Exercise 10

Calculating the binomial coefficients we find out that if k is an odd number then the coefficient is 0. If k is even number between -50 and 50, then the coefficient is Binomial[100, (100-k)/2]. Test

```
{Coefficient[Expand[(x + 1/x)^100], x^4], Binomial[100, (100 - 4) / 2]}
```

```
{93 206 558 875 049 876 949 581 681 100, 93 206 558 875 049 876 949 581 681 100}
```

```
{Coefficient[Expand[(x + 1/x)^100], x^18], Binomial[100, (100 - 18) / 2]}
```

```
{20 116 440 213 369 968 050 635 175 200, 20 116 440 213 369 968 050 635 175 200}
```

To verify all the coefficients I use

Expand$\left[\left(\mathbf{x}+\frac{\mathbf{1}}{\mathbf{x}}\right)^{100}\right]$

$$100\,891\,344\,545\,564\,193\,334\,812\,497\,256 + \frac{1}{x^{100}} + \frac{100}{x^{98}} + \frac{4950}{x^{96}} + \frac{161\,700}{x^{94}} + \frac{3\,921\,225}{x^{92}} + \frac{75\,287\,520}{x^{90}} +$$
$$\frac{1\,192\,052\,400}{x^{88}} + \frac{16\,007\,560\,800}{x^{86}} + \frac{186\,087\,894\,300}{x^{84}} + \frac{1\,902\,231\,808\,400}{x^{82}} + \frac{17\,310\,309\,456\,440}{x^{80}} +$$
$$\frac{141\,629\,804\,643\,600}{x^{78}} + \frac{1\,050\,421\,051\,106\,700}{x^{76}} + \frac{7\,110\,542\,499\,799\,200}{x^{74}} + \frac{44\,186\,942\,677\,323\,600}{x^{72}} +$$
$$\frac{253\,338\,471\,349\,988\,640}{x^{70}} + \frac{1\,345\,860\,629\,046\,814\,650}{x^{68}} + \frac{6\,650\,134\,872\,937\,201\,800}{x^{66}} +$$
$$\frac{30\,664\,510\,802\,988\,208\,300}{x^{64}} + \frac{132\,341\,572\,939\,212\,267\,400}{x^{62}} + \frac{535\,983\,370\,403\,809\,682\,970}{x^{60}} +$$
$$\frac{2\,041\,841\,411\,062\,132\,125\,600}{x^{58}} + \frac{7\,332\,066\,885\,177\,656\,269\,200}{x^{56}} + \frac{24\,865\,270\,306\,254\,660\,391\,200}{x^{54}} +$$
$$\frac{79\,776\,075\,565\,900\,368\,755\,100}{x^{52}} + \frac{242\,519\,269\,720\,337\,121\,015\,504}{x^{50}} + \frac{699\,574\,816\,500\,972\,464\,467\,800}{x^{48}} +$$
$$1\,917\,353\,200\,780\,443\,050\,763\,600 / x^{46} + 4\,998\,813\,702\,034\,726\,525\,205\,100 / x^{44} +$$
$$12\,410\,847\,811\,948\,286\,545\,336\,800 / x^{42} + 29\,372\,339\,821\,610\,944\,823\,963\,760 / x^{40} +$$
$$66\,324\,638\,306\,863\,423\,796\,047\,200 / x^{38} + 143\,012\,501\,349\,174\,257\,560\,226\,775 / x^{36} +$$
$$294\,692\,427\,022\,540\,894\,366\,527\,900 / x^{34} + 580\,717\,429\,720\,889\,409\,486\,981\,450 / x^{32} +$$
$$1\,095\,067\,153\,187\,962\,886\,461\,165\,020 / x^{30} + 1\,977\,204\,582\,144\,932\,989\,443\,770\,175 / x^{28} +$$
$$3\,420\,029\,547\,493\,938\,143\,902\,737\,600 / x^{26} + 5\,670\,048\,986\,634\,686\,922\,786\,117\,600 / x^{24} +$$
$$9\,013\,924\,030\,034\,630\,492\,634\,340\,800 / x^{22} + 13\,746\,234\,145\,802\,811\,501\,267\,369\,720 / x^{20} +$$
$$20\,116\,440\,213\,369\,968\,050\,635\,175\,200 / x^{18} + 28\,258\,808\,871\,162\,574\,166\,368\,460\,400 / x^{16} +$$
$$38\,116\,532\,895\,986\,727\,945\,334\,202\,400 / x^{14} + 49\,378\,235\,797\,073\,715\,747\,364\,762\,200 / x^{12} +$$
$$61\,448\,471\,214\,136\,179\,596\,720\,592\,960 / x^{10} + 73\,470\,998\,190\,814\,997\,343\,905\,056\,800 / x^{8} +$$
$$84\,413\,487\,283\,064\,039\,501\,507\,937\,600 / x^{6} + 93\,206\,558\,875\,049\,876\,949\,581\,681\,100 / x^{4} +$$
$$98\,913\,082\,887\,808\,032\,681\,188\,722\,800 / x^{2} + 98\,913\,082\,887\,808\,032\,681\,188\,722\,800\, x^{2} +$$
$$93\,206\,558\,875\,049\,876\,949\,581\,681\,100\, x^{4} + 84\,413\,487\,283\,064\,039\,501\,507\,937\,600\, x^{6} +$$
$$73\,470\,998\,190\,814\,997\,343\,905\,056\,800\, x^{8} + 61\,448\,471\,214\,136\,179\,596\,720\,592\,960\, x^{10} +$$
$$49\,378\,235\,797\,073\,715\,747\,364\,762\,200\, x^{12} + 38\,116\,532\,895\,986\,727\,945\,334\,202\,400\, x^{14} +$$
$$28\,258\,808\,871\,162\,574\,166\,368\,460\,400\, x^{16} + 20\,116\,440\,213\,369\,968\,050\,635\,175\,200\, x^{18} +$$
$$13\,746\,234\,145\,802\,811\,501\,267\,369\,720\, x^{20} + 9\,013\,924\,030\,034\,630\,492\,634\,340\,800\, x^{22} +$$
$$5\,670\,048\,986\,634\,686\,922\,786\,117\,600\, x^{24} + 3\,420\,029\,547\,493\,938\,143\,902\,737\,600\, x^{26} +$$
$$1\,977\,204\,582\,144\,932\,989\,443\,770\,175\, x^{28} + 1\,095\,067\,153\,187\,962\,886\,461\,165\,020\, x^{30} +$$
$$580\,717\,429\,720\,889\,409\,486\,981\,450\, x^{32} + 294\,692\,427\,022\,540\,894\,366\,527\,900\, x^{34} +$$
$$143\,012\,501\,349\,174\,257\,560\,226\,775\, x^{36} + 66\,324\,638\,306\,863\,423\,796\,047\,200\, x^{38} +$$
$$29\,372\,339\,821\,610\,944\,823\,963\,760\, x^{40} + 12\,410\,847\,811\,948\,286\,545\,336\,800\, x^{42} +$$
$$4\,998\,813\,702\,034\,726\,525\,205\,100\, x^{44} + 1\,917\,353\,200\,780\,443\,050\,763\,600\, x^{46} +$$
$$699\,574\,816\,500\,972\,464\,467\,800\, x^{48} + 242\,519\,269\,720\,337\,121\,015\,504\, x^{50} +$$
$$79\,776\,075\,565\,900\,368\,755\,100\, x^{52} + 24\,865\,270\,306\,254\,660\,391\,200\, x^{54} +$$
$$7\,332\,066\,885\,177\,656\,269\,200\, x^{56} + 2\,041\,841\,411\,062\,132\,125\,600\, x^{58} + 535\,983\,370\,403\,809\,682\,970\, x^{60} +$$
$$132\,341\,572\,939\,212\,267\,400\, x^{62} + 30\,664\,510\,802\,988\,208\,300\, x^{64} + 6\,650\,134\,872\,937\,201\,800\, x^{66} +$$
$$1\,345\,860\,629\,046\,814\,650\, x^{68} + 253\,338\,471\,349\,988\,640\, x^{70} + 44\,186\,942\,677\,323\,600\, x^{72} +$$
$$7\,110\,542\,499\,799\,200\, x^{74} + 1\,050\,421\,051\,106\,700\, x^{76} + 141\,629\,804\,643\,600\, x^{78} +$$
$$17\,310\,309\,456\,440\, x^{80} + 1\,902\,231\,808\,400\, x^{82} + 186\,087\,894\,300\, x^{84} + 16\,007\,560\,800\, x^{86} +$$
$$1\,192\,052\,400\, x^{88} + 75\,287\,520\, x^{90} + 3\,921\,225\, x^{92} + 161\,700\, x^{94} + 4950\, x^{96} + 100\, x^{98} + x^{100}$$

However, to get coefficients I have to turn the expression into a polynomial. Here is a smaller example

```
Expand[x^10 (x + 1/x)^10]
```

$1 + 10\,x^2 + 45\,x^4 + 120\,x^6 + 210\,x^8 + 252\,x^{10} + 210\,x^{12} + 120\,x^{14} + 45\,x^{16} + 10\,x^{18} + x^{20}$

```
CoefficientList[x^10 Expand[(x + 1/x)^10], x]
```

```
{1, 0, 10, 0, 45, 0, 120, 0, 210, 0, 252, 0, 210, 0, 120, 0, 45, 0, 10, 0, 1}
```

```
Table[{CoefficientList[Expand[x^100 (x + 1/x)^100], x][[2 (j + 50 + 1) - 1]], Binomial[100, 50 - j]},
 {j, -50, 50}]
```

```
{{1, 1}, {100, 100}, {4950, 4950}, {161 700, 161 700}, {3 921 225, 3 921 225},
 {75 287 520, 75 287 520}, {1 192 052 400, 1 192 052 400}, {16 007 560 800, 16 007 560 800},
 {186 087 894 300, 186 087 894 300}, {1 902 231 808 400, 1 902 231 808 400},
 {17 310 309 456 440, 17 310 309 456 440}, {141 629 804 643 600, 141 629 804 643 600},
 {1 050 421 051 106 700, 1 050 421 051 106 700}, {7 110 542 499 799 200, 7 110 542 499 799 200},
 {44 186 942 677 323 600, 44 186 942 677 323 600}, {253 338 471 349 988 640, 253 338 471 349 988 640},
 {1 345 860 629 046 814 650, 1 345 860 629 046 814 650},
 {6 650 134 872 937 201 800, 6 650 134 872 937 201 800},
 {30 664 510 802 988 208 300, 30 664 510 802 988 208 300},
 {132 341 572 939 212 267 400, 132 341 572 939 212 267 400},
 {535 983 370 403 809 682 970, 535 983 370 403 809 682 970},
 {2 041 841 411 062 132 125 600, 2 041 841 411 062 132 125 600},
 {7 332 066 885 177 656 269 200, 7 332 066 885 177 656 269 200},
 {24 865 270 306 254 660 391 200, 24 865 270 306 254 660 391 200},
 {79 776 075 565 900 368 755 100, 79 776 075 565 900 368 755 100},
 {242 519 269 720 337 121 015 504, 242 519 269 720 337 121 015 504},
 {699 574 816 500 972 464 467 800, 699 574 816 500 972 464 467 800},
 {1 917 353 200 780 443 050 763 600, 1 917 353 200 780 443 050 763 600},
 {4 998 813 702 034 726 525 205 100, 4 998 813 702 034 726 525 205 100},
 {12 410 847 811 948 286 545 336 800, 12 410 847 811 948 286 545 336 800},
 {29 372 339 821 610 944 823 963 760, 29 372 339 821 610 944 823 963 760},
 {66 324 638 306 863 423 796 047 200, 66 324 638 306 863 423 796 047 200},
 {143 012 501 349 174 257 560 226 775, 143 012 501 349 174 257 560 226 775},
 {294 692 427 022 540 894 366 527 900, 294 692 427 022 540 894 366 527 900},
 {580 717 429 720 889 409 486 981 450, 580 717 429 720 889 409 486 981 450},
 {1 095 067 153 187 962 886 461 165 020, 1 095 067 153 187 962 886 461 165 020},
 {1 977 204 582 144 932 989 443 770 175, 1 977 204 582 144 932 989 443 770 175},
 {3 420 029 547 493 938 143 902 737 600, 3 420 029 547 493 938 143 902 737 600},
 {5 670 048 986 634 686 922 786 117 600, 5 670 048 986 634 686 922 786 117 600},
 {9 013 924 030 034 630 492 634 340 800, 9 013 924 030 034 630 492 634 340 800},
 {13 746 234 145 802 811 501 267 369 720, 13 746 234 145 802 811 501 267 369 720},
 {20 116 440 213 369 968 050 635 175 200, 20 116 440 213 369 968 050 635 175 200},
 {28 258 808 871 162 574 166 368 460 400, 28 258 808 871 162 574 166 368 460 400},
 {38 116 532 895 986 727 945 334 202 400, 38 116 532 895 986 727 945 334 202 400},
 {49 378 235 797 073 715 747 364 762 200, 49 378 235 797 073 715 747 364 762 200},
 {61 448 471 214 136 179 596 720 592 960, 61 448 471 214 136 179 596 720 592 960},
 {73 470 998 190 814 997 343 905 056 800, 73 470 998 190 814 997 343 905 056 800},
 {84 413 487 283 064 039 501 507 937 600, 84 413 487 283 064 039 501 507 937 600},
 {93 206 558 875 049 876 949 581 681 100, 93 206 558 875 049 876 949 581 681 100},
 {98 913 082 887 808 032 681 188 722 800, 98 913 082 887 808 032 681 188 722 800},
 {100 891 344 545 564 193 334 812 497 256, 100 891 344 545 564 193 334 812 497 256},
 {98 913 082 887 808 032 681 188 722 800, 98 913 082 887 808 032 681 188 722 800},
```

```
{93 206 558 875 049 876 949 581 681 100, 93 206 558 875 049 876 949 581 681 100},
{84 413 487 283 064 039 501 507 937 600, 84 413 487 283 064 039 501 507 937 600},
{73 470 998 190 814 997 343 905 056 800, 73 470 998 190 814 997 343 905 056 800},
{61 448 471 214 136 179 596 720 592 960, 61 448 471 214 136 179 596 720 592 960},
{49 378 235 797 073 715 747 364 762 200, 49 378 235 797 073 715 747 364 762 200},
{38 116 532 895 986 727 945 334 202 400, 38 116 532 895 986 727 945 334 202 400},
{28 258 808 871 162 574 166 368 460 400, 28 258 808 871 162 574 166 368 460 400},
{20 116 440 213 369 968 050 635 175 200, 20 116 440 213 369 968 050 635 175 200},
{13 746 234 145 802 811 501 267 369 720, 13 746 234 145 802 811 501 267 369 720},
{9 013 924 030 034 630 492 634 340 800, 9 013 924 030 034 630 492 634 340 800},
{5 670 048 986 634 686 922 786 117 600, 5 670 048 986 634 686 922 786 117 600},
{3 420 029 547 493 938 143 902 737 600, 3 420 029 547 493 938 143 902 737 600},
{1 977 204 582 144 932 989 443 770 175, 1 977 204 582 144 932 989 443 770 175},
{1 095 067 153 187 962 886 461 165 020, 1 095 067 153 187 962 886 461 165 020},
{580 717 429 720 889 409 486 981 450, 580 717 429 720 889 409 486 981 450},
{294 692 427 022 540 894 366 527 900, 294 692 427 022 540 894 366 527 900},
{143 012 501 349 174 257 560 226 775, 143 012 501 349 174 257 560 226 775},
{66 324 638 306 863 423 796 047 200, 66 324 638 306 863 423 796 047 200},
{29 372 339 821 610 944 823 963 760, 29 372 339 821 610 944 823 963 760},
{12 410 847 811 948 286 545 336 800, 12 410 847 811 948 286 545 336 800},
{4 998 813 702 034 726 525 205 100, 4 998 813 702 034 726 525 205 100},
{1 917 353 200 780 443 050 763 600, 1 917 353 200 780 443 050 763 600},
{699 574 816 500 972 464 467 800, 699 574 816 500 972 464 467 800},
{242 519 269 720 337 121 015 504, 242 519 269 720 337 121 015 504},
{79 776 075 565 900 368 755 100, 79 776 075 565 900 368 755 100},
{24 865 270 306 254 660 391 200, 24 865 270 306 254 660 391 200},
{7 332 066 885 177 656 269 200, 7 332 066 885 177 656 269 200},
{2 041 841 411 062 132 125 600, 2 041 841 411 062 132 125 600},
{535 983 370 403 809 682 970, 535 983 370 403 809 682 970},
{132 341 572 939 212 267 400, 132 341 572 939 212 267 400},
{30 664 510 802 988 208 300, 30 664 510 802 988 208 300},
{6 650 134 872 937 201 800, 6 650 134 872 937 201 800},
{1 345 860 629 046 814 650, 1 345 860 629 046 814 650},
{253 338 471 349 988 640, 253 338 471 349 988 640}, {44 186 942 677 323 600, 44 186 942 677 323 600},
{7 110 542 499 799 200, 7 110 542 499 799 200}, {1 050 421 051 106 700, 1 050 421 051 106 700},
{141 629 804 643 600, 141 629 804 643 600}, {17 310 309 456 440, 17 310 309 456 440},
{1 902 231 808 400, 1 902 231 808 400}, {186 087 894 300, 186 087 894 300},
{16 007 560 800, 16 007 560 800}, {1 192 052 400, 1 192 052 400}, {75 287 520, 75 287 520},
{3 921 225, 3 921 225}, {161 700, 161 700}, {4950, 4950}, {100, 100}, {1, 1}}
```

Or, even better,

```
Table[CoefficientList[Expand[x^100 (x + 1/x)^100], x][[2 (j + 50 + 1) - 1]] - Binomial[100, 50 - j],
 {j, -50, 50}]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Exercise 14

We will prove that Binomial[n,Floor[n/2]] = Binomial[n,Ceiling[n/2]].

First check the first few terms.

```
Table[{Binomial[n, Floor[n / 2]], Binomial[n, Ceiling[n / 2]]}, {n, 1, 20}]
```

```
{{1, 1}, {2, 2}, {3, 3}, {6, 6}, {10, 10}, {20, 20}, {35, 35}, {70, 70}, {126, 126},
 {252, 252}, {462, 462}, {924, 924}, {1716, 1716}, {3432, 3432}, {6435, 6435},
 {12 870, 12 870}, {24 310, 24 310}, {48 620, 48 620}, {92 378, 92 378}, {184 756, 184 756}}
```

If n is even, then $n = 2k$ for some positive integer k. In this case Floor[n] = Ceiling[n] = k.

If n is odd, then $n = 2k - 1$ for some positive integer k. In this case Floor[n] = k-1 and Ceiling[n] = k. Since for every j Binomial[n,j] = Binomial[n, n-j], we have Binomial[n,k-1] = Binomial[n, n-k+1] = Binomial[n, 2 k-1 -k+1] = Binomial[n, k].

Next, let 1≤ k < Floor[n/2]. Then 2k < 2 Floor[n/2]. Consequently, 2k+1 ≤ 2 Floor[n/2]. Since these two numbers can not be equal (one is even, the other odd), we have 2k+1 < 2 Floor[n/2]. Clearly 2 Floor[n/2] ≤ n, and thus 2 k + 1 < n. This yields, k+1 < n-k, and hence

$\frac{1}{k+1} > \frac{1}{n-k}$.

Multiplying both sides by $\frac{n!}{k!\,(n-k-1)!}$ we get $\frac{n!}{(k+1)!\,(n-k-1)!} > \frac{n!}{k!\,(n-k)!}$, that is Binomial[n,k+1] > Binomial[n,k].

Exercise 15

The number Binomial[n,k] represents the number of bit strings of length n with exactly k zeros. The number 2^n represents the number of all bit strings of length n. Thus the inequality.

Exercise 16

```
Table[{N[Binomial[n, Floor[n / 2]], 2], N[2^n / (n), 2]}, {n, 2, 100}]
```

$\{\{2.0, 2.0\}, \{3.0, 2.7\}, \{6.0, 4.0\}, \{10., 6.4\}, \{20., 11.\}, \{35., 18.\}, \{70., 32.\},$
$\{1.3\times10^{2}, 57.\}, \{2.5\times10^{2}, 1.0\times10^{2}\}, \{4.6\times10^{2}, 1.9\times10^{2}\}, \{9.2\times10^{2}, 3.4\times10^{2}\},$
$\{1.7\times10^{3}, 6.3\times10^{2}\}, \{3.4\times10^{3}, 1.2\times10^{3}\}, \{6.4\times10^{3}, 2.2\times10^{3}\}, \{1.3\times10^{4}, 4.1\times10^{3}\},$
$\{2.4\times10^{4}, 7.7\times10^{3}\}, \{4.9\times10^{4}, 1.5\times10^{4}\}, \{9.2\times10^{4}, 2.8\times10^{4}\}, \{1.8\times10^{5}, 5.2\times10^{4}\},$
$\{3.5\times10^{5}, 1.0\times10^{5}\}, \{7.1\times10^{5}, 1.9\times10^{5}\}, \{1.4\times10^{6}, 3.6\times10^{5}\}, \{2.7\times10^{6}, 7.0\times10^{5}\},$
$\{5.2\times10^{6}, 1.3\times10^{6}\}, \{1.0\times10^{7}, 2.6\times10^{6}\}, \{2.0\times10^{7}, 5.0\times10^{6}\}, \{4.0\times10^{7}, 9.6\times10^{6}\},$
$\{7.8\times10^{7}, 1.9\times10^{7}\}, \{1.6\times10^{8}, 3.6\times10^{7}\}, \{3.0\times10^{8}, 6.9\times10^{7}\}, \{6.0\times10^{8}, 1.3\times10^{8}\},$
$\{1.2\times10^{9}, 2.6\times10^{8}\}, \{2.3\times10^{9}, 5.1\times10^{8}\}, \{4.5\times10^{9}, 9.8\times10^{8}\}, \{9.1\times10^{9}, 1.9\times10^{9}\},$
$\{1.8\times10^{10}, 3.7\times10^{9}\}, \{3.5\times10^{10}, 7.2\times10^{9}\}, \{6.9\times10^{10}, 1.4\times10^{10}\}, \{1.4\times10^{11}, 2.7\times10^{10}\},$
$\{2.7\times10^{11}, 5.4\times10^{10}\}, \{5.4\times10^{11}, 1.0\times10^{11}\}, \{1.1\times10^{12}, 2.0\times10^{11}\}, \{2.1\times10^{12}, 4.0\times10^{11}\},$
$\{4.1\times10^{12}, 7.8\times10^{11}\}, \{8.2\times10^{12}, 1.5\times10^{12}\}, \{1.6\times10^{13}, 3.0\times10^{12}\}, \{3.2\times10^{13}, 5.9\times10^{12}\},$
$\{6.3\times10^{13}, 1.1\times10^{13}\}, \{1.3\times10^{14}, 2.3\times10^{13}\}, \{2.5\times10^{14}, 4.4\times10^{13}\}, \{5.0\times10^{14}, 8.7\times10^{13}\},$
$\{9.7\times10^{14}, 1.7\times10^{14}\}, \{1.9\times10^{15}, 3.3\times10^{14}\}, \{3.8\times10^{15}, 6.6\times10^{14}\}, \{7.6\times10^{15}, 1.3\times10^{15}\},$
$\{1.5\times10^{16}, 2.5\times10^{15}\}, \{3.0\times10^{16}, 5.0\times10^{15}\}, \{5.9\times10^{16}, 9.8\times10^{15}\}, \{1.2\times10^{17}, 1.9\times10^{16}\},$
$\{2.3\times10^{17}, 3.8\times10^{16}\}, \{4.7\times10^{17}, 7.4\times10^{16}\}, \{9.2\times10^{17}, 1.5\times10^{17}\}, \{1.8\times10^{18}, 2.9\times10^{17}\},$
$\{3.6\times10^{18}, 5.7\times10^{17}\}, \{7.2\times10^{18}, 1.1\times10^{18}\}, \{1.4\times10^{19}, 2.2\times10^{18}\}, \{2.8\times10^{19}, 4.3\times10^{18}\},$
$\{5.6\times10^{19}, 8.6\times10^{18}\}, \{1.1\times10^{20}, 1.7\times10^{19}\}, \{2.2\times10^{20}, 3.3\times10^{19}\}, \{4.4\times10^{20}, 6.6\times10^{19}\},$
$\{8.7\times10^{20}, 1.3\times10^{20}\}, \{1.7\times10^{21}, 2.6\times10^{20}\}, \{3.4\times10^{21}, 5.0\times10^{20}\}, \{6.9\times10^{21}, 9.9\times10^{20}\},$
$\{1.4\times10^{22}, 2.0\times10^{21}\}, \{2.7\times10^{22}, 3.9\times10^{21}\}, \{5.4\times10^{22}, 7.7\times10^{21}\}, \{1.1\times10^{23}, 1.5\times10^{22}\},$
$\{2.1\times10^{23}, 3.0\times10^{22}\}, \{4.2\times10^{23}, 5.9\times10^{22}\}, \{8.4\times10^{23}, 1.2\times10^{23}\}, \{1.7\times10^{24}, 2.3\times10^{23}\},$
$\{3.3\times10^{24}, 4.6\times10^{23}\}, \{6.6\times10^{24}, 9.0\times10^{23}\}, \{1.3\times10^{25}, 1.8\times10^{24}\}, \{2.6\times10^{25}, 3.5\times10^{24}\},$
$\{5.2\times10^{25}, 7.0\times10^{24}\}, \{1.0\times10^{26}, 1.4\times10^{25}\}, \{2.1\times10^{26}, 2.7\times10^{25}\}, \{4.1\times10^{26}, 5.4\times10^{25}\},$
$\{8.1\times10^{26}, 1.1\times10^{26}\}, \{1.6\times10^{27}, 2.1\times10^{26}\}, \{3.2\times10^{27}, 4.2\times10^{26}\}, \{6.4\times10^{27}, 8.3\times10^{26}\},$
$\{1.3\times10^{28}, 1.6\times10^{27}\}, \{2.5\times10^{28}, 3.2\times10^{27}\}, \{5.0\times10^{28}, 6.4\times10^{27}\}, \{1.0\times10^{29}, 1.3\times10^{28}\}\}$

First notice that for $n \geq 2$ we have Binomial[n,Floor[n/2]] ≥ Binomial[n,0] + Binomial[n,1]. Therefore

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{n} + \sum_{k=1}^{n-1} \binom{n}{k} \leq \binom{n}{\text{Floor}[n/2]} + (n-1)\binom{n}{\text{Floor}[n/2]} = n\binom{n}{\text{Floor}[n/2]}$$

Exercise 22

This is counting number of committees with a subcommittees. For example the university faculty elects the faculty senate, then the senate elects its executive committee. If there are n faculty members, the senate has r members and the executive committee of the senate has k members count number of different ways of forming the senate with its executive committee.

One way of counting: Choose the senate, then choose the executive committee from the senate. By the product rule there are Binomial[n,r]*Binomial[r,k] ways to do this.

Another way of counting is: choose the executive committee from the whole faculty, then choose the remaining members of the senate. By the product rule there are Binomial[n,k]*Binomial[n-k,r-k] ways to do this.

Since we count the same set these two numbers must be equal.

Exercise 25

A combinatorial way to prove this identity is to look at a class of 2n males and 2 females and count the number of committees of n+1 members that can be formed. There are Binomial[2n+2,n+1] such committees.

Now we count the number of committees based of number of female members.

There are Binomial[2n,n+1] committees with no female members, there are 2 Binomial[2n,n] with one female member and thee are Binomial[2n,n-1] committees with two female members. Since Binomial[2n,n-1] = Binomial[2n,n+1], the formula follows.

Exercise 26

Consider the formula

$$\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} = \sum_{k=1}^{n} \binom{n}{k}\binom{n}{n+1-k} = \binom{2n}{n+1}$$

Each side counts the number of committees with n+1 member from a class of 2n people. In the sum we assume that there are n males and n females and then count the committees with 1, 2, 3, ..., n males.

Now we need to prove

$$\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+2}{n+1}\Big/2$$

or, simpler,

$$\binom{2n+2}{n+1} = 2\binom{2n}{n+1} + 2\binom{2n}{n}$$

This can be proved by using Pascal's identity twice

$$\binom{2n+2}{n+1} = \binom{2n+1}{n+1} + \binom{2n+1}{n} =$$

$$\binom{2n}{n+1} + \binom{2n}{n} + \binom{2n}{n} + \binom{2n}{n-1} = \binom{2n}{n+1} + 2\binom{2n}{n} + \binom{2n}{2n-(n-1)} = 2\binom{2n}{n+1} + 2\binom{2n}{n}$$

The identity

$$\binom{2n+2}{n+1} = 2\binom{2n}{n+1} + 2\binom{2n}{n}$$

can be proved using counting. (This kind of a proof is called a combinatorial proof.) The number

$$\binom{2n+2}{n+1}$$

counts the number of committees of $n+1$ people selected from a group of $2n+2$ people. Assume that the number 2 in $2n+2$ represents **me** and **my brother**. Now count the number of committees of $n+1$ people selected from a group of $2n+2$ people which have **me** and **my brother** on the committee, the number is

$$\binom{2n}{n-1} = \binom{2n}{2n-n+1} = \binom{2n}{n+1},$$

the number of committees of $n+1$ people selected from a group of $2n+2$ people which have **me** but not **my brother** on the committee, the number is

$$\binom{2n}{n},$$

the number of committees of $n+1$ people selected from a group of $2n+2$ people which have do not have **me** but do have **my brother** on the committee, the number is

$$\binom{2n}{n},$$

the number of committees of $n+1$ people selected from a group of $2n+2$ people which neither have **me**, nor **my brother** on the committee, the number is

$$\binom{2n}{n+1}.$$

These four kinds of committees constitute all the committees. This proves the identity

$$\binom{2n+2}{n+1} = \binom{2n}{n-1} + 2\binom{2n}{n} + \binom{2n}{n+1} = 2\binom{2n}{n+1} + 2\binom{2n}{n}.$$

Exercise 27

Prove

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

```
In[20]:= nn = 7; rr = 5; Sum[Binomial[nn + k, k], {k, 0, rr}] - Binomial[nn + rr + 1, rr]

Out[20]= 0

In[25]:= nn = 17; rr = 45; Sum[Binomial[nn + k, nn], {k, 0, rr}] - Binomial[nn + rr + 1, nn + 1]

Out[25]= 0
```

We are counting bit strings with n+1 ones and r zeros. Those that start with
1 are Binomial[nn+rr,nn]
01 are Binomial[nn+rr-1,nn]
001 are Binomial[nn+rr-2,nn]
0001 are Binomial[nn+rr-3,nn]
...
0 rr 01 are Binomial[nn+rr-rr,nn]