Axioms for the Set $\mathbb{R}$ of Real Numbers

Axiom 1 (**AE**: Addition exists)**.** If $a, b \in \mathbb{R}$, then the sum of a and b, denoted by $a + b$, is a uniquely defined number in $\mathbb{R}$.

Axiom 2 (**AA**: Addition is associative)**.** For all $a, b, c \in \mathbb{R}$ we have $a + (b + c) = (a + b) + c$.

Axiom 3 (**AC**: Addition is commutative)**.** For all $a, b \in \mathbb{R}$ we have $a + b = b + a$.

Axiom 4 (**AZ**: Addition has 0)**.** There is an element 0 in $\mathbb{R}$ such that $0 + a = a + 0 = a$ for all $a \in \mathbb{R}$.

Axiom 5 (**AO**: Addition has opposites)**.** If $a \in \mathbb{R}$, then the equation $a + x = 0$ has a solution $-a \in \mathbb{R}$. The number $-a$ is called the *opposite* of a.

Axiom 6 (**ME**: Multiplication exists)**.** If $a, b \in \mathbb{R}$, then the product of a and b, denoted by ab, is a uniquely defined number in $\mathbb{R}$.

Axiom 7 (**MA**: Multiplication is associative)**.** For all $a, b, c \in \mathbb{R}$ we have $a(bc) = (ab)c$.

Axiom 8 (**MC**: Multiplication is commutative)**.** For all $a, b \in \mathbb{R}$ we have $ab = ba$.

Axiom 9 (**MO**: Multiplication has 1)**.** There is an element $1 \neq 0$ in $\mathbb{R}$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathbb{R}$.

Axiom 10 (**MR**: Multiplication has reciprocals)**.** If $a \in \mathbb{R}$ is such that $a \neq 0$, then the equation $a \cdot x = 1$ has a solution $a^{-1} = \dfrac{1}{a}$ in $\mathbb{R}$. The number $a^{-1} = \dfrac{1}{a}$ is called the *reciprocal* of a.

Axiom 11 (**DL**: Distributive law, the connection between addition and multiplication)**.** For all $a, b, c \in \mathbb{R}$ we have $a(b + c) = ab + ac$.

Axiom 12 (**OE**: Order exists)**.** Given any $a, b \in \mathbb{R}$, exactly one of these statements is true: $a < b$, $a = b$, or $b < a$. (The symbol $a \leq b$ stands for $a < b$ or $a = b$.)

Axiom 13 (**OT**: Order is transitive)**.** Given any $a, b, c \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$.

Axiom 14 (**OA**: Order respects addition)**.** Given any $a, b, c \in \mathbb{R}$, if $a < b$ then $a + c < b + c$.

Axiom 15 (**OM**: Order respects multiplication)**.** Given any $a, b, c \in \mathbb{R}$, if $a < b$ and $0 < c$, then $ac < bc$.

Axiom 16 (**CA**: Completeness Axiom)**.** If A and B are nonempty subsets of $\mathbb{R}$ such that for every $a \in A$ and for every $b \in B$ we have $a \leq b$, then there exists $c \in \mathbb{R}$ such that $a \leq c \leq b$ for all $a \in A$ and all $b \in B$.

The end of axioms.

All statements about real numbers that are studied in beginning mathematical analysis courses can be deduced from these sixteen axioms.

The formulation of the **Completeness Axiom** given as **Axiom 16** is not standard. This formulation I found in the book *Mathematical analysis* by Vladimir Zorich, published by Springer in 2004. Zorich's formulation is easier to state and it is equivalent to the standard formulation of the Completeness Axiom.