The number e is irrational

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In this note $\mathbb{Z}$ denotes the set of all integers and $\mathbb{N}$ denotes the set of all positive integers.

The goal of this note is to prove that e is irrational. By definition, a real number is irrational if it is not rational. By definition a real number r is **rational** if

$$\exists p \in \mathbb{Z} \quad \exists q \in \mathbb{N} \quad \text{such that} \quad r = \frac{p}{q}.$$

Hence, forming the negation of the preceding statement, we can define that $x \in \mathbb{R}$ is **irrational** if the following statement is true:

$$\forall p \in \mathbb{Z} \quad \forall q \in \mathbb{N} \quad \text{we have} \quad x \neq \frac{p}{q}.$$

In this note we use the following definition of e:

$$e = \lim_{n \to +\infty} \sum_{k=0}^{n} \frac{1}{k!}. \tag{1}$$

Since e is defined by the limit in (1), it is to be expected that for a large $m \in \mathbb{N}$ and even larger $n \in \mathbb{N}$ the finite sum

$$\sum_{k=m+1}^{n} \frac{1}{k!}$$

would be a small positive number. But how small? The next lemma gives us a somewhat surprising simple upper estimate.

Lemma 1. *For every $m, n \in \mathbb{N}$ such that $m < n$ we have*

$$\sum_{k=m+1}^{n} \frac{1}{k!} \leq \frac{1}{m!}. \tag{2}$$

Proof. In the sum in (2) we have $k \in \{m+1, \ldots, n\}$. For such k we have

$$(m-1)!(k-1)k \leq k!$$

Comment. To understand this inequality, notice that the product on the left-hand side has $m+1$ terms and that the product on the right-hand side has k terms. All terms are positive integers. Since $k \geq m+1$, all the terms from the left-hand side are present on the right-hand side. Therefore the inequality holds.

The preceding inequality yields

$$\frac{(m-1)!}{k!} \leq \frac{1}{(k-1)k} = \frac{1}{k-1} - \frac{1}{k}. \tag{3}$$

Comment. When the equal terms in the numerator and the denominator are cancelled, the inequality in (3) is in fact a Pizza-Party inequality. The equality in (3) is a preparation for a telescoping sum that will appear in the next step of the proof.

Next we will utilize (3) to prove (2):

$$
\begin{aligned}
\sum_{k=m+1}^{n} \frac{1}{k!} &= \frac{1}{(m-1)!} \sum_{k=m+1}^{n} \frac{(m-1)!}{k!} \\
&\leq \frac{1}{(m-1)!} \sum_{k=m+1}^{n} \frac{1}{(k-1)k} \\
&= \frac{1}{(m-1)!} \sum_{k=m+1}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) \\
&= \frac{1}{(m-1)!} \left(\frac{1}{m} - \frac{1}{m+1} + \cdots + \frac{1}{n-1} - \frac{1}{n} \right) \\
&= \frac{1}{(m-1)!} \left(\frac{1}{m} - \frac{1}{n} \right) \\
&\leq \frac{1}{m!}.
\end{aligned}
$$

Comment. At the equality on the first line, I multiply all numerators and denominators by $(m-1)!$. At the second line, the inequality follows from (3). At the third line, equality follows from (3). The forth line is just the previous sum written with ellipses, so that it is clear that it is a telescopic sum. At the fifth line, the equality is the consequence of telescoping in the previous sum. The last inequality is a consequence of $1/n>0$.

□

Lemma 2. *For every $m, n \in \mathbb{N}$ such that $m \leq n$ we have*

$$
\sum_{k=0}^{m} \frac{1}{k!} \leq \sum_{k=0}^{n} \frac{1}{k!} \leq \frac{2}{(m+1)!} + \sum_{k=0}^{m} \frac{1}{k!}. \tag{4}
$$

Proof. Let $m, n \in \mathbb{N}$ be such that $m \leq n$. If $m = n$ or $m = n-1$ the inequality is clear. Now assume that $m < n-1$. Since the left-side inequality in (4) is clear, we only need to prove

$$
\sum_{k=0}^{n} \frac{1}{k!} \leq \frac{2}{(m+1)!} + \sum_{k=0}^{m} \frac{1}{k!}. \tag{5}
$$

By Lemma 1 we have

$$
\sum_{k=m+2}^{n} \frac{1}{k!} \leq \frac{1}{(m+1)!}.
$$

Adding $\dfrac{1}{(m+1)!}$ to both sides of the preceding inequality we obtain

$$
\sum_{k=m+1}^{n} \frac{1}{k!} = \frac{1}{(m+1)!} + \sum_{k=m+2}^{n} \frac{1}{k!} \leq \frac{2}{(m+1)!}. \tag{6}
$$

Adding $\sum_{k=0}^{m} \frac{1}{k!}$ to both sides of the inequality in (6), get

$$\sum_{k=0}^{m} \frac{1}{k!} + \sum_{k=m+1}^{n} \frac{1}{k!} = \sum_{k=0}^{n} \frac{1}{k!} \leq \frac{2}{(m+1)!} + \sum_{k=0}^{m} \frac{1}{k!}.$$

This proves (5). The lemma is proved. $\square$

The following theorem can be deduced from Theorem 3.18 on page 52 of the class notes. It is a background knowledge in this context.

Theorem 3. *Let $L \in \mathbb{R}$ and let $\{s_n\}$, be a convergent sequence with the limit L. Let $a, b \in \mathbb{R}$ be such that for some $n_0 \in \mathbb{N}$ we have*

$$a \leq s_n \leq b$$

for all $n \in \mathbb{N}$ such that $n \geq n_0$. Then $a \leq L \leq b$.

Applying Theorem 3 to inequality (4) and the definition of e we obtain the following corollary.

Corollary 4. *For all $m \in \mathbb{N}$ we have*

$$\sum_{k=0}^{m} \frac{1}{k!} < e \leq \frac{2}{(m+1)!} + \sum_{k=0}^{m} \frac{1}{k!}. \tag{7}$$

In particular, setting $m = 3$ we deduce

$$\frac{8}{3} < e < \frac{11}{4}. \tag{8}$$

Theorem 5. *The number e is irrational.*

Proof. By the definition of an irrational number we have to prove

$$\forall p \in \mathbb{Z} \quad \forall q \in \mathbb{N} \quad \text{we have} \quad e \neq \frac{p}{q}. \tag{9}$$

Step 1. Let $q \in \mathbb{N}$ be such that $q > 1$. It follows from (7), when we substitute q for m, that

$$0 < q! \left(e - \sum_{k=0}^{q} \frac{1}{k!} \right) \leq q! \frac{2}{(q+1)!} = \frac{2}{q+1} \leq \frac{2}{3}.$$

If $q = 1$, then by (8)

$$0 < 1! \left(e - \sum_{k=0}^{1} \frac{1}{k!} \right) = e - 2 \leq \frac{3}{4}.$$

From the preceding two displayed inequalities we have

$$\forall q \in \mathbb{N} \quad q! \left(e - \sum_{k=0}^{q} \frac{1}{k!} \right) \notin \mathbb{Z}. \tag{10}$$

Step 2. Let $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ be arbitrary. Then

$$q! \left(\frac{p}{q} - \sum_{k=0}^{q} \frac{1}{k!} \right) = p(q-1)! - \sum_{k=0}^{q} \frac{q!}{k!}. \tag{11}$$

Since

$$\forall k \in \{0, 1, \ldots, q\} \qquad \frac{q!}{k!} \in \mathbb{N},$$

equality (11) yields

$$\forall p \in \mathbb{Z} \quad \forall q \in \mathbb{N} \qquad q! \left(\frac{p}{q} - \sum_{k=0}^{q} \frac{1}{k!} \right) \in \mathbb{Z} \tag{12}$$

Step 3. From (10) and (12) we deduce

$$\forall p \in \mathbb{Z} \quad \forall q \in \mathbb{N} \qquad q! \left(e - \sum_{k=0}^{q} \frac{1}{k!} \right) \neq q! \left(\frac{p}{q} - \sum_{k=0}^{q} \frac{1}{k!} \right)$$

Consequently,

$$\forall p \in \mathbb{Z} \quad \forall q \in \mathbb{N} \qquad e \neq \frac{p}{q}.$$

Thus (9) is proved. The proof is complete. $\square$

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