```
<< DiscreteMath`Combinatorica`
```

# Exercises

## Exercise 20

### ■ (a)

```
Binomial[10, 3]
```

```
120
```

### ■ (b)

```
Sum[Binomial[10, k], {k, 0, 4}]
```

```
386
```

$$\frac{2^{10} - \text{Binomial}[10, 5]}{2}$$

```
386
```

### ■ (c)

```
Sum[Binomial[10, k], {k, 7, 10}]
```

```
176
```

### ■ (d)

```
Sum[Binomial[10, k], {k, 3, 10}]
```

```
968
```

$2^{10}$ - Sum[Binomial[10, k], {k, 0, 2}]

```
968
```

# Exercise 21

### ■ a)

```
5!
```

120

### ■ b)

```
4!
```

24

### ■ c)

```
5!
```

120

### ■ d)

```
4!
```

24

### ■ e)

```
3!
```

6

### ■ f)

```
0
```

0

# Exercise 23

How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other?

```
13!
```

```
6227020800
```

You can order men in 8! ways. This leaves 9 positions to place women at. We can choose those 5 positions in 9-choose-5 ways and for each choice we can arrange women in 5! ways:

```
8! Binomial[9, 5] 5!
```

```
609638400
```

Another way to think about this is to first count the bit strings of length 13 with exactly 5 0s and with no consecutive 0s. There are

```
Binomial[9, 5]
```

```
126
```

such strings. Now we place men where 1s are and women where 0s are. There are 8! ways to place men and 5! ways to place women. By the product rule the total number is

```
In[1]:= 8! Binomial[9, 5] 5!

Out[1]= 609638400
```

## ■ Below is an exercise in *Mathematica* list manipulation to confirm the above calculation for the number of bit strings of length 13

All bit strings of length 13:

```
tt1 = PadLeft[#, 13] & /@ Table[IntegerDigits[k, 2], {k, 0, 2^13 - 1}];
```

```
Length[tt1]
```

```
8192
```

```
tt1[[8192]]
```

```
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

```
tt1[[234]]
```

```
{0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1}
```

Separate 0s and 1s by sorting:

```
Split[Sort[tt1[[123]]]]
```

{{0, 0, 0, 0, 0, 0, 0, 0}, {1, 1, 1, 1, 1}}

```
Split[Sort[tt1[[8192]]]]
```

{{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

```
Split[Sort[tt1[[1]]]]
```

{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}

The last two bit strings are problematic so exclude them

```
And[(Length[Split[Sort[#]]] == 2), Length[Split[Sort[#]][[1]]] == 5] &[tt1[[123]]]
```

False

Select the bit strings with 5 0s:

```
tt2 = Select[tt1, And[(Length[Split[Sort[#]]] == 2), Length[Split[Sort[#]][[1]]] == 5] &];
```

```
Length[tt2]
```

1287

```
Binomial[13, 5]
```

1287

```
tt3 = Map[Split, tt2];
```

```
Length[tt3]
```

1287

```
tt3[[123]]
```

{{0, 0}, {1, 1, 1}, {0}, {1, 1}, {0}, {1, 1}, {0}, {1}}

Now somewhat complicated function to select only those strings that have no consecutive 0s:

```
MyF22[lll_] := Apply[And, Map[Or[And[(#[[1]] == 0), Length[#] == 1], (#[[1]] == 1)] &, lll]]
```

```
MyF22[tt3[[123]]]
```

False

```
MyF22[tt3[[445]]]
```

False

```
tt4 = Select[tt3, MyF22[#] &];
```

```
Length[tt4]
```

126

```
Binomial[9, 5]
```

126

## Exercise 25

```
(* a *) 100 99 98 97
```

94109400

```
(* b *) 99 98 97
```

941094

```
(* c *)4 99 98 97
```

3764376

```
(* d *) 99 98 97 96
```

90345024

```
(* e *) 2 ! Binomial[4, 2] 98 97
```

114072

```
(* f *) 3 ! Binomial[4, 3] 97
```

2328

```
(* g *) 4 !
```

24

```
(* h *) 96 95 94 93
```

79727040

```
(* i *) 4 99 98 97
```

3764376

```
(* j *) 2 Binomial[4, 2] 96 95
```

109440

# Exercise 29

## ■ (a)

```
(* there are 98 consecutive triples which can be accompanied to make a 4-
  permutation by 97 remaining numbers,
 the accompaniment being possible in 4 different ways *)

98 * 97 * 4

38024

(* take into account that we count 1,2,
 3,4 twice, and there are 97 such permutations  *)

98 * 97 * 4 - 97

37927
```

## ■ (b)

```
(* similar as (a) *)

98 * 72 * 2 - 97

14015
```

# Exercise 30

## ■ (a)

```
(* no women *) Binomial[9, 5]

126

(* all possible committees *) Binomial[16, 5]

4368

(* difference *)

Binomial[16, 5] - Binomial[9, 5]

4242

(* or count all *)
```

```
(* exactly one woman, exactly two and so on *)
Table[Binomial[7, k] * Binomial[9, 5 - k], {k, 1, 6}]
```

```
{882, 1764, 1260, 315, 21, 0}
```

```
Sum[Binomial[7, k] * Binomial[9, 5 - k], {k, 1, 5}]
```

```
4242
```

### ■ (b)

```
(* modify the last sum *)
```

```
Sum[Binomial[7, k] * Binomial[9, 5 - k], {k, 1, 4}]
```

```
4221
```

```
(* or subtract all women committees *)
```

```
4242 - Binomial[7, 5]
```

```
4221
```

---

## Exercise 31

```
(* total number of six letter words *)
```

```
26^6
```

```
308915776
```

```
(* a *) 6 5 21^5
```

```
122523030
```

```
(* b *) Binomial[6, 2] 5^2 21^4
```

```
72930375
```

```
(* c *) 26^6 - 21^6 (* total - no vowels *)
```

```
223149655
```

```
(* d *) 26^6 - (21^6 + 6 5 21^5) (* total - no vowels - ex one *)
```

```
100626625
```

---

## Exercise 32

```
(* total number of six letter words *)
```

```
26^6
```

308915776

## (a)

```
(* a *) (* calculate no a *) 26^6 - 25^6
```

64775151

```
(* or exactly one a + exactly two a + ... *)
Sum[Binomial[6, k] 25^(6 - k), {k, 1, 6}]
```

64775151

```
(* this is in fact a binomial theorem for (25 +1)^6 *)
```

## (b)

```
(* b *) (* calcualate no a no b *)  24^6
```

191102976

```
(* this is  a or b *) 26^6 -  24^6
```

117812800

```
(* a and b = a + b - (a or b) *) 2 * (26^6 - 25^6) - (26^6 -  24^6)
```

11737502

## (c)

```
25 24 23 22 21
```

6375600

## (d)

```
(* all letters distinct *)
26 25 24 23 22 21
```

165765600

```
(* half of them have a to the left of b *)
```

```
26 25 24 23 22 21 / 2
```

82882800

## Exercise 33

```
(* committees with 3 men 3 women *)

Binomial[10, 3] * Binomial[15, 3]
```

54600

## Exercise 34

```
(* total number of committees of 6 *)

Binomial[10 + 15, 6]
```

177100

```
(* committees with 6 men 0 women *)

Binomial[10, 6] * Binomial[15, 0]
```

210

```
(* committees with 5 men 1 women *)

Binomial[10, 5] * Binomial[15, 1]
```

3780

```
(* committees with 4 men 2 women *)

Binomial[10, 4] * Binomial[15, 2]
```

22050

```
(* committees with 3 men 3 women *)

Binomial[10, 3] * Binomial[15, 3]
```

54600

```
(* committees with 2 men 4 women *)

Binomial[10, 2] * Binomial[15, 4]
```

61425

```
(* committees with 1 men 5 women *)
```

```
Binomial[10, 1] * Binomial[15, 5]
```

```
30030
```

```
(* committees with 0 men 6 women *)
```

```
Binomial[10, 0] * Binomial[15, 6]
```

```
5005
```

```
(* verify the sum *)
```

```
Sum[Binomial[10, k] * Binomial[15, 6 - k], {k, 0, 6}]
```

```
177100
```

```
Binomial[10 + 15, 6]
```

```
177100
```

```
(* the answer 4 women or 5 women or 6 women *)
```

```
Sum[Binomial[10, k] * Binomial[15, 6 - k], {k, 0, 2}]
```

```
96460
```

# Exercise 35

In fact we have only 10 objects, eight glued 01 and two free 1s. The objective is to place these two 1s among these 10 objects. That can be done in

```
Binomial[10, 2]
```

```
45
```

```
s18 = Strings[{0, 1}, 18];
```

```
Length[s18]
```

```
262144
```

```
Length[Select[Partition[s18[[23434]], 2, 1], Function[y, (y == {0, 1})]]] == 8
```

```
False
```

```
s1808 = Select[s18, (Total[#] == 10) &]
```

```
Length[
 Select[s1808, (Length[Select[Partition[#, 2, 1], Function[y, (y == {0, 1})]]] == 8) &]]
```

```
45
```

## Exercise 36

```
Binomial[14, 5]
```

```
2002
```

## Exercise 37

```
2^10 - 2 (1 + 10 + Binomial[10, 2])
```

```
912
```

## Exercise 38

```
Binomial[45, 3] + Binomial[57, 4] + Binomial[69, 5]
```

```
11647713
```

How many possibilities are lost with the restrictions?

```
Binomial[45 + 57 + 69, 3 + 4 + 5] - (Binomial[45, 3] + Binomial[57, 4] + Binomial[69, 5])
```

```
879234828403848567
```

## Exercise 39

```
26 * 25 * 24 * 10 * 9 * 8
```

```
11232000
```

## Exercise 40

```
4 !
```

```
24
```

```
{{a, b, c, d}, {b, c, d, a}, {c, d, a, b}, {d, a, b, c}}
```

```
{{a, b, d, c}, {b, d, c, a}, {d, c, a, b}, {c, a, b, d}}
```

```
6 ! / 6
```

```
120
```

## Exercise 41

Ways to end three

```
Binomial[3, 3](* all G *) +
 (Binomial[3, 2](* 2 G 1 S *) + Binomial[3, 1](* 1 G 2 S *)) +
 (Binomial[3, 1] * Binomial[2, 1] * Binomial[1, 1]) (* 1 G 1 S 1 B *)
```

```
13
```

```
(* all tie *) 1 +
 (* two tie *) Binomial[3, 2] +
 (* clear winer, ties for 2nd and 3rd *) Binomial[3, 1] (Binomial[2, 2]) +
 (* no ties *) 3!
```

```
13
```

## Exercise 42

Ways to end two

```
{{2, 0}, {1, 1}}
```

```
{{2, 0}, {1, 1}}
```

```
2 + 3 - 1
```

```
4
```

Ways to end three

```
Length[{{3, 0, 0}, {2, 1, 0}, {1, 2, 0}, {1, 1, 1}}]
```

```
4
```

```
2 + 2 * 3 + 1 * 4 - 4
```

```
8
```

Ways to end four

```
{{4, 0, 0, 0}, {3, 1, 0, 0}, {2, 2, 0, 0}, {2, 1, 1, 0},
 {1, 3, 0, 0}, {1, 2, 1, 0}, {1, 1, 2, 0}, {1, 1, 1, 1}}
```

```
{{4, 0, 0, 0}, {3, 1, 0, 0}, {2, 2, 0, 0}, {2, 1, 1, 0},
 {1, 3, 0, 0}, {1, 2, 1, 0}, {1, 1, 2, 0}, {1, 1, 1, 1}}
```

```
Length[%]
```

```
8
```

Ways to end five

```
Length[{{5, 0, 0, 0, 0}, {4, 1, 0, 0, 0}, {3, 2, 0, 0, 0}, {3, 1, 1, 0, 0},
  {2, 3, 0, 0, 0}, {2, 2, 1, 0, 0}, {2, 1, 2, 0, 0}, {2, 1, 1, 1, 0},
  {1, 4, 0, 0, 0}, {1, 3, 1, 0, 0}, {1, 2, 3, 0, 0}, {1, 2, 1, 1, 0},
  {1, 1, 3, 0, 0}, {1, 1, 2, 1, 0}, {1, 1, 1, 1, 0}, {1, 1, 1, 1, 1}}]
```

```
16
```

```
Length[First[Split[{2, 1, 2, 0, 0}, And[#1 > 0, #2 > 0] &]]]
```

```
3
```

```
PadRight[{0, 1}, 5]
```

```
{0, 1, 0, 0, 0}
```

```
Clear[My2Lists, lis];
```

```
My2Lists[lis_] := Module[{tl, tla1, tla2, nn,}, tl = Append[lis, 0];
  nn0 = Length[First[Split[lis, And[#1 > 0, #2 > 0] &]]]; nn = Length[tl];
  tla1 = PadRight[PadLeft[{1}, nn0], nn]; tla2 = PadRight[PadLeft[{0, 1}, nn0 + 1], nn];
  {tl + tla1, tl + tla2}
 ]
```

```
My2Lists[{5, 0, 0, 0, 0}]
```

```
{{6, 0, 0, 0, 0, 0}, {5, 1, 0, 0, 0, 0}}
```

```
Flatten[Map[My2Lists[#] &, #], 1] &[{{1}}]
```

```
{{2, 0}, {1, 1}}
```

```
Flatten[Map[My2Lists[#] &, #], 1] &[{{2, 0}, {1, 1}}]
```

```
{{3, 0, 0}, {2, 1, 0}, {1, 2, 0}, {1, 1, 1}}
```

```
? Nest*
```

**System`**

```
Nest                NestList   NestWhileList
NestedScriptRules   NestWhile
```

Nest[f, expr, n] gives an expression with f applied n times to expr. More…

```
Nest[Flatten[Map[My2Lists[#] &, #], 1] &, {{1}}, 5]

{{6, 0, 0, 0, 0, 0}, {5, 1, 0, 0, 0, 0}, {4, 2, 0, 0, 0, 0}, {4, 1, 1, 0, 0, 0},
 {3, 3, 0, 0, 0, 0}, {3, 2, 1, 0, 0, 0}, {3, 1, 2, 0, 0, 0}, {3, 1, 1, 1, 0, 0},
 {2, 4, 0, 0, 0, 0}, {2, 3, 1, 0, 0, 0}, {2, 2, 2, 0, 0, 0}, {2, 2, 1, 1, 0, 0},
 {2, 1, 3, 0, 0, 0}, {2, 1, 2, 1, 0, 0}, {2, 1, 1, 2, 0, 0}, {2, 1, 1, 1, 1, 0},
 {1, 5, 0, 0, 0, 0}, {1, 4, 1, 0, 0, 0}, {1, 3, 2, 0, 0, 0}, {1, 3, 1, 1, 0, 0},
 {1, 2, 3, 0, 0, 0}, {1, 2, 2, 1, 0, 0}, {1, 2, 1, 2, 0, 0}, {1, 2, 1, 1, 1, 0},
 {1, 1, 4, 0, 0, 0}, {1, 1, 3, 1, 0, 0}, {1, 1, 2, 2, 0, 0}, {1, 1, 2, 1, 1, 0},
 {1, 1, 1, 3, 0, 0}, {1, 1, 1, 2, 1, 0}, {1, 1, 1, 1, 2, 0}, {1, 1, 1, 1, 1, 1}}

{1, 3, 2, 0, 0, 0}

Clear[LisFun, lis]; LisFun[lis_] := Module[{nn, nn0},
  nn = Apply[Plus, lis]; nn0 = Length[First[Split[lis, And[#1 > 0, #2 > 0] &]]];
  Product[Binomial[nn - Sum[lis[[i]], {i, 1, j - 1}], lis[[j]]], {j, 1, nn0}]
 ]

LisFun[{1, 3, 2, 0, 0, 0}]

60

6 * Binomial[5, 3] * Binomial[2, 2]

60

Apply[Plus, Map[LisFun[#] &, Nest[Flatten[Map[My2Lists[#] &, #], 1] &, {{1}}, 1]]]

3

Apply[Plus, Map[LisFun[#] &, Nest[Flatten[Map[My2Lists[#] &, #], 1] &, {{1}}, 2]]]

13

Apply[Plus, Map[LisFun[#] &, Nest[Flatten[Map[My2Lists[#] &, #], 1] &, {{1}}, 3]]]

75

Apply[Plus, Map[LisFun[#] &, Nest[Flatten[Map[My2Lists[#] &, #], 1] &, {{1}}, 4]]]

541

Apply[Plus, Map[LisFun[#] &, Nest[Flatten[Map[My2Lists[#] &, #], 1] &, {{1}}, 5]]]

4683

{3, 13, 75, 541, 4683}

Apply[Plus, Map[LisFun[#] &, Nest[Flatten[Map[My2Lists[#] &, #], 1] &, {{1}}, 6]]]

47293

Apply[Plus, Map[LisFun[#] &, Nest[Flatten[Map[My2Lists[#] &, #], 1] &, {{1}}, 7]]]

545835
```

```
Apply[Plus, Map[LisFun[#] &, Nest[Flatten[Map[My2Lists[#] &, #], 1] &, {{1}}, 8]]]
```

```
7087261
```

```
(* all tie *) 1 +
 (* three tie *) Binomial[4, 3] +
 (* two tie first *) Binomial[4, 2] (1 + 2) +
 (* clear winer, ties for 2 nd and 3 rd *)
 Binomial[4, 1] ( Binomial[3, 3] + Binomial[3, 2] + Binomial[3, 1]) +
 (* no ties *) 4 !
```

```
75
```

---

## Exercise 43

Ways to end

```
Flatten[{Table[{k, 0, 0}, {k, 6, 3, -1}], Table[{2, 0, k}, {k, 4, 1, -1}],
  Table[{1, k, 0}, {k, 5, 2, -1}], Table[{1, 1, k}, {k, 4, 1, -1}]}, 1]
```

```
{{6, 0, 0}, {5, 0, 0}, {4, 0, 0}, {3, 0, 0}, {2, 0, 4}, {2, 0, 3}, {2, 0, 2}, {2, 0, 1},
 {1, 5, 0}, {1, 4, 0}, {1, 3, 0}, {1, 2, 0}, {1, 1, 4}, {1, 1, 3}, {1, 1, 2}, {1, 1, 1}}
```

```
Sum[Binomial[6, k], {k, 6, 3, -1}] + Binomial[6, 2] * Sum[Binomial[4, k], {k, 4, 1, -1}] +
 Binomial[6, 1] * Sum[Binomial[5, k], {k, 5, 2, -1}] +
 2 * Binomial[6, 2] * Sum[Binomial[4, k], {k, 4, 1, -1}]
```

```
873
```

```
(* all gold *) Binomial[6, 6] + Binomial[6, 5] + Binomial[6, 4] + Binomial[6, 3] +
 (* two gold, ties for silver *)
 Binomial[6, 2] * (1 + Binomial[4, 3] + Binomial[4, 2] + Binomial[4, 1]) +
 (* one gold, mult silver *)
 Binomial[6, 1] * ( Binomial[5, 5] + Binomial[5, 4] + Binomial[5, 3] + Binomial[5, 2]) +
 (* one gold, one silver, mult bronze *) Binomial[6, 1] * Binomial[5, 1] *
  ( Binomial[4, 4] + Binomial[4, 3] + Binomial[4, 2] + Binomial[4, 1])
```

```
873
```