+1}{x-1}\right| = \frac{x^2+2x+1}{x-1} \leq \frac{16}{1/2} = 32 \quad \text{for all} \quad x\in(3/2,2)\cup(2,5/2). \quad (2.16)$$

Combining (2.15) and (2.16) we get

$$\left|\frac{x^3-x-4}{x-1}-2\right| \leq 32\,|x-2| \quad \text{for all} \quad x\in(3/2,2)\cup(2,5/2).$$

Let $\epsilon>0$ be given. The inequality $32\,|x-2|<\epsilon$ is very easy to solve for $|x-2|$. The solution is $|x-2|<\epsilon/32$. Now we define $\delta(\epsilon)$:

$$\delta(\epsilon) = \min\left\{\frac{\epsilon}{32},\frac{1}{2}\right\}.$$

The remaining piece of the proof is to prove the implication

$$|x-2|<\delta(\epsilon) \qquad \Rightarrow \qquad \left|\frac{x^3-x-4}{x-1}-2\right|<\epsilon.$$

We hope that at this point you can prove this on your own. Write down all the details of your reasoning. □

EXAMPLE 2.19. Prove $\lim\limits_{x\to 4}\sqrt{x}=2$.

SOLUTION. In this example $a=4$, $L=2$, $D=[0,+\infty)$ and $f(x)=\sqrt{x}$. We first deal with (I). Notice that the function $f(x)=\sqrt{x}$ is defined on $[0,+\infty)$. We are interested in the values of x near the point $a=4$. Thus, for δ_0 we can take any positive number which is <4. Since it is useful to have a specific number, we put $\delta_0=1$. (Notice that this implies that from now on in this proof we are interested only in the values of x which are in the set $(3,4)\cup(4,5)$.)

Next we will discover an inequality which will help us find a formula for $\delta(\epsilon)$:

$$\left|\sqrt{x}-2\right| = \left|\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}+2}\right| = \left|\frac{x-4}{\sqrt{x}+2}\right| = \left|\frac{1}{\sqrt{x}+2}\right| |x-4|. \quad (2.17)$$

Now remember that we are interested only in the values of x which are in the set $(3,4)\cup(4,5)$. For $x\in(3,4)\cup(4,5)$ we estimate

$$\left|\frac{1}{\sqrt{x}+2}\right| = \frac{1}{\sqrt{x}+2} \leq \frac{1}{\sqrt{3}+2} \leq \frac{1}{2} \quad \text{for all} \quad x\in(3,4)\cup(4,5). \quad (2.18)$$

Combining (2.17) and (2.18) we get

$$\left|\sqrt{x}-2\right| \leq \frac{1}{2}\,|x-4| \quad \text{for all} \quad x\in(3,4)\cup(4,5).$$

Let $\epsilon>0$ be given. The inequality $\frac{1}{2}\,|x-4|<\epsilon$ is easy to solve for $|x-4|$. The solution is $|x-4|<2\epsilon$. Now define $\delta(\epsilon)$:

$$\delta(\epsilon) = \min\{2\epsilon,1\}.$$

The remaining step of the proof is to prove the implication

$$|x-4| < \min\{2\epsilon, 1\} \quad \Rightarrow \quad \left|\sqrt{x}-2\right| < \epsilon.$$

We hope that at this point you can prove this on your own. As before, please do it and write down the details of your reasoning. □

EXAMPLE 2.20. Prove that for every $a > 0$, $\lim_{x\to a} \frac{1}{x} = \frac{1}{a}$.

SOLUTION. Let $a > 0$ be arbitrary. In this example $L = 1/a$, $f(x) = 1/x$ and $D = \mathbb{R}\setminus\{0\}$. Next, we deal with (I) in Definition 2.13. Since the function $f(x) = 1/x$ is defined on $\mathbb{R}\setminus\{0\}$ and we are interested in the values of x near the point $a > 0$, for δ_0 we can take any positive number which is smaller than a. Since it is useful to have a specific number, we put $\delta_0 = a/2$. (Notice that this implies that from now on in this proof we are interested only in the values of x which are in the set $(a/2, a) \cup (a, 3a/2)$.)

Next we will discover an inequality which will help us find a formula for $\delta(\epsilon)$:

$$\left|\frac{1}{x}-\frac{1}{a}\right| = \left|\frac{a-x}{xa}\right| = \frac{|a-x|}{xa} = \frac{1}{xa}\,|x-a|\,. \tag{2.19}$$

Now remember that we are interested only in the values of x which are in the set $(a/2, a) \cup (a, 3a/2)$. For $x \in (a/2, a) \cup (a, 3a/2)$ we estimate

$$\frac{1}{xa} \leq \frac{1}{(a/2)a} = \frac{2}{a^2} \quad \text{for all} \quad x \in (a/2, a) \cup (a, 3a/2). \tag{2.20}$$

Combining (2.19) and (2.20) we get

$$\left|\frac{1}{x}-\frac{1}{a}\right| \leq \frac{2}{a^2}\,|x-a| \quad \text{for all} \quad x \in (a/2, a) \cup (a, 3a/2).$$

Let $\epsilon > 0$ be given. The inequality $\frac{2}{a^2}\,|x-a| < \epsilon$ is easy to solve for $|x-a|$. The solution is $|x-a| < (a^2/2)\epsilon$. Now define $\delta(\epsilon)$:

$$\delta(\epsilon) = \min\left\{\frac{a^2\epsilon}{2}, \frac{a}{2}\right\}.$$

The remaining step of the proof is to prove the implication

$$|x-a| < \min\left\{\frac{a^2\epsilon}{2}, \frac{a}{2}\right\} \quad \Rightarrow \quad \left|\frac{1}{x}-\frac{1}{a}\right| < \epsilon.$$

We hope that at this point you can prove this on your own. Write down the details of your reasoning. □

EXERCISE 2.21. Find each of the following limits. Prove your claims using Definition 2.13.

(a) $\lim_{x\to 3}(2x+1)$ (b) $\lim_{x\to 1}(-3x-7)$ (c) $\lim_{x\to 1}\left(4x^2+3\right)$

(d) $\lim_{x\to 2}\frac{x}{x-1}$ (e) $\lim_{x\to 3}\frac{x^2-x+2}{x+1}$ (f) $\lim_{x\to 0}x^{1/3}$

(g) $\lim_{x\to 0}\left(\frac{1}{|x|}\right)^{3/\ln|x|}$ (h) $\lim_{x\to 3}\frac{1}{x}$ (i) $\lim_{x\to 1}\frac{1}{x^2+1}$

(j) $\lim_{x\to -2}\frac{x}{x^2+4\,x+3}$

EXERCISE 2.22. Let $f(x) = \dfrac{x+1}{x^2-1}$. Does f have a limit at $a = -1$? Justify your answer.

EXERCISE 2.23. Prove that for every $a > 0$, $\lim_{x\to a} \sqrt{x} = \sqrt{a}$.

2.2.3. Infinite limits at a real number a.

DEFINITION 2.24. Let $a \in \mathbb{R}$ and let $D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ has the limit $+\infty$ as x approaches a if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $\bigl(a - \delta_0, a\bigr) \cup \bigl(a, a + \delta_0\bigr) \subseteq D$.
(II) For every real number $M > 0$ there exists a real number $\delta(M)$ such that $0 < \delta(M) \leq \delta_0$ and

$$0 < |x - a| < \delta(M) \quad \Rightarrow \quad f(x) > M.$$

If the conditions (I) and (II) in Definition 2.24 are satisfied we write $\lim_{x\to a} f(x) = +\infty$.

DEFINITION 2.25. Let $a \in \mathbb{R}$ and let $D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ has the limit $-\infty$ as x approaches a if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $\bigl(a - \delta_0, a\bigr) \cup \bigl(a, a + \delta_0\bigr) \subseteq D$.
(II) For every real number $M < 0$ there exists a real number $\delta(M)$ such that $0 < \delta(M) \leq \delta_0$ and

$$0 < |x - a| < \delta(M) \quad \Rightarrow \quad f(x) < M.$$

If the conditions (I) and (II) in Definition 2.25 are satisfied we write $\lim_{x\to a} f(x) = -\infty$.

EXERCISE 2.26. Find each of the following limits. Prove your claims using the appropriate definition.

(a) $\displaystyle\lim_{x\to 0} \frac{1}{|x|}$ (b) $\displaystyle\lim_{x\to -3} \frac{1}{(x+3)^2}$ (c) $\displaystyle\lim_{x\to 2} \frac{x-3}{x(x-2)^2}$

(d) $\displaystyle\lim_{x\to -1} \frac{x}{(x+1)^4}$ (e) $\displaystyle\lim_{x\to +\infty} \frac{x^2-x+2}{x+1}$ (f) $\displaystyle\lim_{x\to +\infty} \frac{x^2-x}{3-x}$

2.2.4. One-sided limits.

DEFINITION 2.27. Let $a, L \in \mathbb{R}$ and let $D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ has the limit L as x approaches a from the left if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $\bigl(a - \delta_0, a\bigr) \subseteq D$.
(II) For every real number $\epsilon > 0$ there exists a real number $\delta(\epsilon)$ such that $0 < \delta(\epsilon) \leq \delta_0$ and

$$0 < a - x < \delta(\epsilon) \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

If the conditions (I) and (II) in Definition 2.27 are satisfied we write $\lim_{x\uparrow a} f(x) = L$.

DEFINITION 2.28. Let $a, L \in \mathbb{R}$ and let $D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ has the limit $L \in \mathbb{R}$ as x approaches a from the right if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $\bigl(a, a + \delta_0\bigr) \subseteq D$.
(II) For every real number $\epsilon > 0$ there exists a real number $\delta(\epsilon)$ such that $0 < \delta(\epsilon) \leq \delta_0$ and

$$0 < x - a < \delta(\epsilon) \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

If the conditions (I) and (II) in Definition 2.28 are satisfied we write $\lim\limits_{x\downarrow a} f(x) = L$.

Definition 2.29. Let $a \in \mathbb{R}$ and let $D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ has the limit $+\infty$ as x approaches a from the left if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $\bigl(a - \delta_0, a\bigr) \subseteq D$.
(II) For every real number $M > 0$ there exists a real number $\delta(M)$ such that $0 < \delta(M) \leq \delta_0$ and

$$0 < a - x < \delta(M) \quad \Rightarrow \quad f(x) > M.$$

If the conditions (I) and (II) in Definition 2.29 are satisfied we write $\lim\limits_{x\uparrow a} f(x) = +\infty$.

Definition 2.30. Let $a \in \mathbb{R}$ and let $D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ has the limit $+\infty$ as x approaches a from the right if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $\bigl(a, a + \delta_0\bigr) \subseteq D$.
(II) For every real number $M > 0$ there exists a real number $\delta(M)$ such that $0 < \delta(M) \leq \delta_0$ and

$$0 < x - a < \delta(M) \quad \Rightarrow \quad f(x) > M.$$

If the conditions (I) and (II) in Definition 2.30 are satisfied we write $\lim\limits_{x\downarrow a} f(x) = +\infty$.

Definition 2.31. Let $a \in \mathbb{R}$ and let $D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ has the limit $-\infty$ as x approaches a from the left if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $\bigl(a - \delta_0, a\bigr) \subseteq D$.
(II) For every real number $M < 0$ there exists a real number $\delta(M)$ such that $0 < \delta(M) \leq \delta_0$ and

$$0 < a - x < \delta(M) \quad \Rightarrow \quad f(x) < M.$$

If the conditions (I) and (II) in Definition 2.31 are satisfied we write $\lim\limits_{x\uparrow a} f(x) = -\infty$.

Definition 2.32. Let $a \in \mathbb{R}$ and let $D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ has the limit $-\infty$ as x approaches a from the right if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $\bigl(a, a + \delta_0\bigr) \subseteq D$.
(II) For every real number $M < 0$ there exists a real number $\delta(M)$ such that $0 < \delta(M) \leq \delta_0$ and

$$0 < x - a < \delta(M) \quad \Rightarrow \quad f(x) < M.$$

If the conditions (I) and (II) in Definition 2.32 are satisfied we write $\lim\limits_{x\downarrow a} f(x) = -\infty$.

Exercise 2.33. Find each of the following limits. Prove your claims using the appropriate definition.

(a) $\lim_{x\uparrow 5} \dfrac{3x-15}{\sqrt{x^2-10x+25}}$ (b) $\lim_{x\downarrow 5} \dfrac{3x-15}{\sqrt{x^2-10x+25}}$ (c) $\lim_{x\uparrow 2} \dfrac{x-3}{x(x-2)}$

(d) $\lim_{x\downarrow 0} \left(\dfrac{1}{x}-\dfrac{1}{x^2}\right)$ (e) $\lim_{x\uparrow 5} \dfrac{2}{\sqrt{5-x}}$ (f) $\lim_{x\downarrow 5} \dfrac{6}{5-x}$

(g) $\lim_{x\uparrow 3} \dfrac{x+3}{x^2-9}$ (h) $\lim_{x\uparrow -3} \dfrac{x^2}{x^2-9}$ (i) $\lim_{x\downarrow 0} \left(x-\sqrt{x}\right)$

(j) $\lim_{x\to 3} \dfrac{x}{(x-3)^2}$ (k) $\lim_{x\downarrow -1} \dfrac{x^2}{x+1}$ (l) $\lim_{x\to+\infty} \left(x-\sqrt{x}\right)$

2.3. New limits from old

2.3.1. Squeeze theorems. In this section and in Section 2.3.3 we establish general properties of limits which are based on the formal definition of limit. These properties are stated as theorems.

Establishing theorems of this kind involves a major step forward in sophistication. Up to this point we have been trying to show that limits exist directly from the definition. Now for the first time we are going to **assume** that some limit exists (I refer to this in class as a *green* limit.) and try to make use of this information to establish the existence of some other limit (I refer to this in class as a *red* limit.). Remember that to establish the existence of a limit, we had to come up with a procedure for finding $\delta(\epsilon)$ that will work for any $\epsilon > 0$ that is given. If we assume the existence of a limit, then we are assuming the existence of such a procedure, though we may not know explicitly what it is. I refer to this as a *green* $\delta(\epsilon)$. It is this procedure we will need to use in order to construct a new procedure for the limit whose existence we are trying to establish. I refer to this as a *red* $\delta(\epsilon)$.

We start by considering squeeze theorems that resemble the role of BIN in previous sections. The following theorem is the Sandwich Squeeze Theorem.

THEOREM 2.34. *Let f, g and h be given functions and let a and L be real numbers. Suppose that the following three conditions are satisfied.*

(1) $\lim_{x\to a} f(x) = L$.
(2) $\lim_{x\to a} h(x) = L$.
(3) *There exists $\eta_0 > 0$ such that f, g and h are defined on $\big(a-\eta_0, a\big)\cup\big(a, a+\eta_0\big)$ and*

$$f(x) \le g(x) \le h(x) \quad \textit{for all} \quad x \in \big(a-\eta_0, a\big)\cup\big(a, a+\eta_0\big).$$

Then

$$\lim_{x\to a} g(x) = L.$$

PROOF. Here we have three functions and three definitions of limits, one for each function. Therefore we have to deal with three δ-s. We will give them appropriate names that will distinguish them from each other. Let us name them δ_f, δ_g and δ_h.

In the theorem it is assumed that $\lim_{x\to a} f(x) = L$. This means that we are given the fact that for every $\epsilon > 0$ there exists $\delta_f(\epsilon) > 0$ (that is, we are given a function $\delta_f(\epsilon)$) such that

$$0 < |x-a| < \delta_f(\epsilon) \quad \Rightarrow \quad |f(x)-L| < \epsilon. \tag{2.21}$$

In class I refer to these as a green $\delta_f(\cdot)$ and a green implication.

Since the theorem assumes that $\lim_{x\to a} h(x) = L$, we are also given that for every $\epsilon > 0$ there exists $\delta_h(\epsilon) > 0$ such that

$$0 < |x - a| < \delta_h(\epsilon) \quad \Rightarrow \quad |h(x) - L| < \epsilon. \tag{2.22}$$

Again we refer to these as a green $\delta_h(\cdot)$ and a green implication.

We need to prove that $\lim_{x\to a} g(x) = L$. Therefore, following the definition of limit, we have to show that the following conditions are satisfied:

(I) There exists a real number $\delta_{0,g} > 0$ such that $g(x)$ is defined for every x in the set $\big(a - \delta_{0,g}, a\big) \cup \big(a, a + \delta_{0,g}\big)$.

(II) For every real number $\epsilon > 0$ there exists a real number $\delta_g(\epsilon)$ such that $0 < \delta_g(\epsilon) \leq \delta_{0,g}$ and such that

$$0 < |x - a| < \delta_g(\epsilon) \quad \Rightarrow \quad |g(x) - L| < \epsilon. \tag{2.23}$$

Since we have to produce $\delta_{0,g}, \delta_g(\epsilon)$ and we have to prove the last implication, all of these objects are red.

Notice that η_0 in the theorem is green.

The objective here is to use the green objects to produce the red objects. We will do that next. We put:

(I) $\delta_{0,g} = \eta_0$. By the assumption of the theorem $g(x)$ is defined for every x in the set $\big(a - \eta_0, a\big) \cup \big(a, a + \eta_0\big)$.

(II) For every real number $\epsilon > 0$, put

$$\delta_g(\epsilon) = \min\big\{\delta_f(\epsilon), \delta_h(\epsilon), \eta_0\big\}.$$

This is a beautiful expression since the red object is expressed in terms of the green objects.

It remains to prove the red implication (2.23) using the green implications and the assumptions of the theorem.

To prove (2.23), $\boxed{\text{assume that } 0 < |x - a| < \delta_g(\epsilon).}$ Then, clearly, $0 < |x - a| < \eta_0$. This is telling me that $x \neq a$ and that x is no further than η_0 from a. Consequently, $x \in \big(a - \eta_0, a\big) \cup \big(a, a + \eta_0\big)$. Therefore, by the assumption of the theorem

$$f(x) \leq g(x) \leq h(x).$$

Subtracting L from each term in this inequality, we conclude that

$$f(x) - L \leq g(x) - L \leq h(x) - L.$$

Using the property of the absolute value that $-|u| \leq u \leq |u|$ for every real number u, we conclude that

$$-\,|f(x) - L| \leq f(x) - L \leq g(x) - L \leq h(x) - L \leq |h(x) - L|. \tag{2.24}$$

From the assumption $0 < |x - a| < \delta_g(\epsilon)$, we conclude that $0 < |x - a| < \delta_f(\epsilon)$. By the green implication (2.21), this implies that $|f(x) - L| < \epsilon$ and therefore

$$-\,\epsilon < -|f(x) - L|. \tag{2.25}$$

From the assumption $0 < |x - a| < \delta_g(\epsilon)$, we conclude that $0 < |x - a| < \delta_h(\epsilon)$. By the green implication (2.22), this implies that

$$|h(x) - L| < \epsilon. \tag{2.26}$$

Putting together the inequalities (2.24), (2.25) and (2.26), we conclude that

$$-\epsilon < g(x) - L < \epsilon. \tag{2.27}$$

The inequalities in (2.27) are equivalent to

$$|g(x) - L| < \epsilon.$$

This proves that $0 < |x - a| < \delta_g(\epsilon)$ implies $|g(x) - L| < \epsilon$ and this is exactly the red implication (2.23). This completes the proof. □

The following theorem is the Scissors Squeeze Theorem.

THEOREM 2.35. *Let f, g and h be given functions and let $a \in \mathbb{R}$ and $L \in \mathbb{R}$. Assume that*

(1) $\lim\limits_{x \to a} f(x) = L$.
(2) $\lim\limits_{x \to a} h(x) = L$.
(3) *There exists $\eta_0 > 0$ such that f, g and h are defined on $\bigl(a - \eta_0, a\bigr) \cup \bigl(a, a + \eta_0\bigr)$ and*
$$f(x) \le g(x) \le h(x) \quad \textit{for all} \quad x \in \bigl(a - \eta_0, a\bigr),$$
and
$$h(x) \le g(x) \le f(x) \quad \textit{for all} \quad x \in \bigl(a, a + \eta_0\bigr).$$

Then

$$\lim_{x \to a} g(x) = L.$$

2.3.2. Four trigonometric limits. Figure 5 and the numbers that *you can see on it* are essential for getting squeezes for limits involving trigonometric functions. The table to the left of Figure 5 shows the numbers that you should be able to identify on the picture.

Geometric object	Associated number
Circular arc $\overset{\frown}{CB}$	u
Line segment $\overline{OA}$	$\cos u$
Line segment $\overline{AB}$	$\sin u$
Line segment $\overline{AC}$	$1 - \cos u$
Line segment $\overline{CB}$	You calculate
Line segment $\overline{CD}$	$\tan u$
Line segment $\overline{OB}$	1
Line segment $\overline{OC}$	1

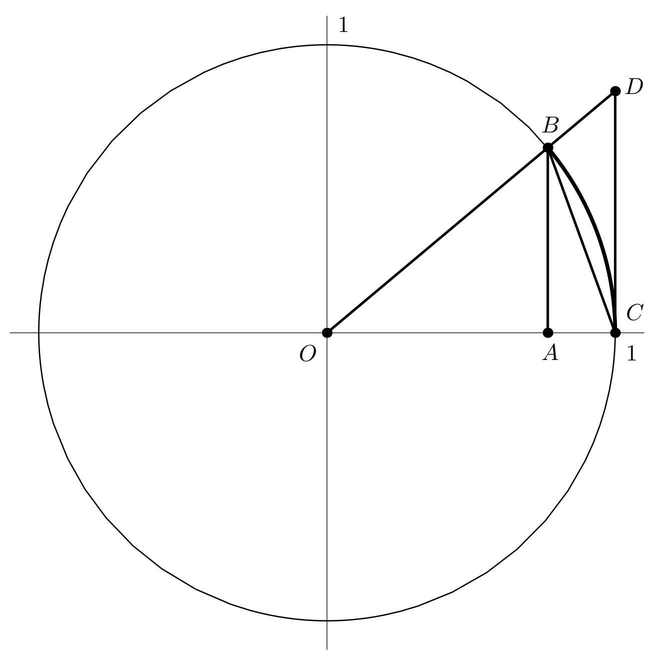

FIG. 5. The unit circle

EXAMPLE 2.36. Prove that $\lim_{x\to 0} \cos x = 1$.

SOLUTION. Set $\eta_0 = 1$. Consider a positive u. Look at Figure 5. The triangle $\triangle ACB$ is a right triangle. Therefore its hypothenuse, the line segment $\overline{CB}$, is longer than its side $\overline{AC}$, whose length equals to $1 - \cos u$. Thus

$$1 - \cos u = \overline{AC} \le \overline{CB}. \tag{2.28}$$

The line segment $\overline{CB}$ is a segment of a straight line, therefore it is shorter than any other curve joining C and B. In particular it is shorter than the circular arc $\overset{\frown}{CB}$ joining the points C and B. The length of the circular arc $\overset{\frown}{CB}$ is u. Thus

$$\overline{CB} \le \quad \overset{\frown}{CB} \ (= \ u \ ). \tag{2.29}$$

Putting together the inequalities (2.28) and (2.29), we conclude that

$$1 - \cos u \le u \qquad \text{for all} \qquad u \in \big(0, 1\big). \tag{2.30}$$

Since the length $\overline{OA} = \cos u$ is smaller than 1, from (2.30) we conclude that

$$0 \le 1 - \cos u \le u \qquad \text{for all} \qquad u \in \big(0, 1\big),$$

or, equivalently,

$$1 - u \le \cos u \le 1 \qquad \text{for all} \qquad u \in \big(0, 1\big),$$

Now we substitute $u = |x|$ and use the fact that $\cos(|x|) = \cos x$ and the preceding inequality becomes

$$1 - |x| \le \cos x \le 1 \qquad \text{for all} \qquad x \in (-1, 1)\,. \tag{2.31}$$

This is a sandwich squeeze for $\cos x$. It is easy to prove that $\lim_{x\to 0} 1 = 1$ and $\lim_{x\to 0}\big(1-|x|\big) = 1$. (Please prove this using the definition!) Now the Sandwich Squeeze Theorem implies that $\lim_{x\to 0} \cos x = 1$.

At the end of the proof here we used the Sandwich Squeeze Theorem. However, we could have also used the definition of limit. To use the definition, we observe that the implication

$$1 - |x| \le \cos x \le 1 \quad \Rightarrow \quad |\cos x - 1| \le |x| \tag{2.32}$$

is true and conclude that

$$\forall x \in (-1, 1) \quad \text{we have} \quad |\cos x - 1| \le |x|. \tag{2.33}$$

Now, for an arbitrary $\epsilon > 0$ we can prove the implication

$$0 < |x - 0| < \min\big\{\epsilon, 1\big\} \quad \Rightarrow \quad |\cos x - 1| < \epsilon.$$

This provides a proof of this limit by using the definition of limit. □

EXAMPLE 2.37. Prove that $\lim_{x\to 0} \sin x = 0$.

SOLUTION. Set $\delta_0 = 1$. Consider a positive u. Look at Figure 5. The triangle $\triangle ACB$ is a right triangle. Therefore its hypothenuse, the line segment $\overline{CB}$, is longer than its side $\overline{AB}$ which equals to $\sin u$. Thus

$$\sin u = \overline{AC} \le \overline{CB}.$$

As in Example 2.36 we have that $\overline{CB} < u$. Therefore,

$$\sin u \le u \qquad \text{for all} \qquad u \in \big(0, 1\big). \tag{2.34}$$

For $x \in \big(-1,1\big)$ we substitute $u = |x|$ and use the fact that $\sin(|x|) = |\sin x|$ and (2.34) becomes

$$|\sin x| \leq |x| \qquad \text{for all} \qquad x \in (-1,1)\,.$$

With the last inequality we use the definition of limit to finish the proof. For an arbitrary $\epsilon > 0$ we can prove the implication

$$0 < |x - 0| < \min\big\{\epsilon, 1\big\} \quad \Rightarrow \quad |\sin x - 0| < \epsilon.$$

□

Example 2.38. Prove that $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Solution. To get a sandwich squeeze for this problem consider the following three areas in Figure 5.

Area 1 The triangle $\triangle OCB$.

Area 2 The sector of the unit disk bounded by the line segments $\overline{CB}$ and $\overline{OB}$ and the circular arc $\overset{\frown}{CB}$ joining the points C and B.

Area 3 The triangle $\triangle OCD$.

The picture tells clearly the inequality between these areas. Write that inequality. Calculate each area in terms of the numbers that appear in the table above. This will lead to the inequality, which when simplified gives

$$\cos u \leq \frac{\sin u}{u} \leq 1 \qquad \text{for all} \qquad u \in \big(0,1\big). \tag{2.35}$$

Using the same idea as in the previous example, the inequality (2.35) leads to

$$\cos x \leq \frac{\sin x}{x} \leq 1 \qquad \text{for all} \qquad x \in (-1,0) \cup (0,1)\,. \tag{2.36}$$

The inequality (2.36) is exactly what we need in the Sandwich Squeeze Theorem. Please fill in all the details of the rest of the proof.

At the end of the proof here we used the Sandwich Squeeze Theorem. However, we could have also used the definition of limit. To use the definition we need one more inequality. We view inequality (2.36) as distances from 1 and conclude that the following inequality is true:

$$\left|\frac{\sin x}{x} - 1\right| \leq |\cos x - 1| \qquad \text{for all} \qquad x \in (-1,0) \cup (0,1)\,.$$

Now we use inequality (2.33) from Example 2.36 and transitivity of order to conclude

$$\left|\frac{\sin x}{x} - 1\right| \leq |x| \qquad \text{for all} \qquad x \in (-1,0) \cup (0,1)\,.$$

Now, for an arbitrary $\epsilon > 0$ we can prove the implication

$$0 < |x - 0| < \min\big\{\epsilon, 1\big\} \quad \Rightarrow \quad \left|\frac{\sin x}{x} - 1\right| < \epsilon.$$

And this proves the stated limit. □

Example 2.39. Prove that $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$.

Solution. To establish squeeze inequalities consider three lengths:

Length 1 The line segment $\overline{AB}$.

Length 2 The line segment $\overline{CB}$.

Length 3 The length of the circular arc $\overset{\frown}{CB}$ joining the points C and B.

The picture tells clearly the inequalities between these three lengths. Write these inequalities. Calculate each length in terms of the numbers that appear in the table above. This will lead to the inequalities, which, when simplified, give

$$\frac{1}{2}\left(\frac{\sin u}{u}\right)^2 \leq \frac{1-\cos u}{u^2} \leq \frac{1}{2} \qquad \text{for all} \qquad u \in \big(0,1\big). \tag{2.37}$$

As in the preceding three examples from inequality (2.37) we deduce

$$\frac{1}{2}\left(\frac{\sin x}{x}\right)^2 \leq \frac{1-\cos x}{x^2} \leq \frac{1}{2} \qquad \text{for all} \qquad x \in (-1,0)\cup(0,1)\,. \tag{2.38}$$

Next we recall two inequalities (2.31) and (2.36) to get

$$1-|x| \leq \cos x \leq \frac{\sin x}{x} \qquad \text{for all} \qquad x \in (-1,0)\cup(0,1)\,.$$

For $x \in (-1,0)\cup(0,1)$ we have $1-|x| \geq 0$, so we can square the first and the last term in the preceding inequalities to get

$$(1-|x|)^2 \leq \left(\frac{\sin x}{x}\right)^2 \qquad \text{for all} \qquad x \in (-1,0)\cup(0,1).$$

Finally, since $(1-|x|)^2 = 1-2|x|+|x|^2 \geq 1-2|x|$ we conclude

$$1-2|x| \leq \left(\frac{\sin x}{x}\right)^2 \qquad \text{for all} \qquad x \in (-1,0)\cup(0,1).$$

Substituting the last inequality in inequality (2.38) we get

$$\frac{1}{2}-|x| \leq \frac{1-\cos x}{x^2} \leq \frac{1}{2} \qquad \text{for all} \qquad x \in (-1,0)\cup(0,1).$$

This is a Sandwich Squeeze for the limit in this example. However, viewing the preceding inequality as distances from $1/2$ we deduce

$$\left|\frac{1-\cos x}{x^2}-\frac{1}{2}\right| \leq |x| \qquad \text{for all} \qquad x \in (-1,0)\cup(0,1).$$

And the last inequality can be used to prove the following implication: for an arbitrary $\epsilon > 0$ we have

$$0 < |x-0| < \min\big\{\epsilon, 1\big\} \quad \Rightarrow \quad \left|\frac{1-\cos x}{x^2}-\frac{1}{2}\right| < \epsilon.$$

proving the limit stated in this example. □

Example 2.40. Prove that $\displaystyle\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$.

Solution. The idea is to use the definition of $\ln$ as an integral and work with areas to get squeeze inequalities. □

2.3.3. Algebra of limits. A nickname that I gave to a function which has a limit L when x approaches a is: f is *constantish* L near a. If we are dealing with constant functions $f(x) = L$ and $g(x) = K$, then clearly the sum $f + g$ of these two functions is a constant function equal to $L+K$. The same is true for the product fg which is the constant function equal to LK. Another question is whether we can talk about the reciprocal $1/f$. If $L \neq 0$, then the reciprocal of f is defined and it equals $1/L$. In this section we will prove that all these properties hold for constantish functions.

THEOREM 2.41. *Let f, g, and h, be functions with domain and range in $\mathbb{R}$. Let a, K and L be real numbers. Assume that*

(1) $\lim\limits_{x\to a} f(x) = K$,
(2) $\lim\limits_{x\to a} g(x) = L$.

Then the following statements hold.

(A) *If $h = f + g$, then $\lim\limits_{x\to a} h(x) = K + L$.*
(B) *If $h = fg$, then $\lim\limits_{x\to a} h(x) = KL$.*
(C) *If $L \neq 0$ and $h = \dfrac{1}{g}$, then $\lim\limits_{x\to a} h(x) = \dfrac{1}{L}$.*
(D) *If $L \neq 0$ and $h = \dfrac{f}{g}$, then $\lim\limits_{x\to a} h(x) = \dfrac{K}{L}$.*

PROOF. The assumption $\lim\limits_{x\to a} f(x) = K$ implies that

green(I-f) There exists (green!) $\delta_{0,f} > 0$ such that $f(x)$ is defined for all x in $\bigl(a - \delta_{0,f}, a\bigr) \cup \bigl(a, a + \delta_{0,f}\bigr)$;
green(II-f) For every $\epsilon > 0$ there exists (green!) $\delta_f(\epsilon)$ such that $0 < \delta_f(\epsilon) \leq \delta_{0,f}$ and such that

$$0 < |x - a| < \delta_f(\epsilon) \quad \Rightarrow \quad |f(x) - K| < \epsilon. \tag{2.39}$$

The assumption $\lim\limits_{x\to a} g(x) = L$ implies that

green(I-g) There exists (green!) $\delta_{0,g} > 0$ such that $g(x)$ is defined for all x in $\bigl(a-\delta_{0,g}, a\bigr) \cup \bigl(a, a + \delta_{0,g}\bigr)$;
green(II-g) For every $\epsilon > 0$ there exists (green!) $\delta_g(\epsilon)$ such that $0 < \delta_g(\epsilon) \leq \delta_{0,g}$ and such that

$$0 < |x - a| < \delta_g(\epsilon) \quad \Rightarrow \quad |g(x) - L| < \epsilon. \tag{2.40}$$

Proof of the statement (A). Remember that $h(x) = f(x) + g(x)$ here. First we list what is red in this proof.

red(I-h) There exists (red!) $\delta_{0,h} > 0$ such that $h(x)$ is defined for all x in $\bigl(a - \delta_{0,h}, a\bigr) \cup \bigl(a, a + \delta_{0,h}\bigr)$;
red(II-h) For every $\epsilon > 0$ there exists (red!) $\delta_h(\epsilon)$ such that $0 < \delta_h(\epsilon) \leq \delta_{0,h}$ and such that

$$0 < |x - a| < \delta_h(\epsilon) \quad \Rightarrow \quad \bigl|h(x) - (K + L)\bigr| < \epsilon. \tag{2.41}$$

I will not elaborate here how I got the idea for $\delta_{0,h}$ and $\delta_h(\epsilon)$, I will just give formulas and convince you that my choice is a correct one. The idea for the formulas comes from the boxed paragraph on page 35. I invite you to enjoy the separation of colors in the following formulas.

Let $\epsilon > 0$ be given. Put

$$\delta_{0,h} = \min\{\delta_{0,f}, \delta_{0,g}\}$$
$$\delta_h(\epsilon) = \min\left\{\delta_f\left(\frac{\epsilon}{2}\right), \delta_g\left(\frac{\epsilon}{2}\right)\right\}$$

Now we have to prove that $h(x)$ is defined for every $x \in \big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$. Assume that $x \in \big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$. Then

$$0 < |x - a| < \delta_{0,h} \leq \min\{\delta_{0,f}, \delta_{0,g}\}. \tag{2.42}$$

It follows from (2.42) that

$$0 < |x - a| < \delta_{0,f},$$

and therefore $x \in \big(a - \delta_{0,f}, a\big) \cup \big(a, a + \delta_{0,f}\big)$. Thus $f(x)$ is defined. It also follows from (2.42) that

$$0 < |x - a| < \delta_{0,g},$$

and therefore $x \in \big(a - \delta_{0,g}, a\big) \cup \big(a, a + \delta_{0,g}\big)$. Thus $g(x)$ is defined. Therefore $h(x) = f(x) + g(x)$ is defined for every $x \in \big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$.

Now we will prove the red implication (2.41). Assume

$$0 < |x - a| < \delta_h(\epsilon) = \min\left\{\delta_f\left(\frac{\epsilon}{2}\right), \delta_g\left(\frac{\epsilon}{2}\right)\right\}. \tag{2.43}$$

Then

$$0 < |x - a| < \delta_f\left(\frac{\epsilon}{2}\right). \tag{2.44}$$

The inequality (2.44) and the implication (2.39) allow me to conclude that

$$|f(x) - K| < \frac{\epsilon}{2}. \tag{2.45}$$

It follows from (2.43) that

$$0 < |x - a| < \delta_g\left(\frac{\epsilon}{2}\right). \tag{2.46}$$

The inequality (2.46) and the implication (2.40) allow me to conclude that

$$|g(x) - L| < \frac{\epsilon}{2}. \tag{2.47}$$

Now we remember that the absolute value has the property that $|u + v| \leq |u| + |v|$. We will apply this to the expression

$$|h(x) - (K + L)| = |f(x) + g(x) - K - L| = |\underbrace{(f(x) - K)}_{u} + \underbrace{(g(x) - L)}_{v}|$$

to get

$$|h(x) - (K + L)| \leq |f(x) - K| + |g(x) - L|. \tag{2.48}$$

This inequality plays a role of a BIN in this abstract proof. It has an unfriendly object on the left and all friendly objects on the right.

The inequalities (2.45), (2.47) and (2.48) imply that

$$|h(x) - (K + L)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{2.49}$$

Reviewing my reasoning above you should be convinced that based on the assumption (2.43) we proved the inequality (2.49). This is exactly the implication (2.41). This completes the proof of the statement (A).

Proof of the statement (B). Remember that $h(x) = f(x)g(x)$ here. We first list what is red in this proof.

red(I-h) There exists (red!) $\delta_{0,h} > 0$ such that $h(x)$ is defined for all x in $\big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$;

red(II-h) For every $\epsilon > 0$ there exists (red!) $\delta_h(\epsilon)$ such that $0 < \delta_h(\epsilon) \le \delta_{0,h}$ and such that

$$0 < |x - a| < \delta_h(\epsilon) \qquad \Rightarrow \qquad |h(x) - KL| < \epsilon. \tag{2.50}$$

I will not elaborate how I got the idea for $\delta_{0,h}$ and $\delta_h(\epsilon)$, I will just give formulas and convince you that my choice is a correct one. The idea for the formulas comes from the boxed paragraph on page 37. Again, I invite you to enjoy the separation of colors in the following formulas.

Let $\epsilon > 0$ be given. Put

$$\begin{aligned} \delta_{0,h} &= \min\left\{\delta_{0,f}, \delta_g(1)\right\} \\ \delta_h(\epsilon) &= \min\left\{\delta_f\left(\frac{\epsilon}{2(|L|+1)}\right), \delta_g\left(\frac{\epsilon}{2(|K|+1)}\right)\right\}. \end{aligned}$$

Now we have to prove that $h(x)$ is defined for every $x \in \big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$. Assume that $x \in \big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$. Then

$$0 < |x - a| < \delta_{0,h} \le \min\left\{\delta_{0,f}, \delta_g(1)\right\}. \tag{2.51}$$

It follows from (2.51) that

$$0 < |x - a| < \delta_{0,f}\ ,$$

and therefore $x \in \big(a - \delta_{0,f}, a\big) \cup \big(a, a + \delta_{0,f}\big)$. Thus $f(x)$ is defined. It also follows from (2.51) that

$$0 < |x - a| < \delta_g(1). \tag{2.52}$$

Since by the assumption (II-g) we know that $\delta_g(1) \le \delta_{0,g}$, the inequality (2.52) implies that

$$0 < |x - a| < \delta_{0,g}.$$

Therefore $x \in \big(a - \delta_{0,g}, a\big) \cup \big(a, a + \delta_{0,g}\big)$. Thus $g(x)$ is defined. Therefore $h(x) = f(x)g(x)$ is defined for every $x \in \big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$.

At this point we will prove another consequence of the inequality (2.52). This inequality and the implication (2.40) allow me to conclude that

$$|g(x) - L| < 1.$$

Therefore

$$-1 < g(x) - L < 1\ ,$$

or, equivalently

$$-1 + L < g(x) < L + 1.$$

Multiplying the last inequality by -1, we conclude that

$$-1 - L < -g(x) < -L + 1.$$

From the last two inequalities we conclude that $\max\{g(x), -g(x)\} < \max\{L+1, -L+1\} = \max\{L, -L\} + 1$. Thus

$$|g(x)| < |L| + 1. \tag{2.53}$$

Now we will prove the red implication (2.50). Assume

$$0 < |x-a| < \delta_h(\epsilon) = \min\left\{\delta_f\left(\frac{\epsilon}{2(|L|+1)}\right), \delta_g\left(\frac{\epsilon}{2(|K|+1)}\right)\right\}. \tag{2.54}$$

Then

$$0 < |x-a| < \delta_f\left(\frac{\epsilon}{2(|L|+1)}\right). \tag{2.55}$$

The inequality (2.55) and the implication (2.39) allow me to conclude that

$$|f(x)-K| < \frac{\epsilon}{2(|L|+1)}. \tag{2.56}$$

It follows from (2.54) that

$$0 < |x-a| < \delta_g\left(\frac{\epsilon}{2(|K|+1)}\right). \tag{2.57}$$

The inequality (2.57) and the implication (2.40) allow me to conclude that

$$|g(x)-L| < \frac{\epsilon}{2(|K|+1)}. \tag{2.58}$$

Now we remember that the absolute value has the property that $|u+v| \leq |u|+|v|$ and that $|uv| = |u||v|$. we will apply these properties to the expression

$$\begin{aligned}|h(x)-KL| = \bigl|f(x)g(x)-KL\bigr| &= \bigl|\underbrace{\bigl(f(x)g(x)-Kg(x)\bigr)}_{u} + \underbrace{\bigl(Kg(x)-KL\bigr)}_{v}\bigr| \\ &\leq \bigl|f(x)g(x)-Kg(x))\bigr| + \bigl|Kg(x)-KL\bigr| \\ &\leq \bigl|g(x)\bigr|\,\bigl|f(x)-K\bigr| + |K|\,\bigl|g(x)-L\bigr|.\end{aligned}$$

Summarizing

$$\bigl|h(x)-KL\bigr| \leq \bigl|g(x)\bigr|\,\bigl|f(x)-K\bigr| + |K|\,\bigl|g(x)-L\bigr|. \tag{2.59}$$

The inequalities (2.53) and (2.59) imply that

$$\bigl|h(x)-KL\bigr| \leq \bigl(|L|+1\bigr)\,\bigl|f(x)-K\bigr| + |K|\,\bigl|g(x)-L\bigr|. \tag{2.60}$$

This inequality plays a role of a BIN in this abstract proof. It has an unfriendly object on the left and all friendly objects on the right.

The inequalities (2.56), (2.58) and (2.60) imply that

$$|h(x)-LK| \leq \bigl(|L|+1\bigr)\frac{\epsilon}{2(|L|+1)} + |K|\frac{\epsilon}{2(|K|+1)} < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon. \tag{2.61}$$

I hope that my reasoning above convinces you that the assumption (2.54) implies the inequality (2.61). This is exactly the implication (2.50). This completes the proof of the part (B).

Proof of the statement (C). Here we assume that $L \neq 0$ and $h(x) = \dfrac{1}{g(x)}$. Next we list what is red in this proof.

red(I-h) There exists (red!) $\delta_{0,h} > 0$ such that $h(x)$ is defined for all x in $\bigl(a-\delta_{0,h}, a\bigr) \cup \bigl(a, a+\delta_{0,h}\bigr)$;

red(II-h) For every $\epsilon > 0$ there exists (red!) $\delta_h(\epsilon)$ such that $0 < \delta_h(\epsilon) \leq \delta_{0,h}$ and such that

$$0 < |x - a| < \delta_h(\epsilon) \quad \Rightarrow \quad \left|\frac{1}{g(x)} - \frac{1}{L}\right| < \epsilon. \tag{2.62}$$

I will not elaborate how I got the idea for $\delta_{0,h}$ and $\delta_h(\epsilon)$, I will just give formulas and convince you that my choice is a correct one. The idea for the formulas comes from the boxed paragraph on page 39. Again, I invite you to enjoy the separation of colors in the following formulas.

Let $\epsilon > 0$ be given. Remember that it is assumed that $|L| > 0$. Put

$$\delta_{0,h} = \delta_g\!\left(\frac{|L|}{2}\right)$$

$$\delta_h(\epsilon) = \min\left\{\delta_g\!\left(\frac{\epsilon L^2}{2}\right), \delta_g\!\left(\frac{|L|}{2}\right)\right\}.$$

Now we have to prove that $h(x)$ is defined for every $x \in \big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$. Assume that $x \in \big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$. Then

$$0 < |x - a| < \delta_{0,h} = \delta_g\!\left(\frac{|L|}{2}\right).$$

This inequality and the implication (2.40) allow me to conclude that

$$|g(x) - L| < \frac{|L|}{2}.$$

Therefore

$$-\frac{|L|}{2} < g(x) - L < \frac{|L|}{2},$$

or, equivalently

$$-\frac{|L|}{2} + L < g(x) < L + \frac{|L|}{2}.$$

Multiplying the last inequality by -1, we conclude that

$$-L - \frac{|L|}{2} < -g(x) < \frac{|L|}{2} - L.$$

From the last two displayed relationships we conclude that

$$\max\{g(x), -g(x)\} > \max\left\{L - \frac{|L|}{2}, -L - \frac{|L|}{2}\right\} = \max\{L, -L\} - \frac{|L|}{2}.$$

Thus

$$|g(x)| > |L| - \frac{|L|}{2} = \frac{|L|}{2} > 0. \tag{2.63}$$

Consequently, $g(x) \neq 0$. Therefore, $h(x) = \dfrac{1}{g(x)}$ is defined for all $x \in \big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$.

Now we will prove the red implication (2.62). Assume

$$0 < |x - a| < \delta_h(\epsilon) = \min\left\{\delta_g\!\left(\frac{\epsilon L^2}{2}\right), \delta_g\!\left(\frac{|L|}{2}\right)\right\}. \tag{2.64}$$

Then

$$0 < |x - a| < \delta_g\!\left(\frac{\epsilon L^2}{2}\right). \tag{2.65}$$

The inequality (2.65) and the implication (2.40) allow me to conclude that

$$|g(x) - L| < \frac{\epsilon L^2}{2}. \tag{2.66}$$

It also follows from (2.64) that

$$0 < |x - a| < \delta_g\left(\frac{|L|}{2}\right).$$

We already proved that this inequality implies (2.63). Therefore

$$\frac{1}{|g(x)|} < \frac{2}{|L|}. \tag{2.67}$$

This inequality is used at the last step in the sequence of inequalities below. In some sense this is an abstract version of a "pizza-party" play.

Using our standard tools, algebra, properties of the absolute value and the inequality (2.67) we get

$$\begin{aligned}\left|h(x) - \frac{1}{L}\right| &= \left|\frac{1}{g(x)} - \frac{1}{L}\right| = \left|\frac{L - g(x)}{g(x)L}\right| = \frac{|L - g(x)|}{|g(x)|\ |L|} \\ &= \frac{|g(x) - L|}{|g(x)|\ |L|} \leq \frac{1}{|g(x)|}\ \frac{|g(x) - L|}{|L|} \leq \frac{2}{|L|}\ \frac{|g(x) - L|}{|L|}.\end{aligned}$$

Summarizing

$$\left|\frac{1}{g(x)} - \frac{1}{L}\right| \leq \frac{2}{L^2}\ |g(x) - L|\,. \tag{2.68}$$

This inequality plays a role of a BIN in this abstract proof. It has an unfriendly object on the left and all friendly objects on the right.

The inequalities (2.66) and (2.68) imply that

$$\left|\frac{1}{g(x)} - \frac{1}{L}\right| \leq \frac{2}{L^2}\ \frac{\epsilon L^2}{2} = \epsilon. \tag{2.69}$$

I hope that the reasoning above convinces you that the assumption (2.64) implies the inequality (2.69). This is exactly the implication (2.62). This completes the proof of the part (C).

Proof of the statement (D). Here we assume that $L \neq 0$ and $h(x) = \dfrac{f(x)}{g(x)}$. We can prove the statement (D) by using the universal power of the statements (B) and (C). First define the functions $g_1(x) = \dfrac{1}{g(x)}$. Then, by the statement (C) we know

$$\lim_{x\to a} g_1(x) = \frac{1}{L}. \tag{2.70}$$

Clearly, $h(x) = f(x)g_1(x)$. Now we can apply the statement (B) to this function h. Taking into account (2.70) the statement (B) implies

$$\lim_{x\to a} h(x) = K\,\frac{1}{L} = \frac{K}{L}.$$

This completes the proof of the statement (D). The theorem is proved. □

EXERCISE 2.42. Use the algebra of limits to give much simpler proofs for most of the limits in the previous exercises and examples.

2.4. Continuous functions

2.4.1. The definition and examples. All this work about limits will now pay off since we will be able to give mathematically rigorous definition of a continuous function.

DEFINITION 2.43. Let D be a nonempty subset of $\mathbb{R}$. A function $f : D \to \mathbb{R}$ is continuous at c if the following two conditions are satisfied:

(i) The function f is defined at c, that is $c \in D$.
(ii) $\lim\limits_{x \to a} f(x) = f(c)$.

To understand Definition 2.43 the reader needs to understand the concept of limit. Since the concept of continuity is fundamental in mathematics it is important to understand the definition of continuity directly, without appealing to the concept of limit.

DEFINITION 2.44. Let D be a nonempty subset of $\mathbb{R}$. A function $f : D \to \mathbb{R}$ is continuous at c if the following two conditions are satisfied:

(I) There exists a $\delta_0 > 0$ such that $(a - \delta_0, a + \delta_0) \subseteq D$.
(II) For every $\epsilon > 0$ there exists $\delta(\epsilon)$ such that $0 < \delta(\epsilon) \leq \delta_0$ and such that

$$|x - c| < \delta(\epsilon) \quad \Rightarrow \quad |f(x) - f(c)| < \epsilon.$$

Definition 2.44 is called ϵ-δ definition of continuity. (The symbol ϵ-δ is read "epsilon-delta.")

DEFINITION 2.45. Let D be a nonempty subset of $\mathbb{R}$. A function $f : D \to \mathbb{R}$ is *continuous on* D if it is continuous at each point in D.

A drawback of the Definition 2.45, together with Definition 2.44, is that it does not apply to functions that are defined on closed intervals. For example, we cannot use Definition 2.44, to prove that the square root function, that is defined on $D = [0, +\infty)$, is continuous at $c = 0$. Why? Since for the square root function to be continuous at $c = 0$, Definition 2.44 requires that there exists $\delta_0 > 0$ such that

$$(0 - \delta_0, 0 + \delta_0) = (-\delta_0, \delta_0) \subseteq [0, +\infty).$$

Such $\delta_0 > 0$ does not exist. So, in the sense of Definition 2.44, the square root function is not continuous at $c = 0$. Since our intuitive sense of continuity expects that the square root function is not continuous at $c = 0$, the above definition needs to be modified. A modification is presented as Definition 2.46 below.

Notice that Definition 2.46 requires that a function is defined on an interval of real numbers. Before stating Definition 2.46 we review nine kinds of intervals of real numbers that one can encounter.

Recall that there are four kinds of finite intervals; with $a, b \in \mathbb{R}$ and $a < b$, the finite intervals are:

$$(a, b), \quad (a, b], \quad [a, b), \quad [a, b].$$

There are four kinds of infinite intervals; with $a \in \mathbb{R}$, the infinite intervals are:

$$(a, +\infty), \quad [a, +\infty), \quad (-\infty, a), \quad (-\infty, a];$$

and also $\mathbb{R}$ is an infinite interval, sometimes written as $(-\infty, +\infty)$.

DEFINITION 2.46. Let $D \subseteq \mathbb{R}$ be an interval. A function $f : D \to \mathbb{R}$ is continuous on D if the following condition is satisfied:

$$\forall c \in D \quad \forall \epsilon > 0 \quad \exists \delta(\epsilon, c) > 0 \quad \text{such that} \quad \forall\, x \in D \quad \text{we have} \quad |x - c| < \delta(\epsilon, c) \quad \Rightarrow \quad |f(x) - f(c)| < \epsilon.$$

Definition 2.46 is also an ϵ-δ definition of continuity. The advantage of Definition 2.46 is that it defines continuity of a function defined on an interval in one statement, not two statements as in Definitions 2.44 and 2.45. If you are working with a function which is defined on an open interval you can use either of the definitions. In fact, these two definitions are equivalent if a function is defined on an open interval. However, in most cases doing a proof using Definition 2.46 might be somewhat easier. It is a prudent proof strategy to always have in mind both definitions. Then, when writing the final proof you write the proof which will satisfy Definition 2.46. A good example of this strategy is Example 2.49 below.

EXAMPLE 2.47. Let K be a real number and define $f(x) = K$ for all $x \in \mathbb{R}$. Use Definition 2.46 to prove that f is continuous on $\mathbb{R}$.

EXAMPLE 2.48. Let $f(x) = x$ for all $x \in \mathbb{R}$. Use Definition 2.46 to prove that f is continuous on $\mathbb{R}$.

EXAMPLE 2.49. Use ϵ-δ definition of continuity, that is Definition 2.46, to prove that the function $f(x) = 1/x$ is continuous on the interval $(0, +\infty)$.

SOLUTION. It is interesting that we start this proof as if we are using Definition 2.44. Then, after we find $\delta(\epsilon, c) > 0$, we do the final proof which proves the statement in Definition 2.46.

Let $c \in (0, +\infty)$, that is let c be an arbitrary positive number. Chose $\delta_0 = c/2$. Since $c > 0$, we conclude that $c/2 > 0$ and $f(x) = 1/x$ is defined for all $x \in \bigl(c/2, 3c/2\bigr)$.

Let $\epsilon > 0$ be arbitrary. Now we have to solve

$$\left|\frac{1}{x} - \frac{1}{c}\right| < \epsilon \quad \text{for} \quad |x - c|.$$

First simplify the expression, using the fact that $x > 0$ and $c > 0$ and rules for the absolute value:

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{c - x}{x\,c}\right| = \frac{|c - x|}{|x|\,|c|} = \frac{|x - c|}{x\,c}.$$

To get a larger expression which will be easy to solve we replace x in the denominator by the smallest possible value for x. That value is $c - c/2 = c/2$. This gives me my BIN:

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x - c|}{x\,c} \leq \frac{|x - c|}{\dfrac{c}{2}\,c} = 2\,\frac{|x - c|}{c^2}.$$

Thus my BIN is $\displaystyle \left|\frac{1}{x} - \frac{1}{c}\right| \leq \frac{2}{c^2}\,|x - c|$ valid for all $x \in \bigl(c/2, 3c/2\bigr)$.

Next we solve the inequality $\dfrac{2}{c^2}\,|x - c| < \epsilon$ for $|x - c|$. Since $c > 0$ we have

$$\frac{2}{c^2}\,|x - c| < \epsilon \quad \Leftrightarrow \quad |x - c| < \frac{c^2\,\epsilon}{2}.$$

We are ready to define

$$\delta(\epsilon) = \min\left\{\frac{c^2\,\epsilon}{2}, \frac{c}{2}\right\}.$$

To finish the proof, it remains to prove the implication

$$\forall c > 0 \;\; \forall x > 0 \qquad |x - c| < \min\left\{\frac{c^2\,\epsilon}{2}, \frac{c}{2}\right\} \quad \Rightarrow \quad \left|\frac{1}{x} - \frac{1}{c}\right| < \epsilon.$$

Using the BIN and the preceding displayed equivalence you can prove this implication as an exercise. □

EXAMPLE 2.50. Use ϵ-δ definition of continuity, that is Definition 2.46, to prove that the function $x \mapsto \sqrt{x}$ is continuous on the interval $(0, +\infty)$.

SOLUTION. It is interesting that we start this proof as if we are using Definition 2.44. Then, after we find $\delta(\epsilon, c) > 0$, we do the final proof which proves the statement in Definition 2.46.

Let $c \in (0, +\infty)$. Chose $\delta_0 = \dfrac{c}{2}$. Since $c > 0$, as before we conclude that $\dfrac{c}{2} > 0$ and the function $x \mapsto \sqrt{x}$ is defined for all $x \in (c/2, 3c/2)$.

Let $\epsilon > 0$ be arbitrary. Now we have to solve

$$\left|\sqrt{x} - \sqrt{c}\right| < \epsilon \quad \text{for} \quad |x - c|.$$

First simplify algebraically the expression, using the fact that $x > 0$ and $c > 0$ and rules for the absolute value and the Pizza-Party to get:

$$\begin{aligned}\left|\sqrt{x} - \sqrt{c}\right| &= \left|\left(\sqrt{x} - \sqrt{c}\right)\,\frac{1}{1}\right| = \left|\left(\sqrt{x} - \sqrt{c}\right)\,\frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}}\right| = \left|\frac{x - c}{\sqrt{x} + \sqrt{c}}\right| \\ &= \frac{|x - c|}{\left|\sqrt{x} + \sqrt{c}\right|} = \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \leq \frac{|x - c|}{\sqrt{c}}\end{aligned}$$

Thus the BIN is: $\quad \left|\sqrt{x} - \sqrt{c}\right| \leq \dfrac{|x - c|}{\sqrt{c}}$, valid for all $x > 0$ and all $c > 0$.

Next we solve $\dfrac{|x - c|}{\sqrt{c}} < \epsilon$ for $|x - c|$. The solution is the following equivalence: Since $c > 0$ we have

$$\frac{|x - c|}{\sqrt{c}} < \epsilon \quad \Leftrightarrow \quad |x - c| < \sqrt{c}\,\epsilon. \tag{2.71}$$

Since the BIN is valid for all $c > 0$ we can define

$$\delta(\epsilon) = \sqrt{c}\,\epsilon.$$

It remains to prove the implication

$$\forall c > 0 \;\; \forall x > 0 \qquad |x - c| < \sqrt{c}\,\epsilon \quad \Rightarrow \quad \left|\sqrt{x} - \sqrt{c}\right| < \epsilon.$$

As usual, this is done using the BIN and the equivalence in (2.71). Let $c > 0$ and $x > 0$ be arbitrary. Assume that $|x - c| < \sqrt{c}\,\epsilon$. By the equivalence in (2.71) we deduce that $\dfrac{|x - c|}{\sqrt{c}} < \epsilon$ holds. By the BIN we have $\left|\sqrt{x} - \sqrt{c}\right| \leq \dfrac{|x - c|}{\sqrt{c}}$. By the transitivity of order from the last two inequalities we deduce that $\left|\sqrt{x} - \sqrt{c}\right| < \epsilon$. □

EXAMPLE 2.51. Let $f(x) = \dfrac{1}{x^2 + 1}$ for all $x \in \mathbb{R}$. Use ϵ-δ definition to prove that f is continuous on its domain.

EXAMPLE 2.52. Let a, b, c be any real numbers. Let $f(x) = ax^2 + bx + c$ for all $x \in \mathbb{R}$. Let v be an arbitrary real number. Prove that f is continuous at v.

EXAMPLE 2.53. Let $f(x) = \sin x$ for all $x \in \mathbb{R}$. Prove that f is continuous at an arbitrary real number a.

EXAMPLE 2.54. Let $f(x) = \cos x$ for all $x \in \mathbb{R}$. Prove that f is continuous at an arbitrary real number a.

HINT FOR Exercises 2.53 and 2.54. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be two points in the xy-plane. Then the length of the line segment $\overline{AB}$ is given by

$$\overline{AB} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Consequently

$$|x_1 - x_2| \leq \overline{AB} \qquad \text{and} \qquad |y_1 - y_2| \leq \overline{AB}.$$

Let u and v be real numbers and set $A = (\cos u, \sin u)$, $B = (\cos v, \sin v)$. The last displayed inequalities now imply

$$|\cos u - \cos v| \leq \overline{AB} \qquad \text{and} \qquad |\sin u - \sin v| \leq \overline{AB}.$$

Recall that the points A and B are on the unit circle. Any two points on the unit circle determine two arcs. Denote by $\widehat{AB}$ the length of the shorter circular arc determined by A and B. Since the shortest path between two points is a straight line we have that $\overline{AB} < \widehat{AB}$. How is the arc length $\widehat{AB}$ related to the numbers u and v? First, if $|u - v| \leq \pi$, then $\widehat{AB} = |u - v|$. Second, if $|u - v| > \pi$, then $\widehat{AB} \leq \pi < |u - v|$. Hence in each case $\widehat{AB} \leq |u - v|$. Thus we have established inequalities

$$|\cos u - \cos v| \leq \overline{AB} \leq \widehat{AB} \leq |u - v|,$$

$$|\sin u - \sin v| \leq \overline{AB} \leq \widehat{AB} \leq |u - v|,$$

for arbitrary real numbers u and v. These inequalities can be used to solve Exercises 2.53 and 2.54. THE END OF THE HINT.

EXAMPLE 2.55. Let $f(x) = \ln x$ for all $x \in (0, +\infty)$. Prove that f is continuous on its domain.

SOLUTION. First we recall the inequality

$$1 - \frac{1}{v} \leq \ln v \leq v - 1 \qquad \text{valid for all} \qquad v > 0, \tag{2.72}$$

which we proved using the integral definition of ln.

An inequality for $|\ln v|$ will be useful in the proof of the continuity below. Such an inequality can be obtained from the inequality in (2.72) by considering two cases:

$$\begin{aligned} |\ln v| &\leq \left\{ \begin{array}{ll} v - 1 & \text{if} \quad 1 \leq v \\ -\left(1 - \frac{1}{v}\right) & \text{if} \quad 0 < v < 1 \end{array} \right\} \\ &= \left\{ \begin{array}{ll} v - 1 & \text{if} \quad 1 \leq v \\ -\frac{v-1}{v} & \text{if} \quad 0 < v < 1 \end{array} \right\} \\ &= \left\{ \begin{array}{ll} |v - 1| & \text{if} \quad 1 \leq v \\ \frac{|v-1|}{v} & \text{if} \quad 0 < v < 1 \end{array} \right\}. \end{aligned}$$

Next we will restrict v to the interval $(1/2, 3/2)$. That is we assume $v \in (1/2, 3/2)$. Then we have that $|v-1|/v \leq 2|v-1|$. Since always $|v-1| \leq 2|v-1|$, we have that

$$|\ln v| \leq 2|v-1| \qquad \text{is valid for all} \qquad v \in (1/2, 3/2). \tag{2.73}$$

Let $a > 0$ be arbitrary. Let $x \in (a/2, 3a/2)$. Then $x/a \in (1/2, 3/2)$ and we can simplify the expression $|\ln x - \ln a|$ which appears in the definition of continuity. In the next sequence of inequalities we first use a property of logarithm, then the inequality in (2.73) and simple algebra to get:

$$\begin{aligned} |\ln x - \ln a| &= \left|\ln \frac{x}{a}\right| \\ &\leq 2\left|\frac{x}{a} - 1\right| \\ &= 2\left|\frac{x-a}{a}\right| \\ &= 2\frac{|x-a|}{a} \\ &= \frac{2}{a}\,|x-a|. \end{aligned}$$

Thus, we proved that

$$|\ln x - \ln a| \leq \frac{2}{a}\,|x-a| \qquad \text{is valid for all} \qquad x \in (a/2, 3a/2). \tag{2.74}$$

To finish the proof of continuity let $\epsilon > 0$ be arbitrary and set

$$\delta(\epsilon) = \min\left\{\frac{a\epsilon}{2}, \frac{a}{2}\right\}.$$

Clearly $\delta(\epsilon) > 0$. Next we will prove the implication

$$|x-a| < \min\left\{\frac{a\epsilon}{2}, \frac{a}{2}\right\} \qquad \Rightarrow \qquad |\ln x - \ln a| < \epsilon.$$

Assume $|x-a| < \min\left\{\frac{a\epsilon}{2}, \frac{a}{2}\right\}$. Then $|x-a| < \frac{a\epsilon}{2}$ and $|x-a| < \frac{a}{2}$. Since $|x-a| < \frac{a}{2}$, we have $x \in (a/2, 3a/2)$ and therefore, by (2.74), we have

$$|\ln x - \ln a| \leq \frac{2}{a}\,|x-a|.$$

Since $|x-a| < \frac{a\epsilon}{2}$ we have

$$\frac{2}{a}\,|x-a| < \epsilon.$$

The last two displayed inequalities yield

$$|\ln x - \ln a| < \epsilon.$$

This completes the proof of the continuity of the logarithm function ln. ☐

Example 2.56. Let $f(x) = e^x$ for all $x \in \mathbb{R}$. Prove that f is continuous at an arbitrary real number a.

Solution. We first substitute $v = \exp u = e^u$ in (2.72) to get

$$1 - \frac{1}{e^u} \leq \ln e^u \leq e^u - 1 \qquad \text{is valid for all} \qquad u \in \mathbb{R}.$$

Simplifying we get

$$1 - \frac{1}{e^u} \leq u \leq e^u - 1.$$

We need a squeeze for e^u. Above we already have one side of the squeeze. That is $u+1 \leq e^u$. To get the other side we transform

$$1 - \frac{1}{e^u} \leq u$$

to

$$1 - u \leq \frac{1}{e^u}.$$

To get a useful inequality we need to take the reciprocals in the last inequality. For that we need $1 - u > 0$. That is we need to assume that $u < 1$. Assuming that $u < 1$ we have

$$e^u \leq \frac{1}{1-u}.$$

Together with $u + 1 \leq e^u$, we proved that

$$u + 1 \leq e^u \leq \frac{1}{1-u} \qquad \text{is valid for all} \qquad u < 1. \tag{2.75}$$

An inequality for $|e^u - 1|$ will be useful in the proof of the continuity below. The inequalities in (2.75) yield that

$$u \leq e^u - 1 \leq \frac{u}{1-u} \qquad \text{is valid for all} \qquad u < 1.$$

To get an inequality for $|e^u - 1|$ we consider two cases:

$$\begin{aligned} |e^u - 1| &\leq \left\{ \begin{array}{ll} \frac{u}{1-u} & \text{if} \quad 0 \leq u < 1 \\ -u & \text{if} \quad u < 0 \end{array} \right\} \\ &= \left\{ \begin{array}{ll} \frac{|u|}{1-u} & \text{if} \quad 0 \leq u < 1 \\ |u| & \text{if} \quad u < 0 \end{array} \right\} \end{aligned}$$

Next we will restrict u to the interval $(-1/2, 1/2)$. That is we assume $u \in (-1/2, 1/2)$. Then we have that $|u|/(1-u) \leq 2|u|$. Since always $|u| \leq 2|u|$, we have that

$$|e^u - 1| \leq 2|u| \qquad \text{is valid for all} \qquad u \in (-1/2, 1/2). \tag{2.76}$$

Let $a > 0$ be arbitrary. Let $x \in (a - 1/2, a + 1/2)$. Then $x - a \in (-1/2, 1/2)$ and we can simplify the expression $|e^x - e^a|$ which appears in the definition of continuity. For that we use a property of the exponential function and (2.76) to get:

$$|e^x - e^a| = e^a |e^{(x-a)} - 1| \leq 2e^a |x - a|.$$

Thus, we proved that

$$|e^x - e^a| \leq 2e^a|x - a| \qquad \text{is valid for all} \qquad x \in (a - 1/2, a + 1/2). \tag{2.77}$$

To finish the proof of the continuity let $\epsilon > 0$ be arbitrary and set

$$\delta(\epsilon) = \min\left\{\frac{\epsilon}{2e^a}, \frac{1}{2}\right\}.$$

Clearly $\delta(\epsilon) > 0$.

Next we will prove the implication

$$|x - a| < \min\left\{\frac{\epsilon}{2e^a}, \frac{1}{2}\right\} \qquad \Rightarrow \qquad |e^x - e^a| < \epsilon.$$

Assume $|x-a| < \min\left\{\frac{\epsilon}{2e^a}, \frac{1}{2}\right\}$. Then $|x-a| < \frac{\epsilon}{2e^a}$ and $|x-a| < \frac{1}{2}$. Since $|x-a| < \frac{1}{2}$, we have $x \in (a-1/2, a+1/2)$ and therefore, by (2.77), we have

$$|e^x - e^a| \leq 2e^a|x-a|.$$

Since $|x-a| < \frac{\epsilon}{2e^a}$ we have

$$2e^a|x-a| < \epsilon.$$

The last two displayed inequalities yield

$$|e^x - e^a| < \epsilon.$$

This completes the proof of the continuity of of the exponential function exp. □

2.4.2. General theorems about continuous functions. The next theorem can be deduced from Theorem 2.41.

THEOREM 2.57 (Algebra of Continuous Functions). *Let f and g be functions and let a be a real number. Assume that f and g are continuous at the point a.*

(a) *If $h = f + g$, then h is continuous at a.*
(b) *If $h = fg$, then h is continuous at a.*
(c) *If $h = \dfrac{f}{g}$ and $g(a) \neq 0$, then h is continuous at a.*

EXAMPLE 2.58. Let $f(x) = \tan x$ for all $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$. Prove that f is continuous at an arbitrary real number a such that $-\dfrac{\pi}{2} < a < \dfrac{\pi}{2}$.

SOLUTION. Use the algebra of continuous functions. □

The following theorem states that a composition of continuous functions is continuous.

THEOREM 2.59. *Let f and g be functions and let a be a real number. Assume that g is continuous at a and that f is continuous at $g(a)$. If $h = f \circ g$, then h is continuous at a.*

PROOF. Assume that the function g is continuous at a. That is assume

(I-g) There exists a $\delta_{0,g} > 0$ such that $g(x)$ is defined for all $x \in (a - \delta_{0,g}, a + \delta_{0,g})$.
(II-g) For every $\epsilon > 0$ there exists $\delta_g(\epsilon)$ such that $0 < \delta_g(\epsilon) \leq \delta_{0,g}$ and such that

$$|x-a| < \delta_g(\epsilon) \quad \Rightarrow \quad |g(x) - g(a)| < \epsilon.$$

Also assume that the function f is continuous at $g(a)$. That is assume

(I-f) There exists a $\delta_{0,f} > 0$ such that $f(x)$ is defined for all $x \in \big(g(a) - \delta_{0,f}, g(a) + \delta_{0,f}\big)$.
(II-f) For every $\epsilon > 0$ there exists $\delta_g(\epsilon)$ such that $0 < \delta_f(\epsilon) \leq \delta_{0,f}$ and such that

$$|u - g(a)| < \delta_f(\epsilon) \quad \Rightarrow \quad |f(u) - f(g(a))| < \epsilon.$$

Let $h = f \circ g$, that is $h(x) = f(g(x))$. I have to prove that h has the following properties: (These items are red.)

(I-h) There exists a $\delta_{0,h} > 0$ such that $h(x)$ is defined for all $x \in \big(a - \delta_{0,h}, a + \delta_{0,h}\big)$.
(II-h) For every $\epsilon > 0$ there exists $\delta_h(\epsilon)$ such that $0 < \delta_h(\epsilon) \leq \delta_{0,h}$ and such that

$$|x-a| < \delta_h(\epsilon) \quad \Rightarrow \quad |h(x) - h(a)| < \epsilon.$$

Where is h guaranteed to be defined? I must make sure that x is such that $|g(x) - g(a)| < \delta_{0,f}$. We can achieve this by using (II-g)!

Put $\delta_{0,h} = \delta_g\big(\delta_{0,f}\big)$. Now assume that $|x - a| < \delta_{0,h}$. By (II-g) it follows that $|g(x) - g(a)| < \delta_{0,f}$. Therefore $g(x) \in \big(g(a) - \delta_{0,f}, g(a) + \delta_{0,f}\big)$. Hence, by (I-f), $f(g(x))$ is defined. Thus we proved that $f(g(x))$ is defined whenever $|x - a| < \delta_{0,h}$.

Let $\epsilon > 0$ be given. Put

$$\delta_h(\epsilon) = \min\big\{\delta_g\big(\delta_f(\epsilon)\big), \delta_g\big(\delta_{0,f}\big)\big\}.$$

Now we prove the red implication in (II-h).

Assume $|x - a| < \delta_h(\epsilon)$. Then $|x - a| < \delta_g\big(\delta_f(\epsilon)\big)$. By the green implication in (II-g), we conclude that

$$|x - a| < \delta_g\big(\delta_f(\epsilon)\big) \quad \Rightarrow \quad |g(x) - g(a)| < \delta_f(\epsilon).$$

Using the green implication in (II-f), we conclude that

$$|g(x) - g(a)| < \delta_f(\epsilon) \quad \Rightarrow \quad |f(g(x)) - f(g(a))| < \epsilon.$$

Thus we proved that the assumption $|x - a| < \delta_h(\epsilon)$ implies that

$$|h(x) - h(a)| = |f(g(x)) - f(g(a))| < \epsilon.$$

This completes the proof. $\square$

CHAPTER 3

Infinite Series

3.1. Sequences of real numbers

3.1.1. Definitions and examples.

Definition 3.1. A sequence of real numbers is a real function whose domain is either the set $\mathbb{N}$ of positive integers or the set $\mathbb{N}_0$ of nonnegative integers.

Let $s : \mathbb{N} \to \mathbb{R}$ be a sequence. Then the values of s are $s(1), s(2), s(3), \ldots, s(n), \ldots$. It is customary to write s_n instead of $s(n)$ in this case. Sometimes a sequence will be specified by listing its first few terms

$$s_1,\ s_2,\ s_3,\ s_4,\ \ldots\ ,$$

and sometimes by listing of all its terms $\{s_n\}_{n\in\mathbb{N}}$ or simply $\{s_n\}$ since domain is clear. One way of specifying a sequence is to give a formula, or recursion formula for its $n-$th term s_n. Notice that in this notation s is the "name" of the sequence and n is the variable.

Some examples of sequences follow.

Example 3.2.
(a) $1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ \ldots;$
(b) $1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 5,\ 5,\ 5,\ 5,\ 5,\ 6,\ 6,\ 6,\ 6,\ 6,\ 6,\ 7,\ 7,\ \ldots;$
(c) $1,\ 1,\ 1,\ 1,\ 1,\ \ldots;$ (the constant sequence)
(d) $\frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{4},\ \frac{3}{4},\ \frac{1}{5},\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5},\ \frac{1}{6},\ \frac{5}{6},\ \frac{1}{7},\ \frac{2}{7},\ \frac{3}{7},\ \frac{4}{7},\ \frac{5}{7},\ \frac{6}{7},\ \frac{1}{8},\ \frac{3}{8},\ \frac{5}{8},\ \frac{7}{8},\ \ldots;$ (What is the range of this sequence?)

Recursively defined sequences

Example 3.3.
(a) $x_1 = 1, \quad x_{n+1} = 1 + \frac{x_n}{4},\ n \in \mathbb{N};$
(b) $x_1 = 2, \quad x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n},\ n \in \mathbb{N};$
(c) $a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2 + a_n}, \quad n \in \mathbb{N};$
(d) $s_1 = 1, \quad s_{n+1} = \sqrt{1 + s_n}, \quad n \in \mathbb{N};$
(e) $x_1 = \frac{9}{10}, \quad x_{n+1} = \frac{9 + x_n}{10}, \quad n \in \mathbb{N}.$
(f) $b_1 = \frac{1}{2}, \quad b_{n+1} = \frac{1}{2\sqrt{1 - b_n^2}}, \quad n \in \mathbb{N}$
(g) $f_0 = 1,\ f_n = n \cdot f_{n-1}, \quad n \in \mathbb{N}.$

The standard notation for the terms of the sequence in (g) $f : \mathbb{N}_0 \to \mathbb{R}$ is $f_n = n!,\ n \in \mathbb{N}_0$.

Below we present more examples of important sequences.

Example 3.4.
(a) For $c \in \mathbb{R}$ set $b_n = c$ for all $n \in \mathbb{N}$. This is a constant sequence.
(b) For $a \in \mathbb{R}$ recursively define the sequence $p_0 = 1$ and $p_n = a p_{n-1}$ for all $n \in \mathbb{N}$. This is the sequence of the powers of a commonly written as $p_n = a^n$ for all $n \in \mathbb{N}_0$.

(c) A remarkable property of the two sequences given in this item is that they converge to the same limit: the famous real number e. The first sequence is given by a formula

$$q_n = \left(1 + \frac{1}{n}\right)^n \quad \text{for all} \quad n \in \mathbb{N},$$

and the second by a recursive definition

$$s_0 = \frac{1}{0!} = 1, \;\; s_n = s_{n-1} + \frac{1}{n!} \;\; \text{with} \;\; n \in \mathbb{N}$$

The n-th term of the preceding sequence is often written as

$$s_n = \sum_{k=0}^{n} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

We will hopefully have time to rigorously prove the claim in this item.

(d) A remarkable property of the two sequences given in this item is that they converge to the same limit: the famous exponential function e^x. Let $x \in \mathbb{R}$ and define the first sequence is given by a formula

$$q_n = \left(1 + \frac{x}{n}\right)^n \quad \text{for all} \quad n \in \mathbb{N},$$

and the second by a recursive definition

$$s_0 = \frac{1}{0!} = 1, \;\; s_n = s_{n-1} + \frac{x^n}{n!} \;\; \text{with} \;\; n \in \mathbb{N}$$

The n-th term of the preceding sequence is often written as

$$s_n = \sum_{k=0}^{n} \frac{x^n}{n!} = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

We might have time to explain why the claim in this item is true.

(e) The recursively defined sequences named s in the last two items are examples of a general recursive pattern which we explain here. Let $a : \mathbb{N}_0 \to \mathbb{R}$ be an arbitrary sequence. An important recursively defined sequence associated with $a : \mathbb{N}_0 \to \mathbb{R}$ is the following sequence:

$$S_0 = a_0, \;\; S_n = S_{n-1} + a_n \;\; \text{with} \;\; n \in \mathbb{N}.$$

The n-th term of the preceding sequence is often written as

$$S_n = \sum_{k=0}^{n} a_n = a_0 + a_1 + a_2 + \cdots + a_n.$$

3.1.2. Convergent sequences.

DEFINITION 3.5. A sequence $s : \mathbb{N} \to \mathbb{R}$ of real numbers *converges to the real number* L if for every $\epsilon > 0$ there exists a real number $N(\epsilon)$ such that

$$\forall n \in \mathbb{N} \quad n > N(\epsilon) \quad \Rightarrow \quad |s_n - L| < \epsilon.$$

If $s : \mathbb{N} \to \mathbb{R}$ converges to L we will write

$$\lim_{n \to +\infty} s_n = L \quad \text{or} \quad s_n \to L \;\; (n \to +\infty).$$

The number L is called the *limit* of the sequence $s : \mathbb{N} \to \mathbb{R}$.

DEFINITION 3.6. A sequence $s : \mathbb{N} \to \mathbb{R}$ *converges* if there exists $L \in \mathbb{R}$ such that $\lim\limits_{n\to+\infty} s_n = L$. In other words, a sequence $s : \mathbb{N} \to \mathbb{R}$ *converges* if

$$\exists\, L \in \mathbb{R} \quad \text{s.t.} \quad \forall\, \epsilon > 0 \quad \exists N(\epsilon) \in \mathbb{R} \quad \text{s.t.} \quad \forall n \in \mathbb{N} \quad n > N(\epsilon) \quad \Rightarrow \quad |s_n - L| < \epsilon.$$

A sequence that does not converge is said to *diverge*.

EXAMPLE 3.7. Let r be a real number such that $|r| < 1$. Prove that $\lim_{n\to+\infty} r^n = 0$.

SOLUTION. First note that if $r = 0$, then $r^n = 0$ for all $n \in \mathbb{N}$, so the given sequence is a constant sequence. Therefore it converges to 0. Assume that $r \in (-1, 0) \cup (0, 1)$, that is $0 < |r| < 1$ and let $\epsilon > 0$ be arbitrary. We need to solve $|r^n - 0| < \epsilon$ for n. First simplify $|r^n - 0| = |r^n| = |r|^n$. Now solve $|r|^n < \epsilon$ by taking ln of both sides of the inequality (note that ln is an increasing function)

$$\ln|r|^n = n \ln|r| < \ln \epsilon.$$

Since $0 < |r| < 1$, we conclude that $\ln|r| < 0$. Therefore the solution is

$$n > \frac{\ln \epsilon}{\ln|r|}.$$

Thus, with $N(\epsilon) = \dfrac{\ln \epsilon}{\ln|r|}$, the implication

$$\forall n \in \mathbb{N} \quad n > N(\epsilon) \quad \Rightarrow \quad |r^n - 0| < \epsilon$$

is valid. □

EXAMPLE 3.8. Prove that $\lim\limits_{n\to+\infty} \dfrac{n^2 - n - 1}{2n^2 - 1} = \dfrac{1}{2}$.

SOLUTION. Let $\epsilon > 0$ be arbitrary. We need to solve $\left|\dfrac{n^2 - n - 1}{2n^2 - 1} - \dfrac{1}{2}\right| < \epsilon$ for n. First simplify:

$$\left|\frac{n^2 - n - 1}{2n^2 - 1} - \frac{1}{2}\right| = \left|\frac{2}{2}\,\frac{n^2 - n - 1}{2n^2 - 1} - \frac{1}{2}\,\frac{2n^2 - 1}{2n^2 - 1}\right| = \left|\frac{-2n - 1}{2\,(2n^2 - 1)}\right| = \frac{2n + 1}{4n^2 - 2}$$

Now invent the BIN:

$$\frac{2n + 1}{4n^2 - 2} \leq \frac{2n + n}{4n^2 - 2n^2} = \frac{3n}{2n^2} = \frac{3}{2n}.$$

Therefore the BIN is:

$$\left|\frac{n^2 - n - 1}{2n^2 - 1} - \frac{1}{2}\right| \leq \frac{3}{2n} \quad \text{valid for all} \quad n \in \mathbb{N}.$$

Solving for n is now easy:

$$\frac{3}{2n} < \epsilon. \qquad \text{The solution is} \quad n > \frac{3}{2\epsilon}.$$

Thus, with $N(\epsilon) = \dfrac{3}{2\epsilon}$, the implication

$$\forall n \in \mathbb{R} \quad n > N(\epsilon) \quad \Rightarrow \quad \left|\frac{n^2 - n - 1}{2n^2 - 1} - \frac{1}{2}\right| < \epsilon$$

is valid. (This implication is proved by using the BIN) □

3.1.3. Theorems about convergent sequences. The procedure of proving limits of sequences is very similar to the procedure for proving limits of functions as x approaches infinity. In fact the following two theorems are true.

THEOREM 3.9. *Let $f : [1, +\infty) \to \mathbb{R}$ be a function and define the sequence $a : \mathbb{N} \to \mathbb{R}$ by*

$$a_n = f(n) \quad \textit{for every} \quad n \in \mathbb{N}.$$

If $\lim\limits_{x\to+\infty} f(x) = L$, then $\lim\limits_{n\to+\infty} a_n = L$.

THEOREM 3.10. *Let $f : (0, 1] \to \mathbb{R}$ be a function which is defined for every $x \in (0, 1]$. Define the sequence $a : \mathbb{N} \to \mathbb{R}$ by*

$$a_n = f(1/n) \quad \textit{for every} \quad n \in \mathbb{N}.$$

If $\lim\limits_{x\downarrow 0} f(x) = L$, then $\lim\limits_{n\to+\infty} a_n = L$.

The above two theorems are useful for proving limits of sequences which are defined by a formula. For example you can prove the following limits by using these two theorems and what we proved in previous sections.

EXERCISE 3.11. Find the following limits. Provide proofs.

(a) $\lim\limits_{n\to+\infty} \sin\left(\dfrac{1}{n}\right)$ (b) $\lim\limits_{n\to+\infty} n\sin\left(\dfrac{1}{n}\right)$ (c) $\lim\limits_{n\to+\infty} \ln\left(1+\dfrac{1}{n}\right)$

(d) $\lim\limits_{n\to+\infty} n\ln\left(1+\dfrac{1}{n}\right)$ (e) $\lim\limits_{n\to+\infty} \cos\left(\dfrac{1}{n}\right)$ (f) $\lim\limits_{n\to+\infty} \dfrac{1}{n}\cos\left(\dfrac{1}{n}\right)$

In the following theorem we prove that the operation of taking the limit of a sequence respects the algebra of real numbers. The theorem is called the Algebra of Limits Theorem.

THEOREM 3.12. *Let $a : \mathbb{N} \to \mathbb{R}$, $b : \mathbb{N} \to \mathbb{R}$ and $c : \mathbb{N} \to \mathbb{R}$ be given sequences. Let K and L be real numbers. Assume that*

(1) $\lim\limits_{x\to+\infty} a_n = K$,
(2) $\lim\limits_{x\to+\infty} b_n = L$.

Then the following statements hold.

(A) *If $c_n = a_n + b_n$, $n \in \mathbb{N}$, then $\lim\limits_{x\to+\infty} c_n = K + L$.*
(B) *If $c_n = a_n b_n$, $n \in \mathbb{N}$, then $\lim\limits_{x\to+\infty} c_n = KL$.*
(C) *If $L \neq 0$ and $c_n = \dfrac{a_n}{b_n}$, $n \in \mathbb{N}$, then $\lim\limits_{x\to+\infty} c_n = \dfrac{K}{L}$.*

PROOF. To prove (A) assume that

$$\lim_{x\to+\infty} a_n = K, \qquad \lim_{x\to+\infty} b_n = L, \quad \text{and} \quad \forall n \in \mathbb{N} \;\; c_n = a_n + b_n.$$

By the definition of limit we have

$$\forall \epsilon > 0 \quad \exists N_a(\epsilon) \in \mathbb{R} \quad \text{such that} \quad \forall n \in \mathbb{N} \quad n > N_a(\epsilon) \Rightarrow |a_n - K| < \epsilon \tag{3.1}$$

and

$$\forall \epsilon > 0 \quad \exists N_b(\epsilon) \in \mathbb{R} \quad \text{such that} \quad \forall n \in \mathbb{N} \quad n > N_b(\epsilon) \Rightarrow |b_n - L| < \epsilon. \tag{3.2}$$

Let $\epsilon > 0$ be arbitrary. Define

$$N_c(\epsilon) = \max\big\{N_a(\epsilon/2), N_b(\epsilon/2)\big\}.$$

Let $n \in \mathbb{N}$ be arbtrary. Assume

$$n > N_c(\epsilon).$$

Then by the definition of $N_c(\epsilon)$ we have

$$n > N_a(\epsilon/2) \quad \text{and} \quad n > N_b(\epsilon/2).$$

By (3.1) we have that

$$n > N_a(\epsilon/2) \quad \Rightarrow \quad |a_n - K| < \epsilon/2.$$

By (3.2) we have that

$$n > N_b(\epsilon/2) \quad \Rightarrow \quad |b_n - L| < \epsilon/2.$$

Therefore

$$n > N_c(\epsilon) \quad \Rightarrow \quad |a_n - K| < \epsilon/2 \quad \text{and} \quad |b_n - L| < \epsilon/2. \tag{3.3}$$

By algebra and the Triangle Inequality we have that for all $n \in \mathbb{N}$ we have

$$\big|c_n - (K + L)\big| = \big|a_n + b_n - K - L\big| = \big|(a_n - K) + (b_n - L)\big| \leq \big|a_n - K\big| + \big|b_n - L\big| \tag{3.4}$$

From (3.3) and (3.4) and the transitivity of order we deduce that

$$n > N_c(\epsilon) \quad \Rightarrow \quad |c_n - (K + L)| < \epsilon.$$

Thus we have proved that the following statement is true

$$\forall \epsilon > 0 \quad \exists N_c(\epsilon) \in \mathbb{R} \quad \text{such that} \quad \forall n \in \mathbb{N} \quad n > N_c(\epsilon) \Rightarrow |c_n - (K + L)| < \epsilon.$$

Therefore

$$\lim_{x \to +\infty} c_n = K + L.$$

□

THEOREM 3.13. *Let $a : \mathbb{N} \to \mathbb{R}$ and $b : \mathbb{N} \to \mathbb{R}$ be given sequences. Let K and L be real numbers. Assume that*

(1) $\lim\limits_{x \to +\infty} a_n = K$.
(2) $\lim\limits_{x \to +\infty} b_n = L$.
(3) *There exists a positive integer n_0 such that*

$$\forall n \in \mathbb{N} \quad \textit{such that} \quad n \geq n_0 \quad \textit{we have} \quad a_n \leq b_n.$$

Then $K \leq L$.

PROOF. Assume (1), (2) and (3). Let $\epsilon > 0$ be arbitrary. Since $\lim\limits_{x \to +\infty} a_n = K$, there exists $N_a(\epsilon)$ such that

$$\forall n \in \mathbb{N} \quad \text{and} \quad n > N_a(\epsilon) \quad \Rightarrow \quad \big|a_n - K\big| < \epsilon.$$

Since $\lim\limits_{x \to +\infty} b_n = L$, there exists $N_b(\epsilon)$ such that

$$\forall n \in \mathbb{N} \quad \text{and} \quad n > N_b(\epsilon) \quad \Rightarrow \quad \big|b_n - L\big| < \epsilon.$$

Choose $m \in \mathbb{N}$ such that $m > \max\big\{n_0, N_a(\epsilon), N_b(\epsilon)\big\}$. Then

$$K - \epsilon < a_m < K + \epsilon$$
$$a_m \leq b_m$$
$$L - \epsilon < b_m < L + \epsilon.$$

Consequently, by the transitivity axiom for real numbers we have

$$K - \epsilon < a_m \leq b_m < L + \epsilon.$$

Hence

$$K - L < 2\epsilon.$$

Now recall that $\epsilon > 0$ was arbitrary. Thus, the inequality $K - L < 2\epsilon$ holds for all $\epsilon > 0$. The following implication is true

$$\forall \epsilon > 0 \quad K - L < 2\epsilon \qquad \Rightarrow \qquad K - L \leq 0.$$

(This implication is proved by proving its contrapositive.) Hence, we conclude that $K - L \leq 0$. □

THEOREM 3.14. *Let $a : \mathbb{N} \to \mathbb{R}$, $b : \mathbb{N} \to \mathbb{R}$ and $s : \mathbb{N} \to \mathbb{R}$ be given sequences. Let L be a real number. Assume the following*

(1) *The sequence $a : \mathbb{N} \to \mathbb{R}$ converges to L.*
(2) *The sequence $b : \mathbb{N} \to \mathbb{R}$ converges to L.*
(3) *There exists a positive integer n_0 such that*

$$a_n \leq s_n \leq b_n \quad \text{for all} \quad n > n_0.$$

Then the sequence $s : \mathbb{N} \to \mathbb{R}$ converges to L.

Prove this theorem.

3.1.4. The Monotone Convergence Theorem. Many limits of sequences cannot be found using theorems from the previous section. For example, the recursively defined sequences (a), (b), (c), (d) and (e) in Example 3.3 converge but it cannot be proved using the methods that we presented so far.

DEFINITION 3.15. (1) A sequence $s : \mathbb{N} \to \mathbb{R}$ is *bounded above* if there exists a real number M such that

$$\forall\, n \in \mathbb{N} \quad s_n \leq M.$$

A number M with the above property is called an *upper bound* of the sequence s.

(2) A sequence $s : \mathbb{N} \to \mathbb{R}$ is *bounded below* if there exists a real number m such that

$$\forall\, n \in \mathbb{N} \quad m \leq s_n.$$

A number m with the above property is called a *lower bound* of the sequence s.

(3) A sequence $s : \mathbb{N} \to \mathbb{R}$ is *bounded* if it is bounded above and bounded below. In other words, a sequence $s : \mathbb{N} \to \mathbb{R}$ is bounded if there exist real numbers m and M such that

$$\forall\, n \in \mathbb{N} \quad m \leq s_n \leq M.$$

THEOREM 3.16. *If a sequence converges, then it is bounded.*

PROOF. Assume that a sequence $a : \mathbb{N} \to \mathbb{R}$ converges to a real number L. We need to prove that there exist real numbers m and M such that

$$\forall n \in \mathbb{N} \quad \text{we have} \quad m \leq a_n \leq M. \tag{3.5}$$

Since $a : \mathbb{N} \to \mathbb{R}$ converges to L, Definition 3.5 yields that for every $\epsilon > 0$ there exists a real number $N(\epsilon)$ such that

$$\forall n \in \mathbb{N} \quad n > N(\epsilon) \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

In particular for $\epsilon = 1 > 0$ there exists a real number $N(1)$ such that

$$\forall n \in \mathbb{N} \quad n > N(1) \quad \Rightarrow \quad |a_n - L| < 1.$$

Since $|a_n - L| < 1$ is equivalent to $L-1 < a_n < L+1$, the preceding implication can be rewritten as

$$\forall n \in \mathbb{N} \quad n > N(1) \quad \Rightarrow \quad L-1 < a_n < L+1. \tag{3.6}$$

Case 1. Assume that $N(1) < 1$. Then for all $n \in \mathbb{N}$ we have $n > N(1)$. Therefore (3.6) yields

$$\forall n \in \mathbb{N} \quad \text{we have} \quad L-1 < a_n < L+1.$$

Thus, we can take $m = L-1$ and $M = L+1$ and (3.5) holds.
Case 2. Assume that $N(1) \geq 1$. Set $n_0 = \lfloor N(1) \rfloor$. Then n_0 is a positive integer with the following property

$$n_0 \leq N(1) < n_0 + 1. \tag{3.7}$$

The preceding inequality suggests a partition of the set $\mathbb{N}$ in two disjoint sets

$$\mathbb{N} = \bigl\{1, 2, \ldots, n_0\bigr\} \cup \bigl\{k \in \mathbb{N} : k > n_0\bigr\}. \tag{3.8}$$

The first set $\bigl\{1, 2, \ldots, n_0\bigr\}$ in the preceding union is finite and has n_0 elements. The second set $\bigl\{k \in \mathbb{N} : k > n_0\bigr\}$ is infinite and consists of the positive integers $n_0+1, n_0+2, \ldots$. It follows from (3.7) that

$$\forall n \in \bigl\{k \in \mathbb{N} : k > n_0\bigr\} \quad \text{we have} \quad n > N(1).$$

Therefore (3.6) yields

$$\forall n \in \bigl\{k \in \mathbb{N} : k > n_0\bigr\} \quad \text{we have} \quad L-1 < a_n < L+1. \tag{3.9}$$

The number $L-1$ is not necessarily a lower bound and $L+1$ is not necessarily an upper bound for the sequence since we do not know whether the relation of $L-1$ and $L+1$ to the terms

$$a_1, a_2, \ldots, a_{n_0}.$$

Since every finite set has a minimum and a maximum we set

$$m = \min\bigl\{a_1, a_2, \ldots, a_{n_0}, L-1\bigr\}$$

and

$$M = \max\bigl\{a_1, a_2, \ldots, a_{n_0}, L+1\bigr\}.$$

Now we can prove that m is a lower bound and M is an upper bound for the sequence $a : \mathbb{N} \to \mathbb{R}$. By the definitions of the minimum and the maximum we have

$$\forall n \in \bigl\{1, 2, \ldots, n_0\bigr\} \quad \text{we have} \quad m \leq a_n \leq M.$$

The definitions of the minimum and the maximum also imply that $m \leq L-1$ and $L+1 \leq M$. Using these inequalities, the transitivity of order and (3.9) we obtain

$$\forall n \in \bigl\{k \in \mathbb{N} : k > n_0\bigr\} \quad \text{we have} \quad m < a_n < M.$$

Because of (3.8) the preceding two displayed statements yield

$$\forall n \in \mathbb{N} \quad \text{we have} \quad m \leq a_n \leq M.$$

Hence the sequence $a : \mathbb{N} \to \mathbb{R}$ is bounded. $\square$

Is the converse of Theorem 3.16 true? The converse is: If a sequence is bounded, then it converges. This statement is not true since there exists a sequence that is bounded and which does not converge. One such sequence is $n \mapsto (-1)^n$ for all $n \in \mathbb{N}$. This sequence is bounded and it is not convergent. Thus, we found a counterexample to the implication: If a sequence is bounded, then it converges.

The next question is whether boundedness and an additional property of a sequence can guarantee convergence. It turns out that such an property is monotonicity defined in the following definition.

Definition 3.17. A sequence $s : \mathbb{N} \to \mathbb{R}$ of real numbers is said to be:

non-decreasing if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$,

non-increasing if $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$.

A sequence with either of these four properties is said to be *monotonic*.

The following theorem is the Monotone Convergence Theorem.

Theorem 3.18 (Monotone Convergence Theorem). *A bounded monotonic sequence converges.*

To prove these theorems we have to resort to the most important property of the set of real numbers: the Completeness Axiom.

The Completeness Axiom. If A and B are nonempty subsets of $\mathbb{R}$ such that for every $a \in A$ and for every $b \in B$ we have $a \leq b$, then there exists $c \in \mathbb{R}$ such that $a \leq c \leq b$ for all $a \in A$ and all $b \in B$.

Proof of Theorem 3.18. Assume that $s : \mathbb{N} \to \mathbb{R}$ is a non-decreasing sequence and that it is bounded above. Since $s : \mathbb{N} \to \mathbb{R}$ is non-decreasing we know that

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_{n-1} \leq s_n \leq s_{n+1} \leq \cdots . \tag{3.10}$$

Let A be the range of the sequence $s : \mathbb{N} \to \mathbb{R}$. That is let

$$A = \bigl\{ s_n : n \in \mathbb{N} \bigr\}.$$

Clearly $A \neq \emptyset$. Let B be the set of all upper bounds of the sequence $s : \mathbb{N} \to \mathbb{R}$. Since the sequence $s : \mathbb{N} \to \mathbb{R}$ is bounded above, the set B is not empty. Let $b \in B$ be arbitrary. Then b is an upper bound for $s : \mathbb{N} \to \mathbb{R}$. Therefore

$$s_n \leq b \qquad \text{for all} \qquad n \in \mathbb{N}.$$

By the definition of A this means

$$a \leq b \qquad \text{for all} \qquad a \in A.$$

Since $b \in B$ was arbitrary we have

$$a \leq b \quad \text{for all} \quad a \in A \quad \text{and for all} \quad b \in B.$$

By the Completeness Axiom there exists $c \in \mathbb{R}$ such that

$$s_n \leq c \leq b \qquad \text{for all} \quad n \in \mathbb{N} \quad \text{and for all} \quad b \in B. \tag{3.11}$$

Thus c is an upper bound for $s : \mathbb{N} \to \mathbb{R}$ and also $c \leq b$ for all upper bounds b of the sequence $s : \mathbb{N} \to \mathbb{R}$. Therefore, for an arbitrary $\epsilon > 0$ the number $c - \epsilon$ (which is $< c$) is not an upper bound of the sequence $s : \mathbb{N} \to \mathbb{R}$. Consequently, there exists a positive integer $N(\epsilon)$ such that

$$c - \epsilon < s_{N(\epsilon)}. \tag{3.12}$$

Let $n \in \mathbb{N}$ be any positive integer which is $> N(\epsilon)$. Then the inequalities (3.10) imply that

$$s_{N(\epsilon)} \leq s_n. \tag{3.13}$$

By (3.11) the number c is an upper bound of $s : \mathbb{N} \to \mathbb{R}$. Hence we have

$$s_n \leq c \qquad \text{for all} \qquad n \in \mathbb{N}. \tag{3.14}$$

Putting together the inequalities (3.12), (3.13) and (3.14) we conclude that

$$c - \epsilon < s_n \leq c \quad \text{ for all } \quad n \in \mathbb{N} \ \text{ such that } \ n > N(\epsilon). \tag{3.15}$$

The relationship (3.15) shows that for $n \in \mathbb{N}$ such that $n > N(\epsilon)$ the distance between numbers s_n and c is $< \epsilon$. In other words

$$\forall n \in \mathbb{N} \quad n > N(\epsilon) \quad \Rightarrow \quad |s_n - c| < \epsilon.$$

This is exactly the implication in Definition 3.5. Thus, we proved that

$$\lim_{n \to +\infty} s_n = c. \qquad \square$$

EXAMPLE 3.19. Prove that the sequence in Example 3.3 (b) converges. That is, prove that the recursively defined sequence $x_1 = 2, \quad x_{n+1} = \dfrac{x_n}{2} + \dfrac{1}{x_n}$, $n \in \mathbb{N}$, converges.

SOLUTION. It is useful to calculate the first few terms of this sequence:

$$x_1 = 2, \ x_2 = \frac{3}{2}, \ x_3 = \frac{17}{12}, \ x_4 = \frac{577}{408}, \ x_5 = \frac{665857}{470832}, \ x_6 = \frac{886731088897}{627013566048}.$$

Notice that the formula $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ gives a positive output x_{n+1} whenever the input x_n is positive. Since $x_1 > 0$ this guaranties that $x_2 > 0$. In turn, the fact that $x_2 > 0$ guaranties that $x_3 > 0$, and so on. This reasoning justifies that $x_n > 0$ for all $n \in \mathbb{N}$. This proves that the sequence $\{x_n\}$ is bounded below by 0.

Next we will prove that $(x_n)^2 \geq 2$ for all $n \in \mathbb{N}$. We consider two cases $n = 1$ and $n > 1$. If $n = 1$, then $(x_1)^2 = 2^2 = 4 \geq 2$. Now assume that $n > 1$. Then $n - 1 \in \mathbb{N}$ and $x_n = \frac{x_{n-1}}{2} + \frac{1}{x_{n-1}}$. Therefore

$$\begin{aligned}
(x_n)^2 &= \left(\frac{x_{n-1}}{2} + \frac{1}{x_{n-1}}\right)^2 \\
&= \frac{(x_{n-1})^2}{4} + 1 + \frac{1}{(x_{n-1})^2} \\
&= 2 + \frac{(x_{n-1})^2}{4} - 1 + \frac{1}{(x_{n-1})^2} \\
&= 2 + \left(\frac{x_{n-1}}{2} - \frac{1}{x_{n-1}}\right)^2 \\
&\geq 2.
\end{aligned}$$

Thus $(x_n)^2 \geq 2$ for all $n \in \mathbb{N}$.

Since $x_n > 0$, $(x_n)^2 \geq 2$ implies $x_n \geq \frac{2}{x_n}$. Further, dividing by 2 we get $\frac{x_n}{2} \geq \frac{1}{x_n}$. Adding $\frac{x_n}{2}$ to the both sides of the last inequality we obtain $x_n \geq \frac{x_n}{2} + \frac{1}{x_n}$. Thus $x_n \geq x_{n+1}$. Here we have proved that $(x_n)^2 \geq 2$ implies $x_n \geq x_{n+1}$. Since $(x_n)^2 \geq 2$ is true for all $n \in \mathbb{N}$, we have proved that $x_n \geq x_{n+1}$ is true for all $n \in \mathbb{N}$.

To summarize, we have proved that $x_n > 0$ for all $n \in \mathbb{N}$ and $x_n \geq x_{n+1}$ is true for all $n \in \mathbb{N}$. That is, the sequence $\{x_n\}$ is bounded below and non-increasing. By the Monotone Convergence Theorem this sequence converges. Denote the limit of $\{x_n\}$ by L.

Next we use the algebra of limits to calculate L. Since $\bigl(x_n\bigr)^2 \geq 2$ for all $n \in \mathbb{N}$, by Theorems 3.12 and 3.13 we have $L^2 \geq 2$. Since $x_n > 0$ for all $n \in \mathbb{N}$, by Theorem 3.13 we have $L \geq 0$. Since $L^2 \geq 2$ and $L \geq 0$ we have $L > 0$. It is not difficult to prove that $\lim_{n\to\infty} x_{n+1} = L$. This fact, Theorem 3.12 and the identity $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ imply $L = \frac{L}{2} + \frac{1}{L}$. Hence $L^2 = 2$. That is $L = \sqrt{2}$.

This example is in fact a proof that there exists a positive real number a such that $a^2 = 2$. □

EXAMPLE 3.20. Prove that the sequence $S_n = \displaystyle\sum_{k=0}^{n} \frac{1}{n!}$, $n \in \mathbb{N}$, converges.

SOLUTION. Let $n \in \mathbb{N}$. We first prove that for $n > 1$ we have

$$\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} < 1.$$

For this we use the fact that for $k > 1$ we have

$$\frac{1}{k!} \leq \frac{1}{(k-1)k} = \frac{1}{k-1} - \frac{1}{k}$$

and therefore

$$\begin{aligned}
&\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{(n-1)!} + \frac{1}{n!} \\
&\qquad \leq \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \left(\frac{1}{n-1} - \frac{1}{n}\right)
\end{aligned}$$

Since the right-hand side of the preceding displayed inequality simplifies to $1 - 1/n$ we have

$$\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} < 1.$$

Therefore for all $n \in \mathbb{N}$

$$S_n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} < 3.$$

This proves that the sequence $\{S_n\}$ is bounded above. Since for every $n \in \mathbb{N}$ we have $S_{n+1} - S_n = \dfrac{1}{(n+1)!} > 0$, the sequence $\{S_n\}$ is increasing. By the Monotone Convergence Theorem $\{S_n\}$ converges.

The limit of the sequence $\{S_n\}$ is the famous number e. □

EXAMPLE 3.21. Prove that the sequence

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n, \qquad n \in \mathbb{N},$$

converges.

SOLUTION. Let $n \in \mathbb{N}$. By the definition of the logarithm function

$$\ln n = \int_1^n \frac{1}{x}\, dx.$$

Since

$$\frac{1}{x} \leq \frac{1}{k} \qquad \text{whenever} \qquad k \leq x \leq k+1,$$

for $n > 1$ we have

$$\ln n = \int_1^2 \frac{1}{x}\, dx + \int_2^3 \frac{1}{x}\, dx + \cdots + \int_{n-1}^n \frac{1}{x}\, dx < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1}.$$

Therefore,

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1} + \frac{1}{n} - \ln n > \frac{1}{n} > 0$$

for all $n \in \mathbb{N}, n > 1$. Since $t_1 = 1 > 0$, this proves that the sequence $\{t_n\}$ is bounded below by 0.

Next we prove that $\{t_n\}$ is decreasing. For arbitrary $n \in \mathbb{N}$ we have

$$\begin{aligned} t_n - t_{n+1} &= \big(\ln(n+1) - \ln n\big) - \frac{1}{n+1} \\ &= \int_n^{n+1} \frac{1}{x}\, dx - \frac{1}{n+1} \\ &= \int_n^{n+1} \left(\frac{1}{x} - \frac{1}{n+1}\right) dx \\ &> 0. \end{aligned}$$

Hence $t_n > t_{n+1}$ for all $n \in \mathbb{N}$.

Since $\{t_n\}$ is bounded below and decreasing it converges by the Monotone Convergence Theorem.

The limit of the sequence $\{t_n\}$ is called *Euler's constant*. It is denoted by γ. Its approximate value to 50 decimal places is

$$\gamma \approx 0.57721566490153286060651209008240243104215933593992.$$

It is not known whether γ is a rational or irrational number. □

3.2. Infinite series of real numbers

3.2.1. Definition and basic examples. The most important application of sequences is the definition of convergence of an infinite series. From the elementary school you have been dealing with addition of numbers. As you know the addition gets harder as you add more and more numbers. For example it would take some time to add

$$S_{100} = 1 + 2 + 3 + 4 + 5 + \cdots + 98 + 99 + 100$$

It gets much easier if you add two of these sums, and pair the numbers in a special way:

$$\begin{aligned} 2\,S_{100} = {} & 1 + 2 + 3 + 4 + \cdots + 97 + 98 + 99 + 100 \\ & 100 + 99 + 98 + 97 + \cdots + 4 + 3 + 2 + 1 \end{aligned}$$

A straightforward observation that each column on the right adds to 101 and that there are 100 such columns yields that

$$2\,S_{100} = 101 \cdot 100, \qquad \text{that is} \qquad S_{100} = \frac{101 \cdot 100}{2} = 5050.$$

This can be generalized to any positive integer n to get the formula

$$S_n = 1 + 2 + 3 + 4 + 5 + \cdots + (n-1) + n = \frac{(n+1)\,n}{2}.$$

This procedure indicates that it would be impossible to find the sum

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

where the last set of $\cdots$ indicates that we continue to add positive integers.

The situation is quite different if we consider the sequence

$$\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \ldots,\ \frac{1}{2^n},\ \ldots$$

and start adding more and more consecutive terms of this sequence:

$$\begin{aligned}
&\frac{1}{2} && = 1 - \frac{1}{2} = \frac{1}{2}\\
&\frac{1}{2} + \frac{1}{4} && = 1 - \frac{1}{4} = \frac{3}{4}\\
&\frac{1}{2} + \frac{1}{4} + \frac{1}{8} && = 1 - \frac{1}{8} = \frac{7}{8}\\
&\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} && = 1 - \frac{1}{16} = \frac{15}{16}\\
&\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} && = 1 - \frac{1}{32} = \frac{31}{32}\\
&\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} && = 1 - \frac{1}{64} = \frac{63}{64}
\end{aligned}$$

These sums are nicely illustrated in Fig. 1. The pictures in Fig. 1 strongly indicate that the sum of infinitely many numbers $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ equals 1. That is

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

Why does this make sense? This makes sense since we have seen above that as we add more and more terms of the sequence

$$\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \ldots,\ \frac{1}{2^n},\ \ldots$$

we are getting closer and closer to 1. Indeed,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

and

$$\lim_{n \to +\infty} \left(1 - \frac{1}{2^n} \right) = 1.$$

This reasoning leads to the definition of convergence of an infinite series:

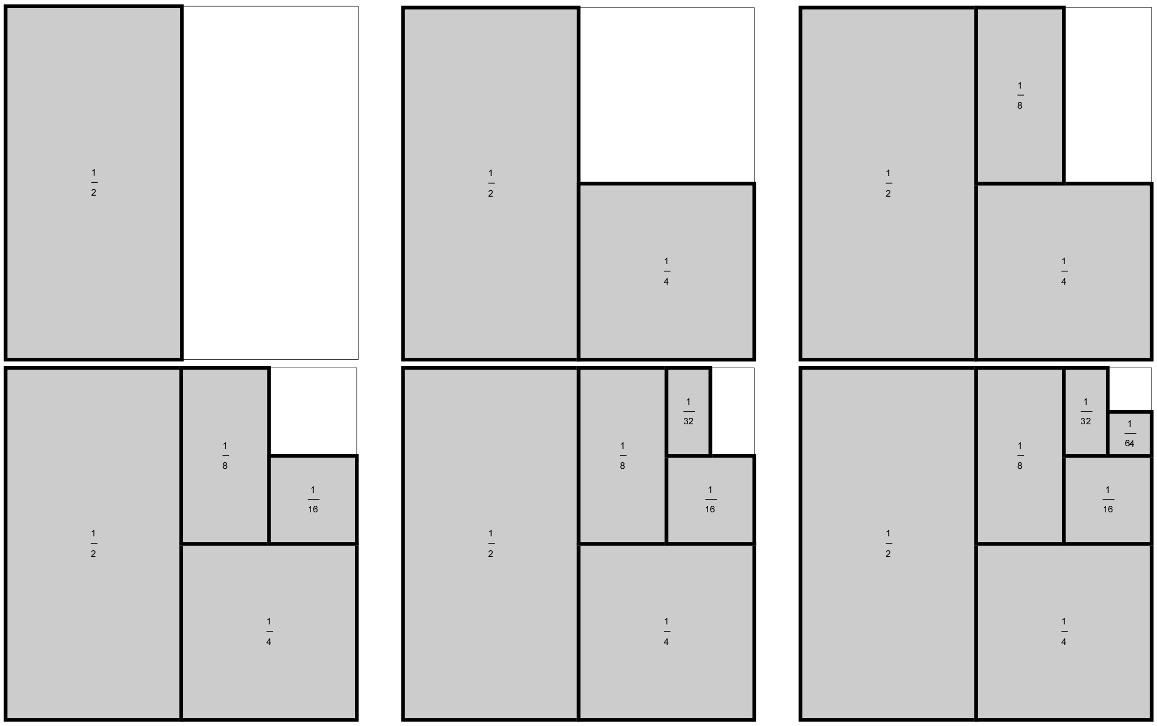

FIG. 1. In this example it seems natural to say that the sum of infinitely many numbers $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ equals 1

DEFINITION 3.22. Let $a : \mathbb{N} \to \mathbb{R}$ be a given sequence. Then the expression

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is called an infinite series. We often abbreviate it by writing

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{+\infty} a_n.$$

For each positive integer n we calculate the (finite) sum of the first n terms of the series

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n.$$

We call S_n a partial sum of the infinite series $\displaystyle\sum_{n=1}^{+\infty} a_n$. (Notice that $\{S_n\}_{n=1}^{+\infty}$ is a new sequence.) If the sequence $\{S_n\}_{n=1}^{+\infty}$ converges to a real number S, that is if

$$\lim_{n\to+\infty} S_n = S,$$

then the infinite series $\sum_{n=1}^{+\infty} a_n$ is said to be *convergent* and we write

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = S \qquad \text{or} \qquad \sum_{n=1}^{+\infty} a_n = S.$$

The number S is called the *sum of the series.*

If the sequence of the partial sums $S : \mathbb{N} \to \mathbb{R}$ does not converge to a real number, then the series is called *divergent.*

In the example above we have

$$a_n = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n,$$
$$S_n = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$
$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\left(1 - \frac{1}{2^n}\right) = 1.$$

Therefore we say that the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = \sum_{n=1}^{+\infty} \frac{1}{2^n}$$

converges and its sum is 1. We write $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{2^n} = 1$.

In our opening example

$$a_n = n,$$
$$S_n = 1 + 2 + 3 + \cdots + n = \frac{(n+1)n}{2}$$
$$\lim_{n\to+\infty} \frac{(n+1)n}{2} \quad \text{does not exist.}$$

Therefore we say that the series

$$1 + 2 + 3 + 4 + \cdots + n + \cdots = \sum_{n=1}^{+\infty} n$$

diverges.

3.2.2. Geometric Series. Let a and r be real numbers. The most important infinite series is

$$a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^n + \cdots = \sum_{n=0}^{+\infty} a\,r^n \tag{3.16}$$

This series is called a geometric series. To determine whether this series converges or not we need to study its partial sums:

$$S_0 = a, \qquad S_1 = a + a\,r,$$
$$S_2 = a + a\,r + a\,r^2, \qquad S_3 = a + a\,r + a\,r^2 + a\,r^3,$$
$$S_4 = a + a\,r + a\,r^2 + a\,r^3 + a\,r^4, \qquad S_5 = a + a\,r + a\,r^2 + a\,r^3 + a\,r^4 + a\,r^5,$$
$$\vdots$$
$$S_n = a + a\,r + a\,r^2 + \cdots + a\,r^{n-1} + a\,r^n$$
$$\vdots$$

Notice that we have already studied the special case when $a = 1$ and $r = \dfrac{1}{2}$. In this special case we found a simple formula for S_n and then we evaluated $\lim\limits_{n\to+\infty} S_n$. It turns out that we can find a simple formula for S_n in the general case as well.

First note that the case $a = 0$ is not interesting, since then all the terms of the geometric series are equal to 0 and the series clearly converges and its sum is 0. Assume that $a \neq 0$. If $r = 1$ then $S_n = n\,a$. Since we assume that $a \neq 0$, $\lim\limits_{n\to+\infty} na$ does not exist. Thus for $r = 1$ the series diverges.

Assume that $r \neq 1$. To find a simple formula for S_n, multiply the long formula for S_n above by r to get:

$$\begin{aligned} S_n &= a + ar + ar^2 + \cdots + ar^{n-1} + ar^n, \\ rS_n &= ar + ar^2 + \cdots + ar^n + ar^{n+1}; \end{aligned}$$

now subtract,

$$S_n - r\,S_n = a - a\,r^{n+1},$$

and solve for S_n:

$$S_n = a\,\frac{1 - r^{n+1}}{1 - r}\,.$$

We already proved that if $|r| < 1$, then $\lim\limits_{n\to+\infty} r^{n+1} = 0$. If $|r| \geq 1$, then $\lim\limits_{n\to+\infty} r^{n+1}$ does not exist. Therefore we conclude that

$$\begin{aligned} &\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} a\,\frac{1 - r^{n+1}}{1 - r} = a\,\frac{1}{1 - r} && \text{for} \quad |r| < 1, \\ &\lim_{n\to+\infty} S_n \qquad \text{does not converge to a real number} && \text{for} \quad |r| \geq 1. \end{aligned}$$

In conclusion

- If $|r| < 1$, then the geometric series $\displaystyle\sum_{n=0}^{+\infty} a\,r^n$ converges and its sum is $a\,\dfrac{1}{1 - r}$.
- If $|r| \geq 1$, then the geometric series $\displaystyle\sum_{n=0}^{+\infty} a\,r^n$ diverges.

Fig. 2 illustrates the sum of a geometric series with $a > 0$ and $0 < r < 1$:

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \frac{a}{1 - r}.$$

In Fig. 2 the terms of a geometric series are represented as areas. As we can see in Fig. 2 the areas of the terms fill in the rectangle whose area is $a/(1 - r)$.

In Fig. 3 we represent the terms of the geometric series by lengths of horizontal line segments. The picture strongly indicates that the total length of infinitely many horizontal line segments is $a/(1 - r)$. The reason for this is that by construction the slope of the hypothenuse CB of the right triangle ABC in Fig. 3 is $(1 - r)$. Since the length of its vertical cathetus AC is a, the length of its horizontal cathetus AB must be $a/(1 - r)$.

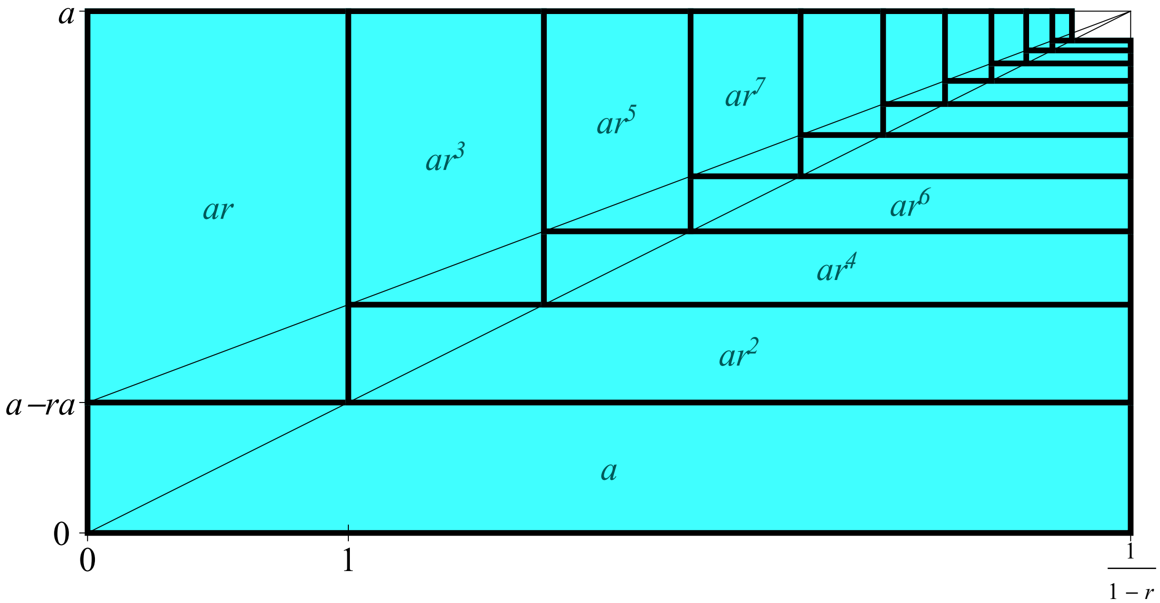

FIG. 2. The width of the rectangle is $1/(1-r)$ and its height is a. The slope of the diagonal is $(1-r)a$. The slope of the line above the diagonal is $r(1-r)a$

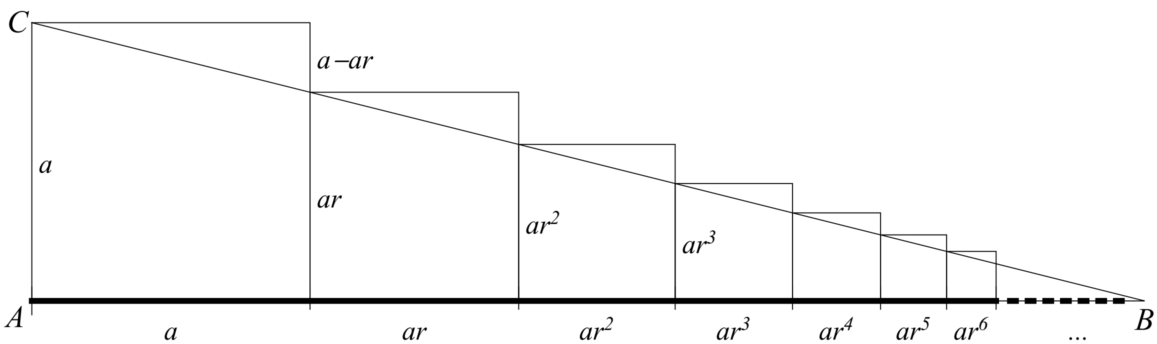

FIG. 3. Consider the right triangle ABC. From the small left-topmost right triangle we calculate that the slope of the hypothenuse CB is $1-r = \frac{\text{rise}}{\text{run}} = \frac{a-ar}{a}$. Since the length of the vertical cathetus AC is a we deduce that the length of the horizontal cathetus AB is $a/(1-r)$.

3.2.3. How to recognize whether an infinite series is a geometric series? Consider for example the infinite series $\sum_{n=1}^{+\infty} \frac{\pi^{n+2}}{e^{2n-1}}$. Here $a_n = \frac{\pi^{n+2}}{e^{2n-1}}$.

Looking at the formula (3.16) we note that the first term of the series is a and that the ratio between any two consecutive terms is r.

For $a_n = \frac{\pi^{n+2}}{e^{2n-1}}$ given above we calculate

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{\pi^{n+1+2}}{e^{2(n+1)-1}}}{\dfrac{\pi^{n+2}}{e^{2n-1}}} = \frac{\pi^{n+3}\, e^{2n-1}}{e^{2n+1}\, \pi^{n+2}} = \frac{\pi}{e^2}\,.$$

Since $\dfrac{a_{n+1}}{a_n}$ is constant, we conclude that the series $\displaystyle\sum_{n=1}^{+\infty} \frac{\pi^{n+2}}{e^{2n-1}}$ is a geometric series with

$$a = a_1 = \frac{\pi^2}{e} \quad \text{and} \quad r = \frac{\pi}{e^2} \quad \text{for all} \quad n \in \mathbb{N}.$$

Since $r = \dfrac{\pi}{e^2} < 1$, we conclude that the sum of this series is

$$\sum_{n=1}^{+\infty} \frac{\pi^{n+2}}{e^{2n-1}} = \frac{\pi^2}{e} \frac{1}{1 - \frac{\pi}{e^2}} = \frac{\pi^2}{e} \frac{e^2}{e^2 - \pi} = \frac{\pi^2 e}{e^2 - \pi}.$$

Thus, to verify whether a given infinite series is a geometric series calculate the ratio of the consecutive terms and see whether it is a constant:

$$\sum_{n=1}^{+\infty} a_n \quad \text{for which} \quad \frac{a_{n+1}}{a_n} = r \quad \text{for all} \quad n \in \mathbb{N} \tag{3.17}$$

is a geometric series. In this case $a = a_1$ (the first term of the series).

3.2.4. Harmonic Series. Harmonic series is the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = \sum_{n=1}^{+\infty} \frac{1}{n}.$$

Again, to explore the convergence of this series we have to study its partial sums:

$$\begin{aligned}
&S_1 = 1, && S_2 = 1 + \frac{1}{2}, \\
&S_3 = 1 + \frac{1}{2} + \frac{1}{3}, && S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \\
&S_5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}, && S_6 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}, \\
&S_7 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}, && S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}, \\
&\quad\vdots \\
&S_n == 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n} \\
&\quad\vdots
\end{aligned}$$

Since $S_{n+1} - S_n = \dfrac{1}{n+1} > 0$ the sequence $\{S_n\}_{n=1}^{+\infty}$ is increasing.

Next we will prove that the sequence $\{S_n\}_{n=1}^{+\infty}$ is not bounded. We will consider only the positive integers which are powers of 2: $2, 4, 8, \ldots, 2^k, \ldots$. The following inequalities hold:

$$S_2 = 1 + \frac{1}{2} \geq 1 + \frac{1}{2} \qquad =1 + 1\,\frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + 2\,\frac{1}{4} \qquad =1 + 2\,\frac{1}{2}$$

$$\begin{aligned} S_8 &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} \\ &\geq 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = 1+\frac{1}{2}+2\,\frac{1}{4}+4\,\frac{1}{8} &&= 1+3\,\frac{1}{2} \\ S_{16} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16} \\ &\geq 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = 1+\frac{1}{2}+2\,\frac{1}{4}+4\,\frac{1}{8}+8\,\frac{1}{16} &&= 1+4\,\frac{1}{2} \end{aligned}$$

Continuing this reasoning we conclude that for every $k \in \mathbb{N}$ the following formula holds:

$$\begin{aligned} S_{2^k} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{8}+\cdots+\frac{1}{2^{k-1}}+\frac{1}{2^{k-1}+1}+\cdots+\frac{1}{2^k} \\ &\geq 1+\frac{1}{2}+2\,\frac{1}{4}+4\,\frac{1}{8}+8\,\frac{1}{16}+\cdots+2^{k-1}\,\frac{1}{2^k} &&= 1+k\,\frac{1}{2} \end{aligned}$$

Thus

$$S_{2^k} \geq 1+k\,\frac{1}{2} \quad \text{for all} \quad k \in \mathbb{N}. \tag{3.18}$$

This formula implies that the sequence $\{S_n\}_{n=1}^{+\infty}$ is not bounded. Namely, let M be an arbitrary real number. We put $j = \max\{2\,\text{floor}(M), 1\}$. Then

$$j \geq 2\,\text{floor}(M) > 2(M-1).$$

Therefore,

$$1 + j\,\frac{1}{2} > M.$$

Together with the inequality (3.18) this implies that

$$S_{2^j} > M.$$

Thus for an arbitrary real number M there exists a positive integer $n = 2^j$ such that $S_n > M$. This proves that the sequence $\{S_n\}_{n=1}^{+\infty}$ is not bounded and therefore it is not convergent.

In conclusion:

- The harmonic series diverges.

3.2.5. Telescoping series. The next example is an example of a series for which we can find a simple formula for the sequence of its partial sums and easily explore the convergence of that sequence. Examples of this kind are called telescoping series.

EXAMPLE 3.23. Prove that the series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$ converges and find its sum.

SOLUTION. We need to examine the series of partial sums of this series:

$$S_n = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}, \quad n \in \mathbb{N}.$$

It turns out that it is easy to find the sum S_n if we use the partial fraction decomposition for each of the terms of the series:

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \quad \text{for all} \quad k \in \mathbb{N}.$$

Now we calculate:

$$\begin{aligned} S_n &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\,(n+1)} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}\,. \end{aligned}$$

Thus $S_n = 1 - \dfrac{1}{n+1}$ for all $n \in \mathbb{N}$. Using the algebra of limits we conclude that

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\left(1 - \frac{1}{n+1}\right) = 1\,.$$

Therefore the series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n\,(n+1)}$ converges and its sum is 1:

$$\sum_{n=1}^{+\infty} \frac{1}{n\,(n+1)} = 1\,.$$

□

EXERCISE 3.24. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

(a) $\displaystyle\sum_{n=1}^{+\infty} 6\left(\frac{2}{3}\right)^{n-1}$ (b) $\displaystyle\sum_{n=1}^{+\infty} \frac{(-2)^{n+3}}{5^{n-1}}$ (c) $\displaystyle\sum_{n=0}^{+\infty} \frac{(\sqrt{2})^n}{2^{n+1}}$ (d) $\displaystyle\sum_{n=1}^{+\infty} \frac{e^{n+3}}{\pi^{n-1}}$

(e) $\displaystyle\sum_{n=1}^{+\infty} \frac{2^{2n-1}}{\pi^n}$ (f) $\displaystyle\sum_{n=1}^{+\infty} \frac{5}{2\,n}$ (g) $\displaystyle\sum_{n=0}^{+\infty} (\sin 1)^n$ (h) $\displaystyle\sum_{n=0}^{+\infty} \frac{2}{n^2+4n+3}$

(i) $\displaystyle\sum_{n=0}^{+\infty} (\cos 1)^n$ (j) $\displaystyle\sum_{n=2}^{+\infty} \frac{2}{n^2-1}$ (k) $\displaystyle\sum_{n=0}^{+\infty} (\tan 1)^n$ (l) $\displaystyle\sum_{n=1}^{+\infty} \ln\left(1 + \frac{1}{n}\right)$

3.2.6. Decimal numbers. A digit is an integer from the set $\mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $d : \mathbb{N} \to \mathbb{D}$ be a sequence of digits. A decimal number with decimal digits $d_1, d_2, d_3, \ldots, d_n, \ldots$ is in fact the infinite series:

$$0.d_1 d_2 d_3 \ldots d_n \ldots \; = \; \sum_{n=1}^{+\infty} \frac{d_n}{10^n}\,.$$

Consider the partial sums:

$$S_n = \sum_{k=1}^{n} \frac{d_k}{10^k} \quad \text{where} \quad n \in \mathbb{N}.$$

Then, for all $n \in \mathbb{N}$ we have

$$S_{n+1} - S_n = \frac{d_{n+1}}{10^{n+1}} \geq 0.$$

Hence $S_n \leq S_{n+1}$ for all $n \in \mathbb{N}$. Thus, the sequence $\{S_n\}$ is nondecreasing. We now prove that $\{S_n\}$ is bounded above by 1. Since $d_k \leq 9$ for all $k \in \mathbb{N}$ we have

$$S_n = \sum_{k=1}^{n} \frac{d_k}{10^k} \leq \sum_{k=1}^{n} \frac{9}{10^k} = \frac{9}{10} \sum_{k=0}^{n-1} \frac{1}{10^k} = \frac{9}{10} \frac{1 - \frac{1}{10^n}}{1 - \frac{1}{10}} \leq \frac{9}{10} \frac{1}{1 - \frac{1}{10}} = 1.$$

for all $n \in \mathbb{N}$.

It turns out that each decimal number with digits that repeat leads to a geometric series. We use the following abbreviation:

$$0.\overline{d_1 d_2 d_3 \dots d_k} = 0.d_1 d_2 d_3 \dots d_k d_1 d_2 d_3 \dots d_k d_1 d_2 d_3 \dots d_k d_1 d_2 d_3 \dots d_k \dots \ .$$

Rather than proving this in general and finding to which rational number the preceding series converges, we leave it to the reader to figure out several examples in the exercises.

Exercise 3.25. Express the following real nubers as ratios of positive integers.
(a) $0.\overline{9} = 0.999\dots$ (b) $0.\overline{7} = 0.777\dots$ (c) $0.\overline{712}$ (d) $0.\overline{5432}$

3.2.7. Basic properties of infinite series. An immediate consequence of the definition of a convergent series is the following theorem

Theorem 3.26. *If a series* $\sum_{n=1}^{+\infty} a_n$ *converges, then* $\lim_{n\to+\infty} a_n = 0$.

Proof. Assume that $\sum_{n=1}^{+\infty} a_n$ is a convergent series. By the definition of convergence of a series its sequence of partial sums $\{S_n\}_{n=1}^{+\infty}$ converges to some number S: $\lim_{n\to+\infty} S_n = S$. Then also $\lim_{n\to+\infty} S_{n-1} = S$. Now using the formula

$$a_n = S_n - S_{n-1}, \quad \text{for all} \quad n \in \mathbb{N} \setminus \{1\},$$

and the algebra of limits we conclude that

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} S_n - \lim_{n\to+\infty} S_{n-1} = S - S = 0.$$

□

Warning: The preceding theorem cannot be used to conclude that a particular series converges. Notice that in this theorem it is assumed that $\sum_{n=1}^{+\infty} a_n$ is a convergent.

On a positive note: Theorem 3.26 can be used to conclude that a given series diverges: If we know that $\lim_{n\to+\infty} a_n = 0$ is not true, then we can conclude that the series $\sum_{n=1}^{+\infty} a_n$ diverges. This is a useful test for divergence.

Theorem 3.27 (The Test for Divergence). *If the sequence* $\{a_n\}_{n=1}^{+\infty}$ *does not converge to* 0, *then the series* $\sum_{n=1}^{+\infty} a_n$ *diverges.*

Example 3.28. Determine whether the infinite series $\sum_{n=1}^{+\infty} \cos\left(\frac{1}{n}\right)$ converges or diverges.

SOLUTION. Just perform the divergence test:

$$\lim_{n\to+\infty} \cos\left(\frac{1}{n}\right) = 1 \neq 0\,.$$

Therefore the series $\displaystyle\sum_{n=1}^{+\infty} \cos\left(\frac{1}{n}\right)$ diverges. □

EXAMPLE 3.29. Determine whether the infinite series $\displaystyle\sum_{n=1}^{+\infty} \frac{n^{(-1)^n}}{n+1}$ converges or diverges.

SOLUTION. Consider the sequence $\displaystyle\left\{\frac{n^{(-1)^n}}{n+1}\right\}_{n=1}^{+\infty}$:

$$\frac{1}{1\cdot 2},\ \frac{2}{3},\ \frac{1}{3\cdot 4},\ \frac{4}{5},\ \frac{1}{5\cdot 6},\ \frac{6}{7},\ \frac{1}{7\cdot 8},\ \frac{8}{9},\ \frac{1}{9\cdot 10},\ \frac{10}{11},\ \frac{1}{11\cdot 12},\ \frac{12}{13},\ \ldots,\ \frac{1}{(2k-1)\cdot 2k},\ \frac{2k}{2k+1},\ \ldots \tag{3.19}$$

Without giving a formal proof we can tell that this sequence diverges. In my informal language the sequence (3.19) is not constantish since it can not decide whether to be close to 0 or 1.

Therefore the series $\displaystyle\sum_{n=1}^{+\infty} \frac{n^{(-1)^n}}{n+1}$ diverges. □

REMARK 3.30. The divergence test can not be used to answer whether the series $\displaystyle\sum_{n=1}^{+\infty} \sin\left(\frac{1}{n}\right)$ converges or diverges. It is clear that $\displaystyle\lim_{n\to+\infty} \sin\left(\frac{1}{n}\right) = 0$. Thus we can not use the test for divergence.

THEOREM 3.31 (The Algebra of Convergent Infinite Series). *Assume that* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *and* $\displaystyle\sum_{n=1}^{+\infty} b_n$ *are convergent series. Let* c *be a real number. Then the series*

$$\sum_{n=1}^{+\infty} c\, a_n, \qquad \sum_{n=1}^{+\infty} (a_n + b_n), \qquad \textit{and} \qquad \sum_{n=1}^{+\infty} (a_n - b_n),$$

are convergent series and the following formulas hold

$$\sum_{n=1}^{+\infty} c\, a_n = c \sum_{n=1}^{+\infty} a_n,$$

$$\sum_{n=1}^{+\infty} (a_n + b_n) = \sum_{n=1}^{+\infty} a_n + \sum_{n=1}^{+\infty} b_n, \quad \textit{and}$$

$$\sum_{n=1}^{+\infty} (a_n - b_n) = \sum_{n=1}^{+\infty} a_n - \sum_{n=1}^{+\infty} b_n\,.$$

REMARK 3.32. The fact that we write $\sum_{n=1}^{+\infty} b_n$ does not necessarily mean that $\sum_{n=1}^{+\infty} b_n$ is a genuine infinite series.

For example, let m be a positive integer and assume that $b_n = 0$ for all $n > m$. Then $\sum_{n=1}^{+\infty} b_n = \sum_{n=1}^{m} b_n$. In this case the series $\sum_{n=1}^{+\infty} b_n$ is clearly convergent. If $\sum_{n=1}^{+\infty} a_n$ is a convergent (genuine) infinite series, then Theorem 3.31 implies that the infinite series $\sum_{n=1}^{+\infty}(a_n + b_n)$ is convergent and

$$\sum_{n=1}^{+\infty}(a_n + b_n) = \sum_{n=1}^{+\infty} a_n + \sum_{n=1}^{m} b_n .$$

This in particular means that the nature of convergence of an infinite series can not be changed by changing finitely many terms of the series.

For example, let m be a positive integer. Then:

The series $\sum_{n=1}^{+\infty} a_n$ converges if and only if the series $\sum_{k=1}^{+\infty} a_{m+k}$ converges.

Moreover, if $\sum_{n=1}^{+\infty} a_n$ converges, then the following formula holds

$$\sum_{n=1}^{+\infty} a_n = \sum_{j=1}^{m} a_j \; + \; \sum_{k=1}^{+\infty} a_{m+k} .$$

EXAMPLE 3.33. Prove that the series $\sum_{n=1}^{+\infty}\left(\frac{\pi}{n(n+1)} - \frac{1}{2^n}\right)$ converges and find its sum.

EXERCISE 3.34. Determine whether the series is convergent or divergent. If a series is convergent find its sum.

(a) $\sum_{n=1}^{+\infty} \frac{n}{n+1}$ (b) $\sum_{n=1}^{+\infty} \arctan n$ (c) $\sum_{n=0}^{+\infty} \frac{3^n + 2^n}{5^{n+1}}$ (d) $\sum_{n=2}^{+\infty}\left(\frac{3}{n^2-1} + \frac{\pi}{e^n}\right)$

(e) $\sum_{n=0}^{+\infty} \frac{e^n + \pi^n}{2^{2n-1}}$ (f) $\sum_{n=1}^{+\infty} n \sin\left(\frac{1}{n}\right)$ (g) $\sum_{n=0}^{+\infty} \frac{(n+1)^2}{n^2+1}$ (h) $\sum_{n=0}^{+\infty} \left((0.9)^n + (0.1)^n\right)$

EXERCISE 3.35. Express the following sums as ratios of positive integers and as repeating decimal numbers.

(a) $0.\overline{47} + 0.\overline{5}$ (b) $0.\overline{499} + 0.\overline{47}$ (c) $0.\overline{499} + 0.\overline{503}$

3.3. Convergence Tests

Warning: All series in the next two subsections have positive terms! Do not use the tests from these sections for series with some negative terms.

3.3.1. Direct Comparison Test. The convergence of the geometric series in Subsection 3.2.2 and the telescopic series in Subsection 3.2.5 was established by calculating the limits of their partial sums. This is not possible for most series. For example we will soon prove that the series

$$\sum_{n=1}^{+\infty} \frac{1}{n^2}$$

converges. To understand why the sum of this series is exactly $\dfrac{\pi^2}{6}$ you need to read a paper that I posted on the class website. You have the background knowledge to understand this proof, but the complete proof is on a longer side.

I hope that you have done your homework and that you proved that the series

$$\sum_{n=2}^{+\infty} \frac{1}{n^2-1}$$

converges and that you found its sum. If you didn't here is a way to do it: (It turns out that this is a telescoping series.)

Let

$$S_n = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots + \frac{1}{n^2-1}.$$

Since $S_{n+1} - S_n = \dfrac{1}{(n+1)^2-1} > 0$ the sequence $\{S_n\}_{n=2}^{+\infty}$ is increasing.

For every $k \in \mathbb{N}$ such that $k > 1$ we have the following partial fractions decomposition

$$\frac{1}{k^2-1} = \frac{1}{(k-1)(k+1)} = \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right).$$

Next we use this formula to simplify the formula for the n-th partial sum

$$\begin{aligned} S_n &= \sum_{k=2}^{n} \frac{1}{k^2-1} = \sum_{k=2}^{n} \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) = \frac{1}{2}\sum_{k=2}^{n}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) \\ &= \frac{1}{2}\left(\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right)\right) \\ &= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right) \\ &= \frac{1}{2}\left(\frac{3}{2} - \frac{2n+1}{n(n+1)}\right) = \frac{3}{4} - \frac{2n+1}{2n(n+1)}. \end{aligned}$$

Using the algebra of limits we calculate

$$\lim_{n\to+\infty} \frac{2n+1}{2n(n+1)} = \lim_{n\to+\infty} \frac{\dfrac{2n+1}{n^2}}{\dfrac{2n(n+1)}{n^2}} = \lim_{n\to+\infty} \frac{\dfrac{2}{n} + \dfrac{1}{n^2}}{2\dfrac{n+1}{n}} = \frac{0+0}{2\cdot 1} = 0.$$

Therefore, using the algebra of limits again, we calculate

$$\lim_{n\to+\infty} S_n = \frac{3}{4} - 0 = \frac{3}{4}.$$

Clearly $S_n < \dfrac{3}{4}$ for all $n \in \mathbb{N} \setminus \{1\}$.

Now consider the series

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

Let

$$T_n = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2}\,.$$

The fact that $T_{n+1} - T_n = \dfrac{1}{(n+1)^2} > 0$ implies that the sequence $\{T_n\}_{n=1}^{+\infty}$ is increasing.

Since

$$\frac{1}{4} < \frac{1}{3}, \quad \frac{1}{9} < \frac{1}{8}, \quad \frac{1}{16} < \frac{1}{15}, \quad \ldots, \quad \frac{1}{n^2} < \frac{1}{n^2-1},$$

we conclude that

$$T_n = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} < 1 + \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots + \frac{1}{n^2-1} = 1 + S_n < 1 + \frac{3}{4}\,.$$

Thus $T_n < \dfrac{7}{4}$ for all $n \in \mathbb{N} \setminus \{1\}$. Since the sequence $\{T_n\}_{n=1}^{+\infty}$ is increasing and bounded above it converges by Theorem 3.18. Thus the series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2}$ converges and its sum is $< \dfrac{7}{4}$.

The principle demonstrated in the above example is the core of the following comparison theorem.

Theorem 3.36 (The Direct Comparison Test). *Let* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *and* $\displaystyle\sum_{n=1}^{+\infty} b_n$ *be infinite series with positive terms. Assume that*

$$a_n \leq b_n \qquad \textit{for all} \quad n \in \mathbb{N}.$$

(a) *If* $\displaystyle\sum_{n=1}^{+\infty} b_n$ *converges, then* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *converges and* $\displaystyle\sum_{n=1}^{+\infty} a_n \leq \sum_{n=1}^{+\infty} b_n$.

(b) *If* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *diverges, then* $\displaystyle\sum_{n=1}^{+\infty} b_n$ *diverges.*

3.3.2. Limit Comparison Test. Sometimes the following comparison theorem is easier to use.

Theorem 3.37 (The Limit Comparison Test). *Let* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *and* $\displaystyle\sum_{n=1}^{+\infty} b_n$ *be infinite series with positive terms. Assume that*

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = L.$$

If $\displaystyle\sum_{n=1}^{+\infty} b_n$ *converges, then* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *converges. Or, equivalently, if* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *diverges, then* $\displaystyle\sum_{n=1}^{+\infty} b_n$ *diverges.*

Example 3.38. Determine whether the series $\displaystyle\sum_{n=1}^{+\infty} \frac{n+1}{\sqrt{1+n^6}}$ converges or diverges.

Solution. The dominant term in the numerator is n and the dominant term in the denominator is $\sqrt{n^6} = n^3$. This suggests that this series behaves as the convergent series $\sum_{n=1}^{+\infty} \frac{1}{n^2}$. Since we are trying to prove convergence we will take

$$a_n = \frac{n+1}{\sqrt{1+n^6}} \quad \text{and} \quad b_n = \frac{1}{n^2}$$

in the Limit Comparison Test. Now calculate:

$$\lim_{n\to+\infty} \frac{\frac{n+1}{\sqrt{1+n^6}}}{\frac{1}{n^2}} = \lim_{n\to+\infty} \frac{n^2(n+1)}{\sqrt{1+n^6}} = \lim_{n\to+\infty} \frac{\frac{n^2(n+1)}{n^3}}{\frac{\sqrt{1+n^6}}{n^3}} = \lim_{n\to+\infty} \frac{1+\frac{1}{n}}{\sqrt{\frac{1}{n^6}+1}} = 1\,.$$

In the last step we used the algebra of limits and the fact that

$$\lim_{n\to+\infty} \sqrt{\frac{1}{n^6}+1} = 1$$

which needs a proof by definition.

Since we proved that $\lim_{n\to+\infty} \frac{\frac{n+1}{\sqrt{1+n^6}}}{\frac{1}{n^2}} = 1$ and since we know that $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ is convergent, the Limit Comparison Test implies that the series $\sum_{n=1}^{+\infty} \frac{n+1}{\sqrt{1+n^6}}$ converges. □

3.3.3. Integral Comparison Test. In the next theorem we compare an infinite series with an improper integral of a positive function. Here it is presumed that we know how to determine the convergence or divergence of the improper integral involved.

Theorem 3.39 (The Integral Test). *Suppose that $x \mapsto f(x)$ is a continuous positive, decreasing function defined on the interval $[1, +\infty)$. Assume that $a_n = f(n)$ for all $n \in \mathbb{N}$. Then the following statements are equivalent*

(a) *The integral* $\int_1^{+\infty} f(x)\,dx$ *converges.*

(b) *The series* $\sum_{n=1}^{+\infty} a_n$ *converges.*

At this point we assume that you are familiar with improper integrals and that you know how to decide whether an improper integral converges or diverges.

We will use this test in two different forms:

- Prove that the integral $\int_1^{+\infty} f(x)\,dx$ converges. Conclude that the series $\sum_{n=1}^{+\infty} a_n$ converges.
- Prove that the integral $\int_1^{+\infty} f(x)\,dx$ diverges. Conclude that the series $\sum_{n=1}^{+\infty} a_n$ diverges.

EXAMPLE 3.40 (Convergence of p-series). Let p be a real number. The p-series $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

SOLUTION. Let $n > 1$. Then the function $x \mapsto n^x$ is an increasing function. Therefore, if $p < 1$, then $n^p < n$. Consequently,

$$\frac{1}{n^p} > \frac{1}{n}, \quad \text{for all} \quad n > 1 \quad \text{and} \quad p < 1\,.$$

Since the series $\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges, the Comparison Test implies that the series $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ diverges for all $p \leq 1$.

Now assume that $p > 1$. Consider the function $f(x) = \frac{1}{x^p}$, $x > 0$. This function is a continuous, decreasing, positive function. Let me calculate the improper integral involved in the Integral Test for convergence:

$$\begin{aligned}\int_1^{+\infty} \frac{1}{x^p}\,dx &= \lim_{t\to+\infty} \int_1^t \frac{1}{x^p}\,dx = \lim_{t\to+\infty} \frac{1}{1-p}\,\frac{1}{x^{p-1}}\,\Bigg|_1^t \\ &= \frac{1}{1-p} \lim_{t\to+\infty} \left(\frac{1}{t^{p-1}} - 1\right) = \frac{1}{1-p}(-1) = \frac{1}{p-1}\end{aligned}$$

Thus this improper integral converges. Notice that the condition $p > 1$ was essential to conclude that $\lim_{t\to+\infty} \frac{1}{t^{p-1}} = 0$. Since $\frac{1}{n^p} = f(n)$ for all $n \in \mathbb{N}$, the Integral Test implies that the series $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ converges for $p > 1$. □

REMARK 3.41. We have not proved this for all $p > 1$ the function $f(x) = \frac{1}{x^p}$, $x > 0$, is continuous. One way to prove that for an arbitrary $a \in \mathbb{R}$ the function $x \mapsto x^a$, $x > 0$ is continuous is to use the identity

$$x^a = e^{a \ln x}, \qquad x > 0.$$

This identity shows that the function $x \mapsto x^a$, $x > 0$ is a composition of the function $\exp(x) = e^x$, $x \in \mathbb{R}$ and the function $x \mapsto a \ln x$, $x > 0$. The later function is continuous by the algebra of continuous functions: It is a product of a constant a and a continuous function ln. We proved that exp is continuous. By Theorem 2.59 a composition of continuous function is continuous. Consequently $x \mapsto x^a$, $x > 0$ is continuous.

EXERCISE 3.42. Determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{+\infty} \frac{1}{n\sqrt{n}}$	(b) $\sum_{n=1}^{+\infty} n e^{-n^2}$	(c) $\sum_{n=2}^{+\infty} \frac{1}{n \ln n}$	(d) $\sum_{n=1}^{+\infty} \frac{\ln n}{n\sqrt{n}}$
(e) $\sum_{n=2}^{+\infty} \frac{1}{n(\ln n)^b}$	(f) $\sum_{n=1}^{+\infty} \frac{1}{n!}$	(g) $\sum_{n=1}^{+\infty} \sin\left(\frac{1}{n}\right)$	(h) $\sum_{n=2}^{+\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$
(i) $\sum_{n=1}^{+\infty} \frac{1}{n} \cos\left(\frac{1}{n}\right)$	(j) $\sum_{n=0}^{+\infty} \frac{\pi + e^n}{e + \pi^n}$	(k) $\sum_{n=1}^{+\infty} \frac{n!}{n^n}$	(l) $\sum_{n=0}^{+\infty} \frac{n^2+1}{\sqrt{n^7+n^3+1}}$

For the series in (e) find all numbers b for which the series converges.

EXERCISE 3.43. A digit is a number from the set $\{0,1,2,3,4,5,6,7,8,9\}$. A decimal number with digits $d_1, d_2, d_3, \ldots, d_n, \ldots$ is in fact the infinite series:

$$0.d_1d_2d_3\ldots d_n\ldots \;=\; \sum_{n=1}^{+\infty} \frac{d_n}{10^n}\,.$$

Use a theorem from this section to prove that the series above always converges.

3.3.4. Ratio and root tests. Warning: All series in this section have positive terms! Do not use the tests from this section for series with negative terms.

In Subsection 3.2.3 we pointed out (see (3.17)) that a series

$$\sum_{n=1}^{+\infty} a_n \quad \text{for which} \quad \frac{a_{n+1}}{a_n} = r \quad \text{for all} \quad \in \mathbb{N}$$

is a **geometric series**. Consequently, if $|r| < 1$ this series is convergent, and it is divergent if $|r| \geq 1$.

Testing the series $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{3^n - 2^{n+1}}$ using this criteria leads to the ratio

$$\frac{\dfrac{1}{3^{n+1}-2^{n+2}}}{\dfrac{1}{3^n-2^{n+1}}} = \frac{3^n-2^{n+1}}{3^{n+1}-2^{n+2}} = \frac{3^n\left(1-2\left(\dfrac{2}{3}\right)^n\right)}{3^{n+1}\left(1-2\left(\dfrac{2}{3}\right)^n\right)} = \frac{1}{3}\;\frac{1-2\left(\dfrac{2}{3}\right)^n}{1-2\left(\dfrac{2}{3}\right)^{n+1}}$$

which certainly is not constant, but it is "constantish." I propose that series for which the ratio a_{n+1}/a_n is not constant but constantish, should be called "geometrish." The following theorem tells that convergence and divergence of these series is determined similarly to geometric series.

THEOREM 3.44 (The Ratio Test). *Assume that* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *is a series with positive terms and that*

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = R.$$

Then

(a) *If* $R < 1$, *then the series converges.*
(b) *If* $R > 1$, *then the series diverges.*

Another way to recognize a geometric series is:

$$\text{A series} \quad \sum_{n=1}^{+\infty} a_n \quad \text{for which} \quad \sqrt[n]{\frac{a_{n+1}}{a_1}} = r \quad \text{for all} \quad \in \mathbb{N}$$

is a **geometric series**. Consequently, if $|r| < 1$ this series is convergent, and it is divergent if $|r| \geq 1$.

Testing the series $\sum_{n=0}^{+\infty} \left(\frac{1+n}{1+2n}\right)^n$ using this criteria leads to the root

$$\sqrt[n]{\left(\frac{1+n}{1+2n}\right)^n} = \frac{1+n}{1+2n} = \frac{\frac{1}{n}+1}{\frac{1}{n}+2}$$

which certainly is not constant, but it is "constantish."

THEOREM 3.45 (The Root Test). *Assume that* $\sum_{n=1}^{+\infty} a_n$ *is a series with positive terms and that*

$$\lim_{n\to+\infty} \sqrt[n]{a_n} = R\,.$$

Then

(a) *If* $R < 1$, *then the series converges.*
(b) *If* $R > 1$, *then the series diverges.*

REMARK 3.46. Notice that in both the ratio test and the root test if the limit $R = 1$ we can conclude neither divergence nor convergence. In this case the test is inconclusive.

EXERCISE 3.47. Determine whether the series is convergent or divergent.

(a) $\sum_{n=2}^{+\infty} \frac{1}{2^n-3}$	(b) $\sum_{n=1}^{+\infty} \left(\frac{n+2}{2n-1}\right)^n$	(c) $\sum_{n=1}^{+\infty} \frac{4^n}{3^{2n-1}}$	(d) $\sum_{n=1}^{+\infty} \frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)}$
(e) $\sum_{n=1}^{+\infty} \frac{3^n n^2}{n!}$	(f) $\sum_{n=1}^{+\infty} e^{-n} n!$	(g) $\sum_{n=1}^{+\infty} \frac{e^{1/n}}{n^2}$	(h) $\sum_{n=1}^{+\infty} \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)}$
(i) $\sum_{n=1}^{+\infty} \frac{(n!)^2}{(2n)!}$	(j) $\sum_{n=1}^{+\infty} \frac{2\,n^{2n}}{(3n^2+1)^n}$	(k) $\sum_{n=1}^{+\infty} \frac{2^{3n}}{3^{2n}}$	(l) $\sum_{n=1}^{+\infty} \frac{1}{(\arctan n)^n}$
(m) $\sum_{n=1}^{+\infty} \frac{n^2}{2^n}$	(n) $\sum_{n=1}^{+\infty} \frac{(n+1)^2}{n2^n}$	(o) $\sum_{n=1}^{+\infty} \frac{a^n}{n!}$	(p) $\sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$

For some of the problems you might need to use tests from previous sections.

■ I intentionally start a new page here. ■

3.3.5. Alternating infinite series. In the previous two sections we considered only series with positive terms. In this section we consider series with both positive and negative terms which alternate: positive, negative, positive, etc. Such series are called **alternating series**. For example

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots+(-1)^{n+1}\frac{1}{n}+\cdots=\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n} \tag{3.20}$$

$$1-1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}-\frac{1}{3}+\frac{1}{7}-\frac{1}{4}+\frac{1}{8}-\frac{1}{5}+\frac{1}{9}-\frac{1}{6}+\cdots=\sum_{n=1}^{+\infty}\frac{4(-1)^{n+1}}{n\big(3+(-1)^{n+1}\big)} \tag{3.21}$$

$$2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\frac{6}{5}-\frac{7}{6}+\cdots+(-1)^{n+1}\frac{n+1}{n}+\cdots=\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{n+1}{n} \tag{3.22}$$

THEOREM 3.48 (The Alternating Series Test). *If the alternating series*

$$a_1-a_2+a_3-a_4+\cdots+(-1)^{n+1}a_n+\cdots=\sum_{n=1}^{+\infty}(-1)^{n+1}a_n$$

satisfies the following three conditions:

(i) *for all* $n\in\mathbb{N}$ *we have* $a_n>0$;
(ii) *for all* $n\in\mathbb{N}$ *we have* $a_{n+1}\leq a_n$,
(iii) $\lim\limits_{n\to+\infty}a_n=0$,

then the series $\sum\limits_{n=1}^{+\infty}(-1)^{n+1}a_n$ *converges.*

PROOF. Assume that a sequence $\{a_n\}$ satisfies (i), (ii) and (iii).

By the definition of convergence the assumption (iii) implies that for every $\epsilon>0$ there exists $N_a(\epsilon)$ such that

$$\forall n\in\mathbb{N}\quad n>N_a(\epsilon)\qquad\Rightarrow\qquad |a_n-0|<\epsilon.$$

Since $a_n>0$, the last implication can be simplified as follows

$$\forall n\in\mathbb{N}\quad n>N_a(\epsilon)\qquad\Rightarrow\qquad a_n<\epsilon. \tag{3.23}$$

We need to show that the sequence of partial sums

$$S_n=a_1-a_1-a_2+a_3-a_4+\cdots+(-1)^{n+1}a_n,\qquad n\in\mathbb{N},$$

converges.

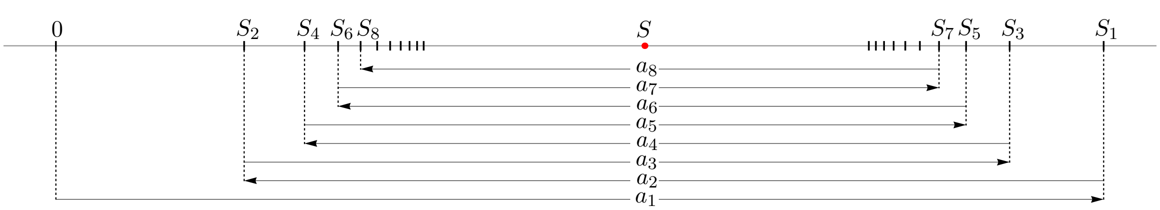

FIG. 4. The partial sums of an alternating series on a number line

As Figure 4 suggests, each even-indexed partial sum is less than each odd-indexed partial sum. That is

$$\forall j \in \mathbb{N} \;\; \forall k \in \mathbb{N} \qquad S_{2j} < S_{2k-1} \tag{3.24}$$

Next we will prove this claim. Let k and j be arbitrary positive integers. First assume $k \leq j$. Then $2k-1 < 2j$ and

$$S_{2j} - S_{2k-1} = \sum_{i=2k}^{2j} (-1)^{i+1} a_i = \big(-a_{2k} + a_{2k+1}\big) + \cdots + \big(-a_{2j-2} + a_{2j-1}\big) - a_{2j}.$$

In the last sum each of the numbers in parenthesis is nonpositive. Therefore,

$$S_{2j} - S_{2k-1} = \sum_{i=2k}^{2j} (-1)^{i+1} a_i \leq -a_{2j} < 0.$$

Hence $S_{2j} < S_{2k-1}$ in this case. Now assume that $k > j$. Then $2k-1 > 2j$ and

$$S_{2k-1} - S_{2j} = \sum_{i=2j+1}^{2k-1} (-1)^{i+1} a_i = \big(a_{2j+1} - a_{2j+2}\big) + \cdots + \big(a_{2k-3} - a_{2k-2}\big) + a_{2k-1}.$$

In the last sum each of the numbers in parenthesis is nonnegative. Therefore,

$$S_{2k-1} - S_{2j} = \sum_{i=2j+1}^{2k-1} (-1)^{i+1} a_i \geq a_{2k-1} > 0.$$

Hence $S_{2j} < S_{2k-1}$ in this case as well. This completes the proof of (3.24).

Define

$$A = \big\{ S_{2j} \, : \, j \in \mathbb{N} \big\} \quad \text{and} \quad B = \big\{ S_{2k-1} \, : \, k \in \mathbb{N} \big\}.$$

With this notation (3.24) can be restated as

$$\forall a \in A \;\; \forall b \in B \qquad a < b.$$

Since clearly $A \neq \emptyset$ and $B \neq \emptyset$ we can apply the Completeness Axiom to the sets A and B. By the Completeness Axiom there exists $c \in \mathbb{R}$ such that

$$\forall a \in A \;\; \forall b \in B \qquad a \leq c \leq b.$$

The last inequality in fact says

$$\forall j \in \mathbb{N} \;\; \forall k \in \mathbb{N} \qquad S_{2j} \leq c \leq S_{2k-1}. \tag{3.25}$$

Let $n \in \mathbb{N}$ be arbitrary. If n is even, then (3.25) yields

$$S_n \leq c \leq S_{n+1} = S_n + a_{n+1}.$$

Therefore $c - S_n \leq a_{n+1}$. If n is odd, then (3.25) yields

$$S_n - a_{n+1} = S_{n+1} \leq c \leq S_n.$$

Therefore, $S_n - c \leq a_{n+1}$. Thus, for all $n \in \mathbb{N}$ we have

$$|S_n - c| \leq a_{n+1}. \tag{3.26}$$

Now, using (3.26) and (3.23) we can prove $\lim_{n\to+\infty} S_n = c$. Let $\epsilon > 0$ be arbitrary. Set $N(\epsilon) = N_a(\epsilon)$. Assume $n \in \mathbb{N}$ and $n > N(\epsilon) = N_a(\epsilon)$. Then also $n+1 \in \mathbb{N}$ and $n+1 > N_a(\epsilon)$. By the implication in (3.23) we conclude $a_{n+1} < \epsilon$. This inequlity, together with (3.26), implies $|S_n - c| < \epsilon$. Thus, we proved that

$$\forall \epsilon > 0 \quad \exists\, N(\epsilon) \in \mathbb{R} \quad \forall n \in \mathbb{N} \;\; n > N(\epsilon) \quad \Rightarrow \quad |S_n - c| < \epsilon.$$

This proves that the sequence $\{S_n\}_{n=1}^{+\infty}$ converges and hence the alternating series converges. □

EXAMPLE 3.49. Prove that the series in (3.20) converges. The series in (3.20) is called the *alternating harmonic series*.

SOLUTION. We verify three conditions of the Alternating Series Test. Here, $a_n = 1/n$, $n \in \mathbb{N}$. First, for all $n \in \mathbb{N}$ we have $n > 0$ and hence $1/n > 0$. Second, since for all $n \in \mathbb{N}$ we have $n+1 > n$, by pizza-party principle $1/(n+1) < 1/n$. Third, $\lim\limits_{n\to+\infty}(1/n) = 0$ is easy to prove. Thus the Alternating Series Test implies that the Alternating Harmonic Series converges. □

REMARK 3.50. There is a nice geometric argument that

$$\sum_{k=1}^{+\infty}(-1)^{k+1}\frac{1}{k} = \ln 2.$$

This argument is based on the fact that the even-indexed partial sums of the Alternating Harmonic Series are in fact right Riemann sums of the integral $\int_1^2 (1/x)dx$:

$$\begin{aligned}
S_{2n} &= \sum_{k=1}^{2n}(-1)^{k+1}\frac{1}{k}\\
&= \sum_{j=1}^{n}\frac{1}{2j-1} - \sum_{j=1}^{n}\frac{1}{2j}\\
&= \sum_{j=1}^{n}\frac{1}{2j-1} + \sum_{j=1}^{n}\frac{1}{2j} - 2\sum_{j=1}^{n}\frac{1}{2j}\\
&= \sum_{k=1}^{2n}\frac{1}{k} - \sum_{j=1}^{n}\frac{1}{j}\\
&= \sum_{k=1}^{n}\frac{1}{n+k}\\
&= \sum_{k=1}^{n}\frac{1}{n}\,\frac{1}{1+\frac{k}{n}}.
\end{aligned}$$

And, similarly, the odd-indexed partial sums of the Alternating Harmonic Series are left Riemann sums of the integral $\int_1^2 (1/x)dx$:

$$S_{2n-1} = \sum_{k=1}^{2n-1}(-1)^{k+1}\tfrac{1}{k} = \sum_{j=1}^{n}\tfrac{1}{2j-1} - \sum_{j=1}^{n-1}\tfrac{1}{2j} = \sum_{j=1}^{n}\tfrac{1}{2j-1} + \sum_{j=1}^{n-1}\tfrac{1}{2j} - 2\sum_{j=1}^{n-1}\tfrac{1}{2j} = \sum_{k=1}^{2n-1}\tfrac{1}{k} - \sum_{j=1}^{n-1}\tfrac{1}{j} = \sum_{k=0}^{n-1}\tfrac{1}{n+k} = \sum_{k=0}^{n-1}\tfrac{1}{n}\,\tfrac{1}{1+\frac{k}{n}}.$$

The details of the argument I will post on the class website.

REMARK 3.51. The Alternating Series Test does not apply to the series in (3.21) since the sequence of numbers

$$1,\ 1,\ \frac{1}{3},\ \frac{1}{2},\ \frac{1}{5},\ \frac{1}{3},\ \frac{1}{7},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{5},\ \frac{1}{9},\ \frac{1}{6},\ \ldots,\ \frac{4}{n\big(3+(-1)^{n+1}\big)},\ \ldots$$

is not non-increasing. Further exploration of the series in (3.21) would show that it diverges.

The Alternating Series Test does not apply to the series in (3.22) since this series does not satisfy the condition (ii):

$$\lim_{n\to+\infty} \frac{n+1}{n} = 1 \neq 0\,.$$

Again this series is divergent by the Test for Divergence.

EXERCISE 3.52. Determine whether the given series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{+\infty} \cos\Big(n\pi + \frac{1}{n}\Big)$ (b) $\displaystyle\sum_{n=0}^{+\infty} \sin\Big(n\frac{\pi}{2}\Big)$ (c) $\displaystyle\sum_{n=1}^{+\infty} \sin\Big(n\pi - \frac{1}{n}\Big)$

(d) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n}\cos\Big(n\pi + \frac{1}{n}\Big)$ (e) $\displaystyle\sum_{n=1}^{+\infty} \ln\Big(1 - \frac{(-1)^n}{n}\Big)$ (f) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n}\sin\Big(n\frac{\pi}{2}\Big)$

(g) $\displaystyle\sum_{n=1}^{+\infty} \sin\Big(n\frac{\pi}{2} + \frac{1}{n}\Big)$ (h) $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n-(-1)^n}$ (i) $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{2n-(-1)^n}$

Several of the exercises in the next section use the Alternating Series Test for convergence. Do those exercises as well.

3.3.6. Absolute and Conditional Convergence. In the previous section we proved that the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + (-1)^{n+1}\frac{1}{n} + \cdots = \sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n} \quad \text{converges.} \tag{3.27}$$

Later on we will see that the sum of this series is $\ln 2$.

Talking about infinite series in class I have often used the analogy with an infinite column in a spreadsheet and finding its sum. A series with positive and negative terms one can interpret as balancing a checkbook with (infinitely) many deposits and withdrawals. Looking at the alternating harmonic series (3.27) we see a sequence of alternating deposits and withdrawals, infinitely many of them. What we proved in the previous section tells that under two conditions on the deposits and withdrawals, although it has infinitely many transactions, this checkbook can be balanced.

Now comes the first surprising fact! Let's calculate how much has been deposited to this account:

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2n-1} + \cdots = \sum_{n=1}^{+\infty} \frac{1}{2n-1}.$$

Since for all $k \in \mathbb{N}$ it holds

$$\frac{1}{2k} < \frac{1}{2k-1},$$

we have that

$$\frac{1}{2}H_n = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k} < \sum_{k=1}^{n}\frac{1}{2k-1}$$

for all $n \in \mathbb{N}$. As the sequence $\{H_n\}$ of harmonic numbers is unbounded, we deduce that the sequence of partial sums of the infinite series $\sum_{n=1}^{+\infty}\frac{1}{2n-1}$ is unbounded. Therefore this series diverges.

Looking at the withdrawals we see

$$-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\cdots-\frac{1}{2n}-\cdots=-\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n}.$$

Again this is a divergent series. In the language of a bank account, we are encountering a suspicious situation: We have an account to which an "infinite" amount of money has been deposited and an "infinite" amount of money has been withdrawn. A simpler way to look at this is to look at the total amount of money that went through this account (one can call this amount the total "activity" in the account):

$$\sum_{n=1}^{+\infty}\left|(-1)^{n+1}\frac{1}{n}\right| = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{n}+\cdots \tag{3.28}$$

This is the harmonic series which is divergent.

Since we know that an infinite amount of money has been deposited to this account we might want to get into the spending mood sooner. So, we rearrange the deposits and the withdrawals; we do two withdrawals after each deposit, keeping the amounts the same. This results in the following infinite series:

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\frac{1}{7}-\frac{1}{14}-\frac{1}{16}+\cdots. \tag{3.29}$$

In any real life checking account just rearranging the deposits and the withdrawals might result in an occasional low balance but the final balance will remain the same. Amazingly this is not always the case with infinite series! (This is the second surprising fact!) For example, the series in (3.29) and the series in (3.27) have identical terms which are arranged differently; in Example 3.49 we proved that the series (3.27) converges and next we will show that the series (3.29) also converges but to a different number.

To be specific, denote the terms of the series (3.29) by $b_n, n \in \mathbb{N}$. Then

$$b_{3k-2}=\frac{1}{2k-1}, \quad b_{3k-1}=-\frac{1}{4k-2}, \quad b_{3k}=-\frac{1}{4k}, \quad k\in\mathbb{N}.$$

It is clear that the series (3.29) has the same terms as the alternating harmonic series. The terms of the alternating harmonic series have been reordered. For $k \in \mathbb{N}$, the term at the positions $2k-1$ (odd-indexed terms) in the alternating harmonic series is at the position $3k-2$ in the series (3.29), the term which is at the position $4k-2$ (a "half" of the even-index terms) in the alternating harmonic series is at the position $3k-1$ in the series (3.29) and the term which is at the position $4k$ (another "half" of the even-index terms) in the alternating harmonic series is at the position $3k$ in the series (3.29).

The following calculation indicates that the sum of the series in (3.29) is $1/2$ of the sum of the alternating harmonic series in (3.27). Let us calculate the $3n$-th partial sum of the series (3.29). Since this sum has $3n$ terms, one-third (exactly n) of them are positive and two-thirds (exactly $2n$) of them are negative. Since this is a finite sum we can rearrange terms as we please. Here is the calculation

$$\begin{aligned} S_{3n} &= 1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\cdots+\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n} \\ &= \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\cdots+\frac{1}{4n-2}-\frac{1}{4n} \end{aligned}$$

$$= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}\right)$$

Hence, $3n$-th partial sum of the series (3.29) is identical to one-half of the $2n$-th partial sum of the alternating harmonic series. Since the sum of the alternating harmonic series is $\ln 2$ we have

$$\lim_{n\to+\infty} S_{3n} = \frac{\ln 2}{2}.$$

Since

$$S_{3n+1} = S_{3n} + \frac{1}{2n+1} \quad \text{and} \quad S_{3n+2} = S_{3n} + \frac{1}{2n+1} - \frac{1}{4n+2} = S_{3n} + \frac{1}{4n+2},$$

we conclude that

$$\lim_{n\to+\infty} S_{3n+1} = \lim_{n\to+\infty} S_{3n+2} = \lim_{n\to+\infty} S_{3n} = \frac{\ln 2}{2}.$$

From the last three equalities one can prove rigorously that

$$\lim_{n\to+\infty} S_n = \frac{\ln 2}{2}.$$

This proves that the series (3.29) converges to $(\ln 2)/2$. That is, just a rearrangement of the terms changed the sum.

This is a remarkable observation: a change of order of summation can change the sum of an infinite series. This feature is closely related to the fact that the total activity of the account expressed in (3.28) is a divergent series. This is a motivation for the following definition.

Definition 3.53. A convergent series $\sum_{n=1}^{+\infty} a_n$ is called **conditionally convergent** if the series of the absolute values of its terms $\sum_{n=1}^{+\infty} |a_n|$ is divergent.

Definition 3.54. A series $\sum_{n=1}^{+\infty} a_n$ is called **absolutely convergent** if the series of the absolute values of its terms $\sum_{n=1}^{+\infty} |a_n|$ is convergent.

Example 3.55. Prove that the series

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \cdots + (-1)^{n+1}\frac{1}{n^2} + \cdots = \sum_{n=1}^{+\infty} (-1)^{n+1}\frac{1}{n^2}$$

is absolutely convergent.

Solution. By the definition of absolute convergence we need to determine the convergence of the series

$$\sum_{n=1}^{+\infty} \left|(-1)^{n+1}\frac{1}{n^2}\right| = \sum_{n=1}^{+\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots$$

This is a p-series with $p = 2$. Therefore this series converges. (Notice that in Subsection 3.3.1 we proved that this series converges by comparing it to a telescoping series.) □

REMARK 3.56. One can interpreted the series in Example 3.55 as a checking account with infinitely many alternating deposits and withdrawals. In this case the total activity of the account is a convergent series. Consequently the total amount deposited

$$1+\frac{1}{9}+\frac{1}{25}+\cdots+\frac{1}{(2n-1)^2}+\cdots=\sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2} \tag{3.30}$$

and the total amount withdrawn

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\cdots+\frac{1}{(2n)^2}+\cdots=\sum_{n=1}^{+\infty}\frac{1}{(2n)^2}=\frac{1}{4}\sum_{n=1}^{+\infty}\frac{1}{n^2} \tag{3.31}$$

are both convergent series. As we can see, the total amount withdrawn is $1/4$ of the total activity of the account. We mentioned before that (this is proved in a paper posted on the class website)

$$\sum_{n=1}^{+\infty}\frac{1}{n^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\cdots=\frac{\pi^2}{6}.$$

Therefore

$$\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n^2}=1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\cdots=\frac{3}{4}\frac{\pi^2}{6}-\frac{1}{4}\frac{\pi^2}{6}=\frac{1}{2}\frac{\pi^2}{6}=\frac{\pi^2}{12}$$

THEOREM 3.57. *If a series* $\sum_{n=1}^{+\infty} a_n$ *is absolutely convergent, then it is convergent.*

PROOF. Assume that $\sum_{n=1}^{+\infty} a_n$ is absolutely convergent, that is assume that $\sum_{n=1}^{+\infty} |a_n|$ is convergent. Then the algebra of convergent series the series $\sum_{n=1}^{+\infty} 2\,|a_n|$ is convergent. Since $-|a_n| \leq a_n \leq |a_n|$, we conclude that

$$0 \leq a_n + |a_n| \leq 2\,|a_n| \quad \text{for all} \quad n \in \mathbb{N}.$$

By the Comparison Test it follows that the series $\sum_{n=1}^{+\infty}\big(a_n+|a_n|\big)$ is convergent. The algebra of convergent series implies that the series

$$\sum_{n=1}^{+\infty}\Big(\big(a_n+|a_n|\big)-|a_n|\Big)=\sum_{n=1}^{+\infty}a_n$$

is also convergent. □

The following stronger versions of the Ratio and the Root test can be applied to any series to determine whether a series converges absolutely or it diverges.

THEOREM 3.58 (The Ratio Test). *Let* $\sum_{n=1}^{+\infty} a_n$ *be a series for which* $\lim_{n\to+\infty}\frac{|a_{n+1}|}{|a_n|}=R$. *Then*

(a) *If* $R<1$, *then the series converges absolutely.*

(b) *If* $R > 1$, *then the series diverges.*

THEOREM 3.59 (The Root Test). *Let* $\sum_{n=1}^{+\infty} a_n$ *be a series for which* $\lim_{n\to+\infty} \sqrt[n]{|a_n|} = R$. *Then*

(a) *If* $R < 1$, *then the series converges absolutely.*
(b) *If* $R > 1$, *then the series diverges.*

Notice that if the root or the ratio test apply to a series, then series either converges absolutely or diverges. This implies that if a series converges conditionally, then either

$$\lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = 1 \quad \text{or} \quad \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} \text{ does not exist,}$$

and also

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|} = 1 \quad \text{or} \quad \lim_{n\to+\infty} \sqrt[n]{|a_n|} \text{ does not exist.}$$

In other words, the root and the ratio test cannot lead to a conclusion that a series converges conditionally.

It turns out that our only tool which can be used to conclude conditional convergence is the alternating series test.

EXERCISE 3.60. Determine whether the given series converges conditionally, converges absolutely or diverges.

(a) $\sum_{n=0}^{+\infty} \frac{\cos(n\pi)}{n^2+1}$ (b) $\sum_{n=0}^{+\infty} \frac{\sin(n\pi/2)}{n+1}$ (c) $\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ (d) $\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

(e) $\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^p}$ (f) $\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{e^{1/n}}{n}$ (g) $\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{n^n}{n!}$ (h) $\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{\sqrt{n}}{n+1}$

(i) $\sum_{n=2}^{+\infty} \frac{(-1)^n}{\ln n}$ (j) $\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{\ln n}{n}$ (k) $\sum_{n=1}^{+\infty} (-1)^{n+1}e^{1/n}$ (l) $\sum_{n=1}^{+\infty} (-1)^{n+1}\ln\frac{n+1}{n}$

In problem (e) determine all the values of p for which the series converges absolutely, converges conditionally and diverges.

EXERCISE 3.61. Determine whether the given series converges conditionally, converges absolutely or diverges.

(a) $\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{(\sin n)^2}{n^2}$ (b) $\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{4}{2n+3+(-1)^n}$

(c) $\sum_{n=1}^{+\infty}(-1)^{n+1}\cos\left(\frac{1}{n}\right)$ (d) $\sum_{n=1}^{+\infty}(-1)^{n+1}\sin\left(\frac{1}{n}\right)$

3.4. Infinite Series of functions

3.4.1. Power Series. The most important series is the **geometric series**:

$$a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^n + \cdots = \sum_{n=0}^{+\infty} a\,r^n\,.$$

If $-1 < r < 1$ the geometric series converges. Moreover, we proved

$$\sum_{n=0}^{+\infty} a\,r^n = a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^n + \cdots = \frac{a}{1-r} \quad \text{for} \quad -1 < r < 1\,. \tag{3.32}$$

Replacing r by x and letting $a = 1$ we can rewrite the formula in (3.32) as

$$\sum_{n=0}^{+\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1-x} \quad \text{for} \quad -1 < x < 1\,. \tag{3.33}$$

The formula (3.33) can be viewed as a representation of the function

$$f(x) = \frac{1}{1-x}, \quad -1 < x < 1,$$

as an infinite series of powers of x: $1 = x^0, x, x^2, x^3, \ldots$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{+\infty} x^n \quad \text{for} \quad -1 < x < 1\,.$$

You will agree that the (non-negative) integer powers of x are very simple functions. Therefore, it is natural to explore the following question:

Q1: Which functions can be represented as infinite series of constant multiples of (non-negative) integer powers of x?

In other words: Which functions $x \mapsto f(x)$ can be represented as

$$f(x) = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots + a_n\,x^n + \cdots = \sum_{n=0}^{+\infty} a_n\,x^n \quad \text{for} \quad ? < x < ?\,.$$

The infinite series

$$a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots + a_n\,x^n + \cdots = \sum_{n=0}^{+\infty} a_n\,x^n \tag{3.34}$$

is called a **power series**.

The basic question to ask about a power series is:

Q2: For which real numbers x does the power series converge?

Since we are working with the powers of x and since there is no restriction on the signs of a_n and x, we can use Theorems 3.58 and 3.59 (the ratio and root test) to determine the absolute convergence of the power series (3.34). To apply Theorem 3.58 we calculate

$$\lim_{n\to+\infty} \frac{|a_{n+1}|\,|x|^{n+1}}{|a_n|\,|x|^n} = \lim_{n\to+\infty} \frac{|a_{n+1}|\,|x|}{|a_n|} = |x| \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|}\,.$$

Assume that

$$\lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = L. \tag{3.35}$$

If $L = 0$, then Theorem 3.58 implies that the series (3.34) converges for all real numbers x. If $L > 0$, then Theorem 3.58 implies that the series (3.34)

$$\begin{aligned} \text{converges absolutely for} \quad |x|\,L < 1, \quad \text{that is for} \quad -\frac{1}{L} < x < \frac{1}{L} \\ \text{diverges for} \quad |x|\,L > 1, \quad \text{that is for} \quad x < -\frac{1}{L} \ \text{ or } \ x > \frac{1}{L} \end{aligned}$$

If the limit in (3.35) does not exist, then no conclusion about the convergence or divergence can be deduced.

To apply Theorem 3.59 we calculate

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|\,|x|^n} = |x| \lim_{n\to+\infty} \sqrt[n]{|a_n|}\,.$$

Assume that

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|} = L. \tag{3.36}$$

If $L = 0$, then Theorem 3.59 implies that the series (3.34) converges for all real numbers x. If $L > 0$, then Theorem 3.59 implies that the series (3.34)

$$\begin{aligned} \text{converges absolutely for} \quad |x|\,L < 1, \quad \text{that is for} \quad -\frac{1}{L} < x < \frac{1}{L} \\ \text{diverges for} \quad |x|\,L > 1, \quad \text{that is for} \quad x < -\frac{1}{L} \ \text{ or } \ x > \frac{1}{L} \end{aligned}$$

If the limit in (3.36) does not exist, then no conclusion about the convergence or divergence can be deduced.

Example 3.62. Consider the power series

$$\frac{1}{0!} + \frac{1}{1!}\,x + \frac{1}{2!}\,x^2 + \frac{1}{3!}\,x^3 + \cdots + \frac{1}{n!}\,x^n + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}\,x^n.$$

In this example $a_n = 1/n!$, $n \in \mathbb{N}\cup\{0\}$. We calculate

$$L = \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to+\infty} \frac{1}{n+1} = 0.$$

Consequently the given power series converges absolutely for every $x \in \mathbb{R}$.

Example 3.63. Consider the power series

$$1 + 2x + 3\,x^2 + 4\,x^3 + \cdots + (n+1)\,x^n + \cdots = \sum_{n=0}^{\infty} (n+1)\,x^n.$$

Here $a_n = n+1$, $n \in \mathbb{N}\cup\{0\}$ and we calculate

$$L = \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{n+2}{n+1} = 1.$$

Consequently the given power series converges absolutely for every $x \in (-1,1)$. Clearly the series diverges for $x = -1$ and for $x = 1$.

EXAMPLE 3.64. Consider the power series

$$x - \frac{1}{2}\,x^2 + \frac{1}{3}\,x^3 - \frac{1}{4}\,x^4 + \cdots + (-1)^{n+1}\frac{1}{n}\,x^n + \cdots = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{n}\,x^n.$$

Here $a_0 = 0$ and $a_n = (-1)^{n+1}1/n$, for all $n \in \mathbb{N}$. We calculate

$$L = \lim_{n\to+\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to+\infty}\frac{n}{n+1} = 1.$$

Consequently the given power series converges absolutely for every $x \in (-1, 1)$. Clearly the series diverges for $x = -1$ and converges conditionally for $x = 1$.

EXAMPLE 3.65. Consider the power series

$$1 + \frac{1}{2}x + \frac{1}{2^2}\,x^2 + \frac{1}{2^3}\,x^3 + \cdots + \frac{1}{2^n}\,x^n + \cdots = \sum_{n=0}^{\infty}\frac{1}{2^n}\,x^n. \tag{3.37}$$

Here $a_n = 2^{-n}, n \in \mathbb{N} \cup \{0\}$. We calculate

$$L = \lim_{n\to+\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty}\frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} = \lim_{n\to+\infty}\frac{1}{2} = \frac{1}{2}.$$

Consequently the given power series converges absolutely for every $x \in (-2, 2)$. Clearly the series diverges for $x = -2$ and for $x = 2$.

Notice that we can actually calculate the sum of this series using the following substitution (or you can call this a trick). Substitute $u = x/2$ in (3.37). Then (3.37) becomes

$$1 + u + u^2 + u^3 + \cdots + u^n + \cdots = \sum_{n=0}^{\infty} u^n. \tag{3.38}$$

We know that the sum of the series in (3.38) is $1/(1-u)$ for $u \in (-1, 1)$, that is,

$$1 + u + u^2 + u^3 + \cdots + u^n + \cdots = \sum_{n=0}^{\infty} u^n = \frac{1}{1-u}, \quad u \in (-1, 1).$$

Substituting back $u = x/2$ we get:

$$1 + \frac{1}{2}x + \frac{1}{2^2}\,x^2 + \frac{1}{2^3}\,x^3 + \cdots + \frac{1}{2^n}\,x^n + \cdots = \sum_{n=0}^{\infty}\frac{1}{2^n}\,x^n = \frac{2}{2-x}, \quad x \in (-2, 2).$$

EXAMPLE 3.66. Consider the power series

$$\frac{1}{1}x + \frac{1}{4}\,x^2 + \frac{1}{9}\,x^3 + \cdots + \frac{1}{n^2}\,x^n + \cdots = \sum_{n=1}^{\infty}\frac{1}{n^2}\,x^n.$$

We calculate

$$L = \lim_{n\to+\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty}\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to+\infty}\frac{n^2}{(n+1)^2} = 1.$$

Consequently the given power series converges absolutely for every $x \in (-1, 1)$. For $x = 1$ we get the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Therefore, for $x = 1$ the given power series converges. For $x = -1$ we get the alternating series which converges absolutely. Therefore the given power series converges absolutely on $[-1, 1]$.

The following theorem answers the question **Q2** above.

THEOREM 3.67. *Let*

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{+\infty} a_n x^n$$

be a power series. Then one of the following three cases holds.

(A) *The power series converges absolutely for all $x \in \mathbb{R}$.*
(B) *There exists $r > 0$ such that the power series converges absolutely for all $x \in (-R, R)$ and diverges for all x such that $|x| > R$.*
(C) *The power series diverges for all $x \neq 0$. For $x = 0$ it is trivial that the power series converges.*

The set on which a power series converges is called the *interval of convergence*. The number $R > 0$ in Theorem 3.67 (B) is called the *radius of convergence*. In the case (A) in Theorem 3.67 we write $R = +\infty$. In the case (C) in Theorem 3.67 we write $R = 0$.

REMARK 3.68. In the case (B) in Theorem 3.67 the convergence of the power series at the points $x = R$ and $x = -R$ must be determined by studying the infinite series

$$\sum_{n=0}^{+\infty} a_n R^n \quad \text{and} \quad \sum_{n=0}^{+\infty} a_n (-R)^n.$$

Examples in this section show that the interval of convergence of a power series can have any of these four forms $(-R, R)$, $(-R, R]$, $[-R, R)$ and $[-R, R]$.

3.4.2. Functions Represented as Power Series. The following theorem lists properties of functions defined by a power series.

THEOREM 3.69. *Let $R > 0$ be the radius of convergence of the power series*

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{+\infty} a_n x^n.$$

Then the function f defined on $(-R, R)$ by

$$f(x) := a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{+\infty} a_n x^n, \quad -R < x < R,$$

has the following three properties:

(a) *The function f is continuous on $(-R, R)$.*
(b) *The derivative $f'(x)$ exists for all $x \in (-R, R)$ and*

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + (n+1)a_{n+1} x^n + \cdots = \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n.$$

(c) *The function f has derivatives of all orders $1, 2, 3, \ldots,$ at all points of $(-R, R)$. In particular*

$$f(0) = a_0,\ f'(0) = a_1,\ f''(0) = 2\,a_2,\ f'''(0) = 3 \cdot 2\,a_3,\ \ldots,\ f^{(n)}(0) = n!\,a_n,\ \ldots. \tag{3.39}$$

(d) *For all* $x \in (-R, R)$ *we have*

$$\int_0^x f(t)dt = a_0 x + \frac{a_1}{2}\, x^2 + \frac{a_2}{3}\, x^3 + \cdots + \frac{a_{n-1}}{n}\, x^n + \frac{a_n}{n+1}\, x^{n+1} + \cdots = \sum_{n=1}^{+\infty} \frac{a_{n-1}}{n}\, x^n.$$

THEOREM 3.70. *Let* $R > 0$ *be the radius of convergence of the power series*

$$a_0 + a_1\, x + a_2\, x^2 + a_3\, x^3 + \cdots + a_n\, x^n + \cdots = \sum_{n=0}^{+\infty} a_n\, x^n.$$

Let f *be the function defined on* $(-R, R)$ *by*

$$f(x) := a_0 + a_1\, x + a_2\, x^2 + a_3\, x^3 + \cdots + a_n\, x^n + \cdots = \sum_{n=0}^{+\infty} a_n\, x^n, \quad -R < x < R.$$

If the series

$$\sum_{n=0}^{+\infty} a_n\, R^n$$

converges, then the limit $\lim_{x \uparrow R} f(x)$ *exists and*

$$\lim_{x \uparrow R} f(x) = \sum_{n=0}^{+\infty} a_n\, R^n.$$

If the series

$$\sum_{n=0}^{+\infty} a_n\, (-R)^n$$

converges, then the limit $\lim_{x \downarrow R} f(x)$ *exists and*

$$\lim_{x \downarrow R} f(x) = \sum_{n=0}^{+\infty} a_n\, (-R)^n.$$

EXAMPLE 3.71. By (3.33) we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \quad \text{for all} \quad -1 < x < 1. \tag{3.40}$$

Thus the function $f(x) = 1/(1-x)$ defined for $x \in (-1, 1)$ can be represented by a power series. Applying Theorem 3.69 we get

$$\frac{1}{(1-x)^2} = 1 + 2\, x + 3\, x^2 + 4\, x^3 + \cdots + n\, x^{n-1} + (n+1)\, x^n + \cdots \quad \text{for all} \quad -1 < x < 1.$$

EXAMPLE 3.72. Substituting $-x$ for x in (3.40) we get

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots \quad \text{for all} \quad -1 < x < 1\,. \tag{3.41}$$

Thus the function $f(x) = 1/(1+x)$ defined for $x \in (-1, 1)$ can be represented by a power series. Applying Theorem 3.69 (d) we get

$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + (-1)^{n+1}\frac{1}{n}x^n + \cdots \quad \text{for all} \quad -1 < x < 1\,.$$

For $x = 1$ the above series is the alternating harmonic series which converges conditionally. By Theorem 3.70 we have

$$\lim_{x \uparrow 1} \ln(1 + x) = \sum_{n=0}^{+\infty} (-1)^{n+1} \frac{1}{n}.$$

Since the function $\ln(1 + x)$ is continuous at $x = 1$ we have

$$\lim_{x \uparrow 1} \ln(1 + x) = \ln 2.$$

Thus we found the sum of the alternating harmonic series

$$\sum_{n=0}^{+\infty} (-1)^{n+1} \frac{1}{n} = \ln 2.$$

EXAMPLE 3.73. Substituting x^2 for x in (3.41) we get

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots \quad \text{for all} \quad -1 < x < 1.$$

Thus the function $f(x) = 1/(1 + x^2)$ defined for $x \in (-1, 1)$ can be represented by a power series. Applying Theorem 3.69 (d) for all $x \in (-1, 1)$ we get

$$\arctan(x) = \int_0^x \frac{1}{1 + t^2} dt = x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 + \cdots + (-1)^{n+1} \frac{1}{2n - 1} x^{2n-1} + \cdots .$$

With $x = 1$ the above series is

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{n+1} \frac{1}{2n - 1} + \cdots$$

is a conditionally convergent alternating series. By Theorem 3.70 we have

$$\lim_{x \uparrow 1} \arctan x = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1}{2n - 1}.$$

We did not prove it, but it can be proved that $\arctan x$ is a continuous function. Therefore

$$\lim_{x \uparrow 1} \arctan x = \arctan 1 = \frac{\pi}{2}.$$

Thus we found a representation of π as an infinite sum:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{n+1} \frac{1}{2n - 1} + \cdots = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1}{2n - 1}.$$

3.4.3. Taylor series at 0 (Maclaurin series). In the preceding section we found power series representations for several well known functions. It turns out that all well known functions can be represented as power series. The key step in finding the power series representation of elementary functions are formulas (3.39) which establish the relationship between the coefficients a_n, $n = 0, 1, 2, \ldots,$ of a power series and the derivatives of the function f which is represented by that power series. We rewrite formulas (3.39) as

$$a_0 = f(0), \quad a_1 = f'(0), \quad a_2 = \frac{1}{2!} f''(0), \quad a_3 = \frac{1}{3!} f^{(3)}(0), \ \ldots, \ a_n = \frac{1}{n!} f^{(n)}(0), \ldots . \quad (3.42)$$

Let $a > 0$ and let f be a function defined on $(-a, a)$. Assume that f has all derivatives on $(-a, a)$. Then the series power series

$$f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f^{(3)}(0)x^3 + \cdots + \frac{1}{n!}f^{(n)}(0)x^n + \cdots = \sum_{n=0}^{+infty} \frac{1}{n!}f^{(n)}(0)x^n$$

is called *Taylor series at* 0 or *Maclaurin series* of f.

Using formulas (3.42) it is not difficult to calculate a Maclaurin series for a given function. The difficulties arise in proving that the function defined by such power series is identical to the given function. Fortunately this is true for all well known functions.

EXAMPLE 3.74. Let $f(x) = e^x = \exp(x)$, $x \in \mathbb{R}$. Then $f^{(n)}(x) = e^x$ for all $n = 0, 1, 2, \ldots$. Therefore the coefficients of the Maclaurin series for the function exp are $a_n = 1/n!$ and it can be proved that for all $x \in \mathbb{R}$ we have

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots .$$

EXAMPLE 3.75. Let $f(x) = \sin(x)$, $x \in \mathbb{R}$. Then

$$f'(x) = \cos(x), \quad f''(x) = -\sin(x), \quad f^{(3)}(x) = -\cos(x), \quad f^{(4)}(x) = \sin(x).$$

Consequently,

$$f^{(2k)}(0) = 0, \quad f^{(2k+1)}(0) = (-1)^k, \quad \text{for all} \quad k \in \mathbb{N} \cup \{0\}.$$

Therefore the coefficients of the Maclaurin series for the function sin are

$$a_{2k} = 0, \quad a_{2k+1} = (-1)^k \frac{1}{(2k+1)!}, \quad \text{for all} \quad k \in \mathbb{N} \cup \{0\}.$$

It can be proved that for all $x \in \mathbb{R}$ we have

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + (-1)^k \frac{1}{(2k+1)!}x^{2k+1} + \cdots .$$

EXAMPLE 3.76. Let $f(x) = \cos(x)$, $x \in \mathbb{R}$. Then

$$f'(x) = -\sin(x), \quad f''(x) = -\cos(x), \quad f^{(3)}(x) = \sin(x), \quad f^{(4)}(x) = \cos(x).$$

Consequently,

$$f^{(2k)}(0) = (-1)^k, \quad f^{(2k+1)}(0) = 0, \quad \text{for all} \quad k \in \mathbb{N} \cup \{0\}.$$

Therefore the coefficients of the Maclaurin series for the function cos are

$$a_{2k} = (-1)^k \frac{1}{(2k)!}, \quad a_{2k+1} = 0 \quad \text{for all} \quad k \in \mathbb{N} \cup \{0\}.$$

It can be proved that for all $x \in \mathbb{R}$ we have

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + (-1)^k \frac{1}{(2k)!}x^{2k} + \cdots .$$

EXAMPLE 3.77 (The Binomial Series). Let $\alpha \in \mathbb{R}$. Let $f(x) = (1+x)^\alpha$, $x \in (-1, 1)$. Then

$$\begin{aligned} f'(x) &= \alpha(1+x)^{\alpha-1}, \\ f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2}, \\ f^{(3)}(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}, \end{aligned}$$

$$\vdots$$

$$f^{(n)}(x) = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$$

$$\vdots$$

Therefore the coefficients of the Maclaurin series for the function f are

$$a_0 = 1, \quad a_n = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, \quad n \in \mathbb{N}.$$

It can be proved that for all $x \in (-1,1)$ we have

$$(1+x)^\alpha = 1+\frac{\alpha}{1!}x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n+\cdots.$$

This series is called *binomial series*. The reason for this name is that for $\alpha \in \mathbb{N}$ the binomial series becomes a polynomial:

$$\begin{aligned}
(1+x)^1 &= 1 + x\\
(1+x)^2 &= 1 + 2x + x^2\\
(1+x)^3 &= 1 + 3x + 3x^2 + x^3\\
(1+x)^4 &= 1 + 4x + 6x^2 + 4x^3 + x^4\\
(1+x)^5 &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5\\
(1+x)^6 &= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6\\
&\vdots
\end{aligned}$$

$$(1+x)^m = \sum_{k=0}^{m} \binom{m}{k} x^k, \quad \text{were} \quad m \in \mathbb{N} \quad \text{and} \quad \binom{m}{k} = \frac{m!}{k!(m-k)!}$$

The last formula is called the *binomial theorem*. The coefficients

$$\binom{m}{k} = \frac{m!}{k!(m-k)!} = \frac{m(m-1)\cdots(m-k+1)}{k!} \quad \text{with} \quad m,k \in \mathbb{N}, \quad 0 \le k \le m,$$

are called *binomial coefficients*. With a general $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$ the coefficients

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

are called *generalized binomial coefficients*. By definition $\binom{\alpha}{0} = 1$. With this notation the binomial series can be written as

$$(1+x)^\alpha = \sum_{k=0}^{+\infty} \binom{\alpha}{k} x^k \quad \text{for qll} \quad x \in (-1,1). \tag{3.43}$$

Notice that formula (3.40) is a special case of (3.43), since

$$\binom{-1}{k} = \frac{(-1)(-2)\cdots(-1-k+1)}{k!} = \frac{(-1)^k k!}{k!} = (-1)^k.$$

Notice also that differentiating (3.40) leads to

$$(1+x)^{-2} = 1 + \sum_{k=1}^{+\infty} (-1)^k (k+1)\, x^k \quad \text{for all} \quad -1 < x < 1\,.$$

This is a binomial series with $\alpha = -2$. To verify this we calculate

$$\binom{-2}{k} = \frac{(-2)(-3)\cdots(-2-k+1)}{k!} = \frac{(-1)^k(k+1)!}{k!} = (-1)^k(k+1).$$

For $\alpha = 1/2$ the expression

$$\begin{aligned}\binom{1/2}{k} &= \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-k+1\right)}{k!}\\ &= \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2k-3}{2}\right)}{k!}\\ &= \frac{(-1)^{k-1}1\cdot 3\cdot\,\cdots\,(2k-3)}{2^k\,k!}\end{aligned}$$

Thus

$$\sqrt{1+x} = 1+\frac{1}{2}x-\frac{1}{2^2 2!}x^2+\frac{1\cdot 3}{2^3 3!}x^3-\frac{1\cdot 3\cdot 5}{2^4 4!}x^4+\frac{1\cdot 3\cdot 5\cdot 7}{2^5 5!}x^5+\cdots \quad \text{for all} \quad -1<x<1\,.$$

EXAMPLE 3.78. Let $f(x) = \arcsin(x),\ x \in [-1,1]$. To calculate the Maclaurin series for arcsin we notice that

$$\frac{d}{dx}\big(\arcsin(x)\big) = \frac{1}{\sqrt{1-x^2}}, \qquad x \in (-1,1).$$

Now calculate the Maclaurin series for the last function using the binomial series with $\alpha = -1/2$. For $\alpha = -1/2$ and $k \in \mathbb{N}$, we calculate

$$\begin{aligned}\binom{-1/2}{k} &= \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-k+1\right)}{k!}\\ &= \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2k-1}{2}\right)}{k!}\\ &= (-1)^k\frac{1\cdot 3\cdot\,\cdots\,\cdot(2k-1)}{2^k\,k!}\end{aligned}$$

Thus

$$\frac{1}{\sqrt{1+x}} = 1-\frac{1}{2}x+\frac{1\cdot 3}{2^2 2!}x^2+\frac{1\cdot 3\cdot 5}{2^3 3!}x^3-\frac{1\cdot 3\cdot 5\cdot 7}{2^4 4!}x^4+\cdots \quad \text{for all} \quad -1<x<1\,,$$

that is,

$$\frac{1}{\sqrt{1+x}} = 1+\sum_{k=1}^{+\infty}(-1)^k\frac{1\cdot 3\cdot\,\cdots\,\cdot(2k-1)}{2^k\,k!}\;x^k,$$

or using the notation of double factorials

$$\frac{1}{\sqrt{1+x}} = 1+\sum_{k=1}^{+\infty}(-1)^k\frac{(2k-1)!!}{(2k)!!}\;x^k.$$

Substituting $-x^2$ instead of x in the above formula we get

$$\frac{1}{\sqrt{1-x^2}} = 1+\sum_{k=1}^{+\infty}\frac{(2k-1)!!}{(2k)!!}\;x^{2k}, \quad \text{for all} \quad -1<x<1.$$

Since

$$\int_0^x \frac{1}{\sqrt{1-t^2}}dt = \arcsin(x),$$

integrating the last power series we get

$$\arcsin(x) = x + \sum_{k=1}^{+\infty} \frac{(2k-1)!!}{(2k+1)(2k)!!}\, x^{2k+1} = \sum_{k=0}^{+\infty} \frac{\binom{2k}{k}}{4^k(2k+1)}\, x^{2k+1}, \quad \text{for all} \quad -1 < x < 1$$

It is interesting to note that the above expansion holds at both endpoints $x = -1$ and $x = 1$. To prove this we need to recall Theorem 3.69 (a) and prove that the series

$$1 + \sum_{k=1}^{+\infty} \frac{(2k-1)!!}{(2k+1)(2k)!!}$$

converges. This series converges by The Comparison Test. (**Hint:** Prove by mathematical induction that $\dfrac{(2k-1)!!}{(2k)!!} < \dfrac{1}{\sqrt[3]{k}}$ for all $k \in \mathbb{N}$.) As a consequence we obtain that

$$1 + \sum_{k=1}^{+\infty} \frac{(2k-1)!!}{(2k+1)(2k)!!} = \sum_{k=0}^{+\infty} \frac{\binom{2k}{k}}{4^k(2k+1)} = \frac{\pi}{2}.$$

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