# Periodic extension

Below is an implementation of the concept of periodic extension in Mathematica. The function **PerExt** is a function of five variables: **x** this is an arbitrary real number, **ff** is a function defined on a half-open interval **[aa, bb)** (here we assume that **aa < bb**) and we consider **aa** and **bb** as variable in this function. It is important to note that **ff** must be given as a *Pure Function* function

In[113]:=
```
Clear[PerExt, ff, aa, bb, x];
PerExt[x_, ff_, aa_, bb_] :=
 ff[x - (Floor[(x - aa)/(bb - aa)]) (bb - aa)]
```

Example of a pure function

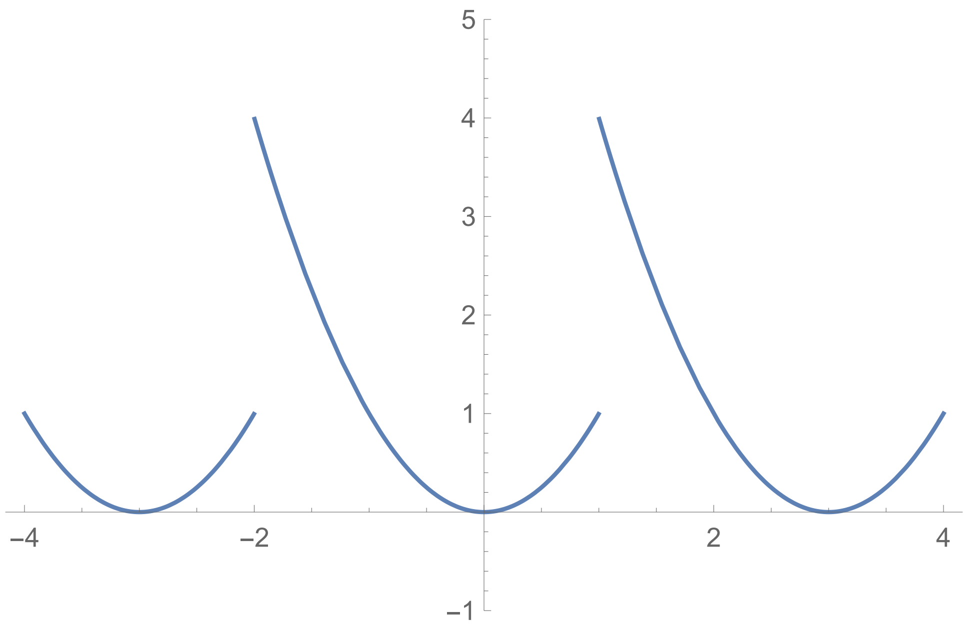
In[114]:=
```
(#^2) &[4]
```

Out[114]= 16

The square function defined on the interval [-2,1] and then periodically extended:

In[115]:=
```
Plot[PerExt[x, (#^2) &, -2, 1], {x, -4, 4},
 PlotRange → {-1, 5}, Exclusions → Range[-32, 10, 3]]
```

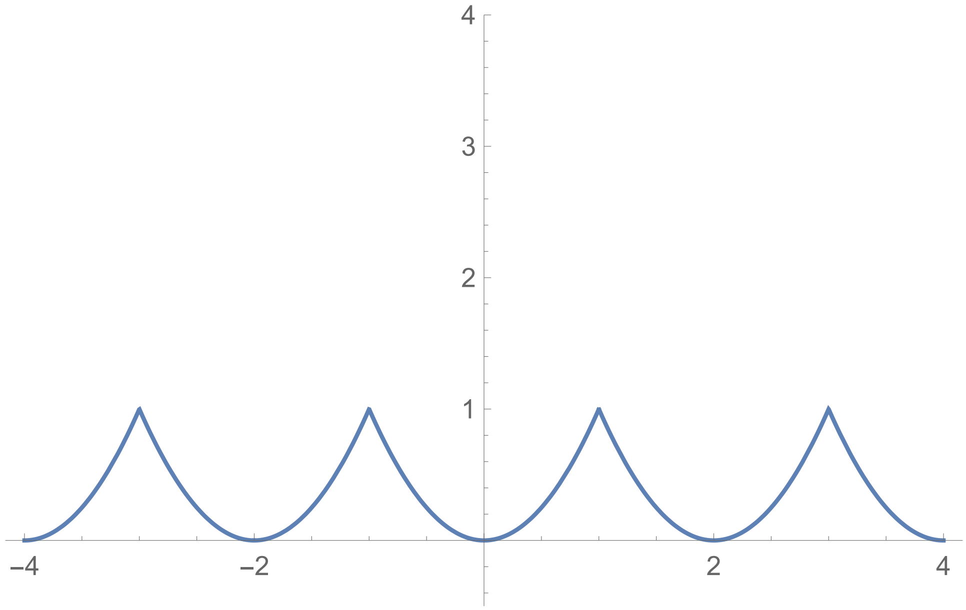
Out[115]=




Or, with a continuous periodic extension, the square function defined on [-1,1], and periodically extended:

In[116]:=
```
Plot[PerExt[x, (#^2) &, -1, 1], {x, -4, 4},
 PlotRange → {-0.5, 4}, Exclusions → Range[-32, 10, 3]]
```

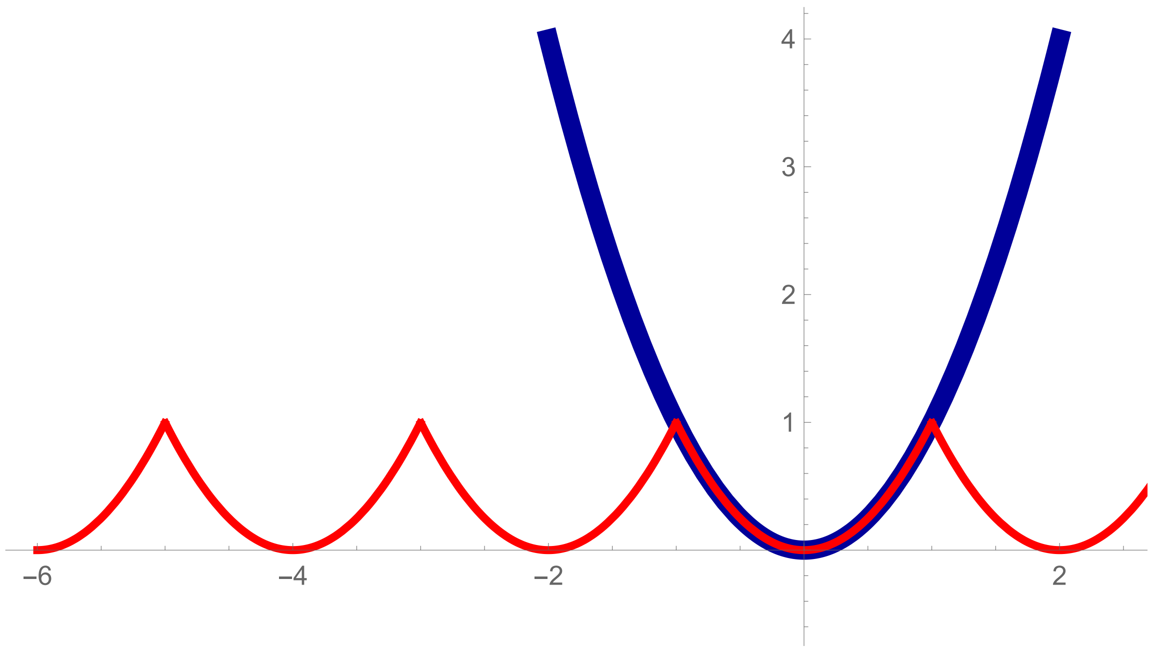
Out[116]=




In the above example the given function is $x^2$. Below we show this function and its periodic extension with period 2:

```
In[117]:= Show[Plot[(#^2) &[x], {x, -2, 2},
    PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.012]}}],
  Plot[PerExt[x, (#^2) &, -1, 1], {x, -6, 6},
    PlotStyle → {{RGBColor[1, 0, 0], Thickness[0.005]}}],
  PlotRange → {-0.5, 4}, AspectRatio → Automatic,
  ImageSize → 600]
```

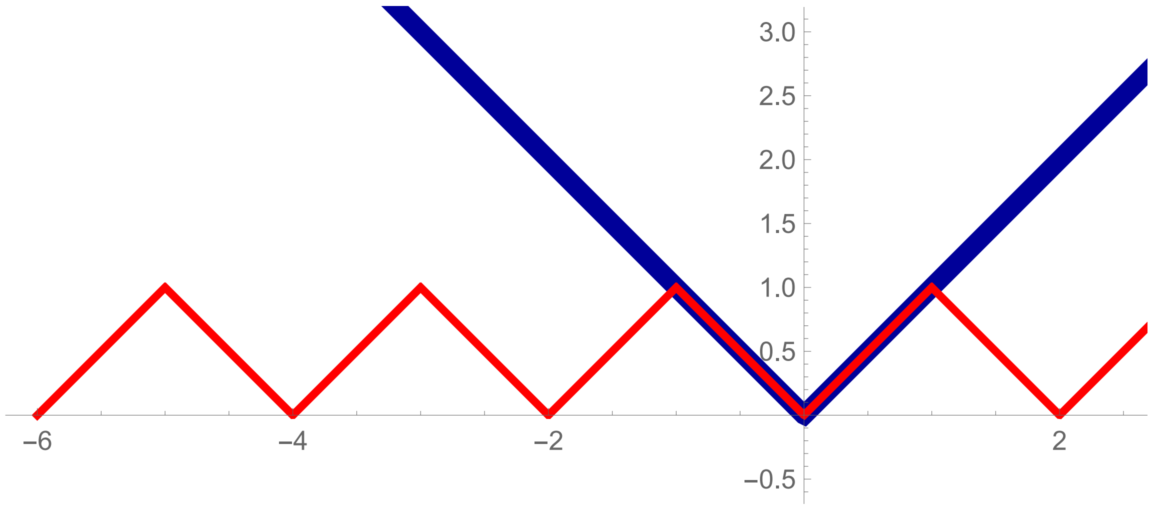
Out[117]=




More examples:

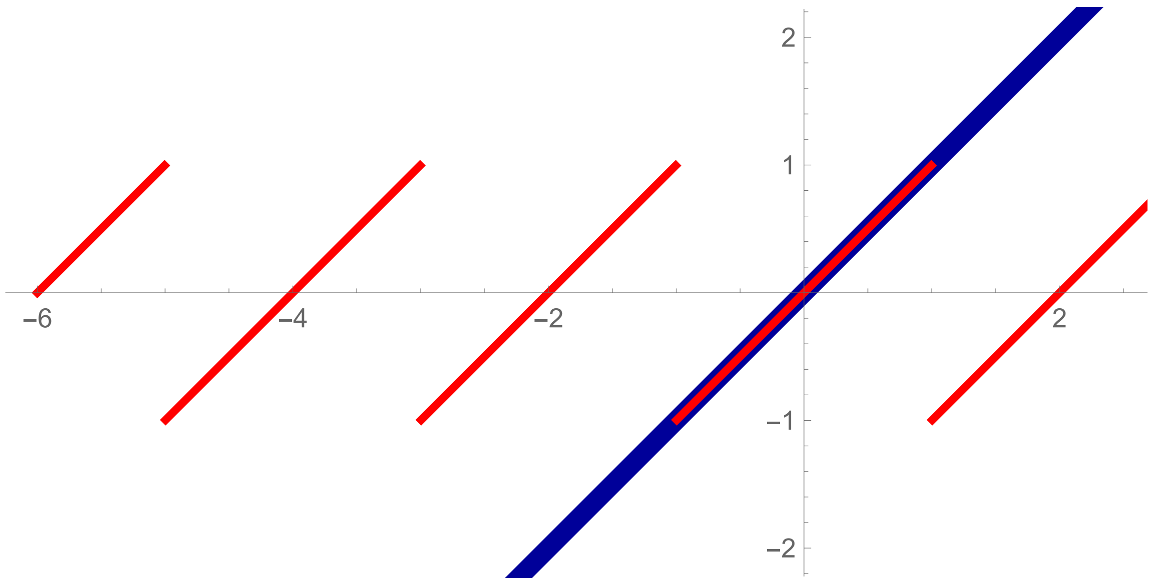
In[118]:=
```
Show[Plot[(Abs[#]) &[x], {x, -5, 5},
   PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.012]}}],
  Plot[PerExt[x, (Abs[#]) &, -1, 1], {x, -6, 6},
   PlotStyle → {{RGBColor[1, 0, 0], Thickness[0.005]}}],
  PlotRange → {-0.5, 3}, AspectRatio → Automatic,
  ImageSize → 600]
```

Out[118]=



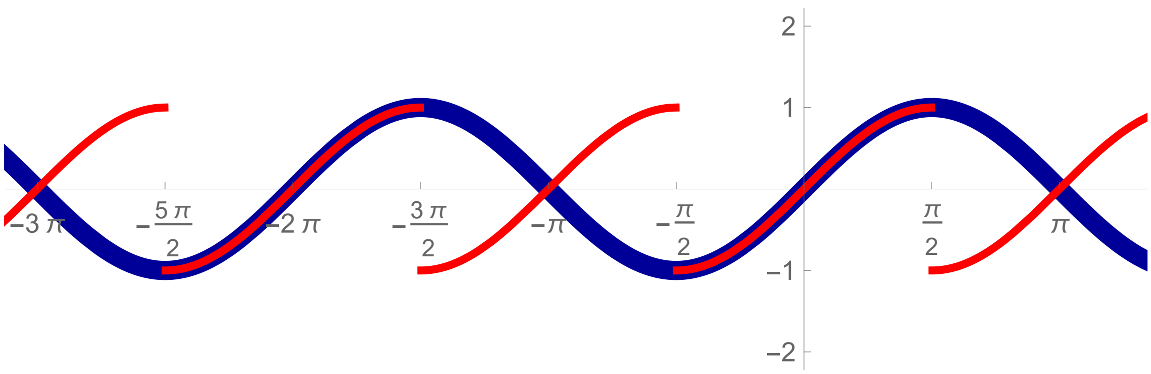
In[119]:=
```
Show[Plot[(#) &[x], {x, -5, 5},
  PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.012]}}],
 Plot[PerExt[x, (#) &, -1, 1], {x, -6, 6},
  PlotStyle → {{RGBColor[1, 0, 0], Thickness[0.005]}}],
 PlotRange → {-2, 2}, AspectRatio → Automatic,
 ImageSize → 600]
```

Out[119]=

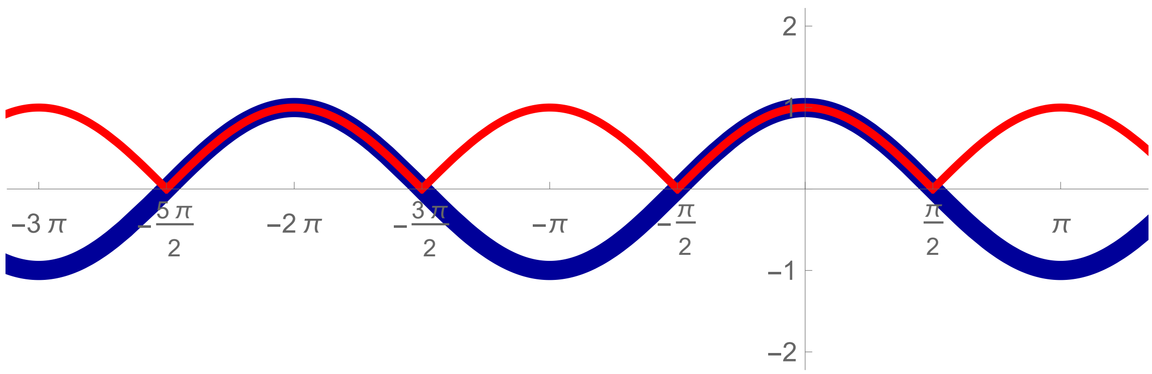


In[120]:=
```
Show[Plot[(Sin[#]) &[x], {x, -5 Pi, 5 Pi},
  PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.012]}}],
 Plot[PerExt[x, (Sin[#]) &, -Pi/2, Pi/2], {x, -4 Pi, 4 Pi},
  PlotStyle → {{RGBColor[1, 0, 0], Thickness[0.005]}}],
 PlotRange → {{-3 Pi, 3 Pi}, {-2, 2}},
 Ticks → {Range[-6 Pi, 6 Pi, Pi / 2], Range[-2, 2, 1]},
 AspectRatio → Automatic, ImageSize → 600]
```

Out[120]=

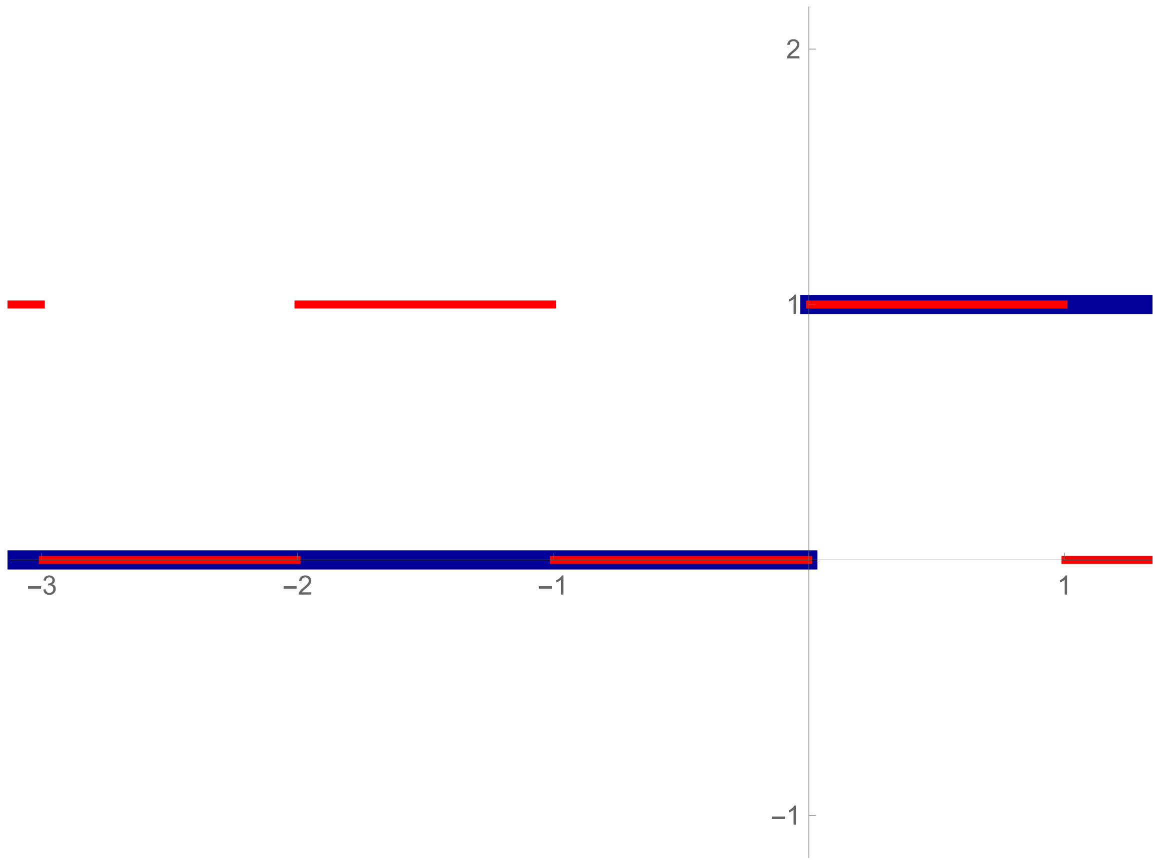


In[121]:=
```
Show[Plot[(Cos[#]) &[x], {x, -5 Pi, 5 Pi},
  PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.012]}}],
 Plot[PerExt[x, (Cos[#]) &, -Pi/2, Pi/2], {x, -4 Pi, 4 Pi},
  PlotStyle → {{RGBColor[1, 0, 0], Thickness[0.005]}}],
 PlotRange → {{-3 Pi, 3 Pi}, {-2, 2}},
 Ticks → {Range[-6 Pi, 6 Pi, Pi / 2], Range[-2, 2, 1]},
 AspectRatio → Automatic, ImageSize → 600]
```

Out[121]=


In[122]:=
```
Show[Plot[(UnitStep[#]) &[x], {x, -6, 6},
  PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.012]}},
  Exclusions → Automatic],
 Plot[PerExt[x, (UnitStep[#]) &, -1, 1], {x, -6, 6},
  PlotStyle → {{RGBColor[1, 0, 0], Thickness[0.005]}}],
 PlotRange → {{-3 , 3}, {-1, 2}},
 Ticks → {Range[-6, 6, 1], Range[-2, 2, 1]},
 AspectRatio → Automatic, ImageSize → 600]
```

Out[122]=



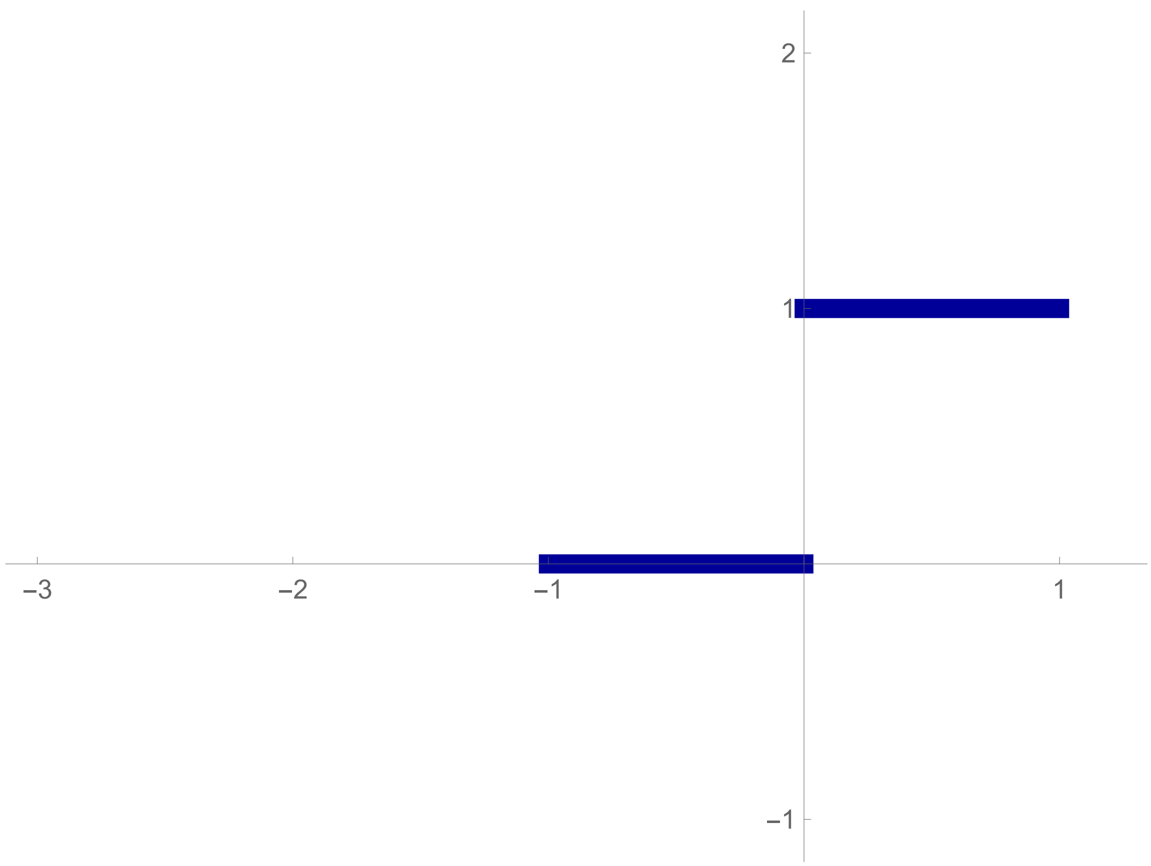
# Fourier series

## Example 1, the unit step function on the interval (-1,1)

Let us find the Fourier series of the function

In[123]:=
```
Show[Plot[(UnitStep[#]) &[x], {x, -1, 1},
  PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.012]}},
  Exclusions → {0}], PlotRange → {{-3 , 3}, {-1, 2}},
 Ticks → {Range[-6, 6, 1], Range[-2, 2, 1]},
 AspectRatio → Automatic, ImageSize → 600]
```

Out[123]=




The coefficient $a_0$ is

In[124]:= 
```
1/2 Integrate[1, {x, 0, 1}]
```

Out[124]= $\frac{1}{2}$

The coefficients $a_k,\ k \in \mathbb{N}$ are

In[125]:= 
```
FullSimplify[1/1 Integrate[Cos[k Pi x], {x, 0, 1}],
 And[k ∈ Integers, k > 0]]
```

Out[125]= 0

The coefficients $b_k,\ k \in \mathbb{N}$ are

In[126]:= 
```
FullSimplify[1/1 Integrate[Sin[k Pi x], {x, 0, 1}],
 And[k ∈ Integers, k > 0]]
```

Out[126]= $\frac{1 + (-1)^{1+k}}{k \pi}$

Thus the partial sum with 40 terms of the Fourier Series is

In[127]:=
```
nn = 20;
1/2 + Sum[2/((2 k - 1) Pi) Sin[(2 k - 1) Pi x], {k, 1, nn}]
```

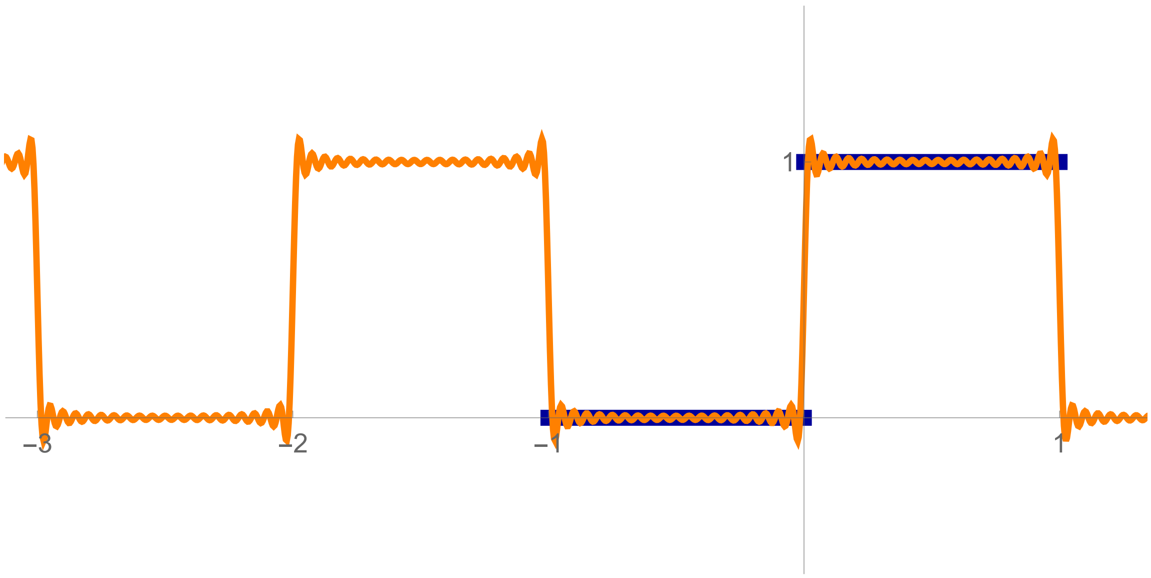
Out[127]=

$$\frac{1}{2}+\frac{2\,\mathrm{Sin}[\pi\,x]}{\pi}+\frac{2\,\mathrm{Sin}[3\,\pi\,x]}{3\,\pi}+\frac{2\,\mathrm{Sin}[5\,\pi\,x]}{5\,\pi}+\frac{2\,\mathrm{Sin}[7\,\pi\,x]}{7\,\pi}+$$
$$\frac{2\,\mathrm{Sin}[9\,\pi\,x]}{9\,\pi}+\frac{2\,\mathrm{Sin}[11\,\pi\,x]}{11\,\pi}+\frac{2\,\mathrm{Sin}[13\,\pi\,x]}{13\,\pi}+\frac{2\,\mathrm{Sin}[15\,\pi\,x]}{15\,\pi}+$$
$$\frac{2\,\mathrm{Sin}[17\,\pi\,x]}{17\,\pi}+\frac{2\,\mathrm{Sin}[19\,\pi\,x]}{19\,\pi}+\frac{2\,\mathrm{Sin}[21\,\pi\,x]}{21\,\pi}+\frac{2\,\mathrm{Sin}[23\,\pi\,x]}{23\,\pi}+$$
$$\frac{2\,\mathrm{Sin}[25\,\pi\,x]}{25\,\pi}+\frac{2\,\mathrm{Sin}[27\,\pi\,x]}{27\,\pi}+\frac{2\,\mathrm{Sin}[29\,\pi\,x]}{29\,\pi}+\frac{2\,\mathrm{Sin}[31\,\pi\,x]}{31\,\pi}+$$
$$\frac{2\,\mathrm{Sin}[33\,\pi\,x]}{33\,\pi}+\frac{2\,\mathrm{Sin}[35\,\pi\,x]}{35\,\pi}+\frac{2\,\mathrm{Sin}[37\,\pi\,x]}{37\,\pi}+\frac{2\,\mathrm{Sin}[39\,\pi\,x]}{39\,\pi}$$

Verify this with graphs

In[128]:=
```
Show[Plot[(UnitStep[#]) &[x], {x, -1, 1},
  PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.01]}},
  Exclusions → {0}],
 Plot[1/2 + (2 Sin[π x])/π + (2 Sin[3 π x])/(3 π) + (2 Sin[5 π x])/(5 π) +
   (2 Sin[7 π x])/(7 π) + (2 Sin[9 π x])/(9 π) + (2 Sin[11 π x])/(11 π) + (2 Sin[13 π x])/(13 π) +
   (2 Sin[15 π x])/(15 π) + (2 Sin[17 π x])/(17 π) + (2 Sin[19 π x])/(19 π) +
   (2 Sin[21 π x])/(21 π) + (2 Sin[23 π x])/(23 π) + (2 Sin[25 π x])/(25 π) +
   (2 Sin[27 π x])/(27 π) + (2 Sin[29 π x])/(29 π) + (2 Sin[31 π x])/(31 π) +
   (2 Sin[33 π x])/(33 π) + (2 Sin[35 π x])/(35 π) + (2 Sin[37 π x])/(37 π) + (2 Sin[39 π x])/(39 π),
  {x, -6, 6},
  PlotStyle → {{RGBColor[1, 0.5, 0], Thickness[0.004]}}],
 PlotRange → {{-3 , 3}, {-0.5, 1.5}},
 Ticks → {Range[-6, 6, 1], Range[-2, 2, 1]},
 AspectRatio → Automatic, ImageSize → 600]
```

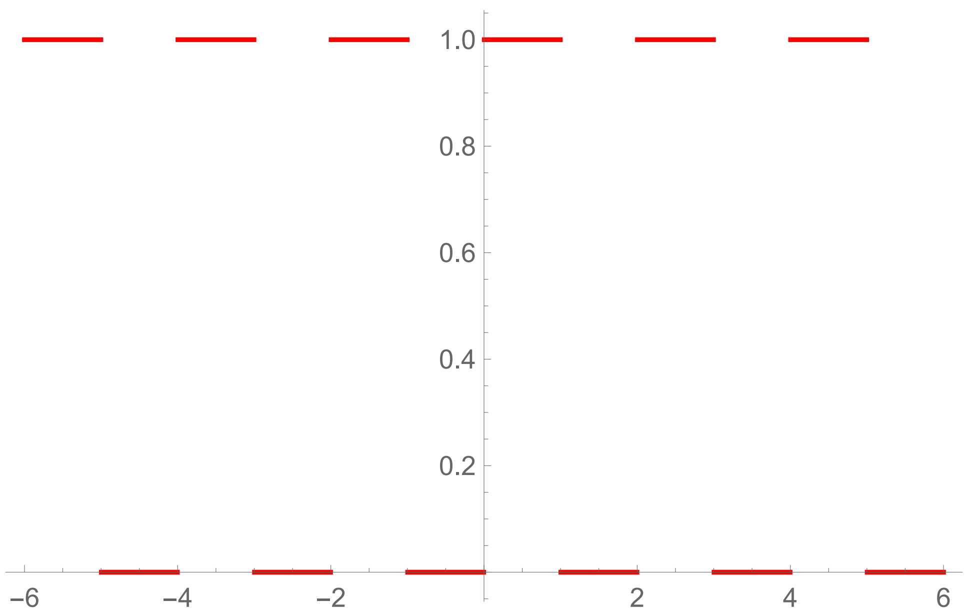
Out[128]=




It is nice to include the periodic extension

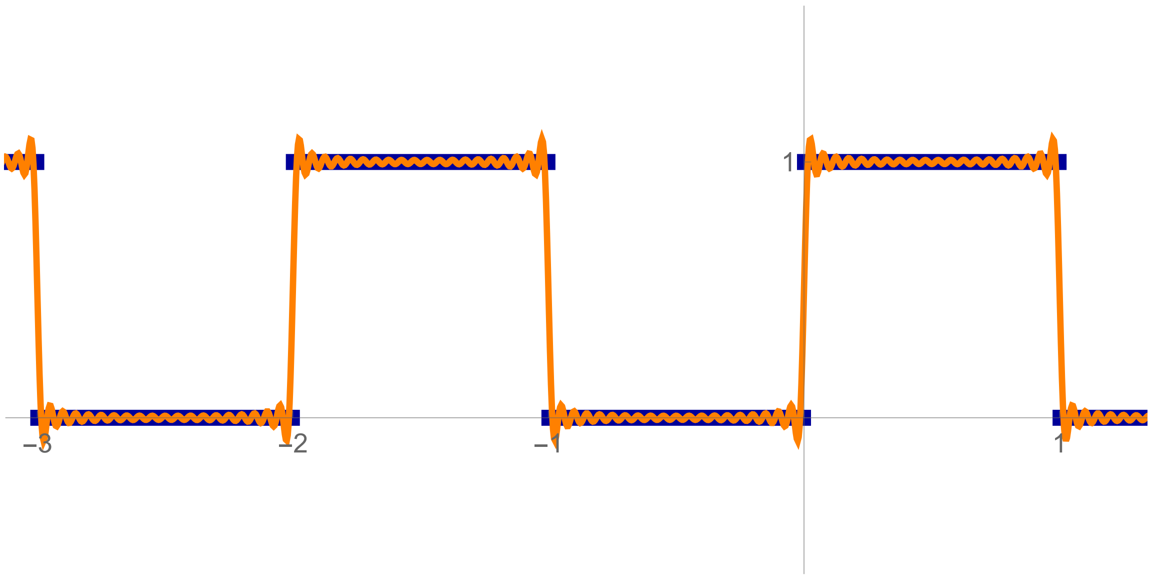
In[129]:=
```
Plot[PerExt[x, (UnitStep[#]) &, -1, 1], {x, -6, 6},
 PlotStyle → {{RGBColor[1, 0, 0], Thickness[0.005]}}]
```

Out[129]=



In[130]:=
```
Show[Plot[PerExt[x, (UnitStep[#]) &, -1, 1], {x, -6, 6},
  PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.01]}},
  Exclusions → Range[-10, 10, 1]],
 Plot[1/2 + (2 Sin[π x])/π + (2 Sin[3 π x])/(3 π) + (2 Sin[5 π x])/(5 π) +
   (2 Sin[7 π x])/(7 π) + (2 Sin[9 π x])/(9 π) + (2 Sin[11 π x])/(11 π) + (2 Sin[13 π x])/(13 π) +
   (2 Sin[15 π x])/(15 π) + (2 Sin[17 π x])/(17 π) + (2 Sin[19 π x])/(19 π) +
   (2 Sin[21 π x])/(21 π) + (2 Sin[23 π x])/(23 π) + (2 Sin[25 π x])/(25 π) +
   (2 Sin[27 π x])/(27 π) + (2 Sin[29 π x])/(29 π) + (2 Sin[31 π x])/(31 π) +
   (2 Sin[33 π x])/(33 π) + (2 Sin[35 π x])/(35 π) + (2 Sin[37 π x])/(37 π) + (2 Sin[39 π x])/(39 π),
  {x, -6, 6},
  PlotStyle → {{RGBColor[1, 0.5, 0], Thickness[0.004]}}],
 PlotRange → {{-3 , 3}, {-0.5, 1.5}},
 Ticks → {Range[-6, 6, 1], Range[-2, 2, 1]},
 AspectRatio → Automatic, ImageSize → 600]
```

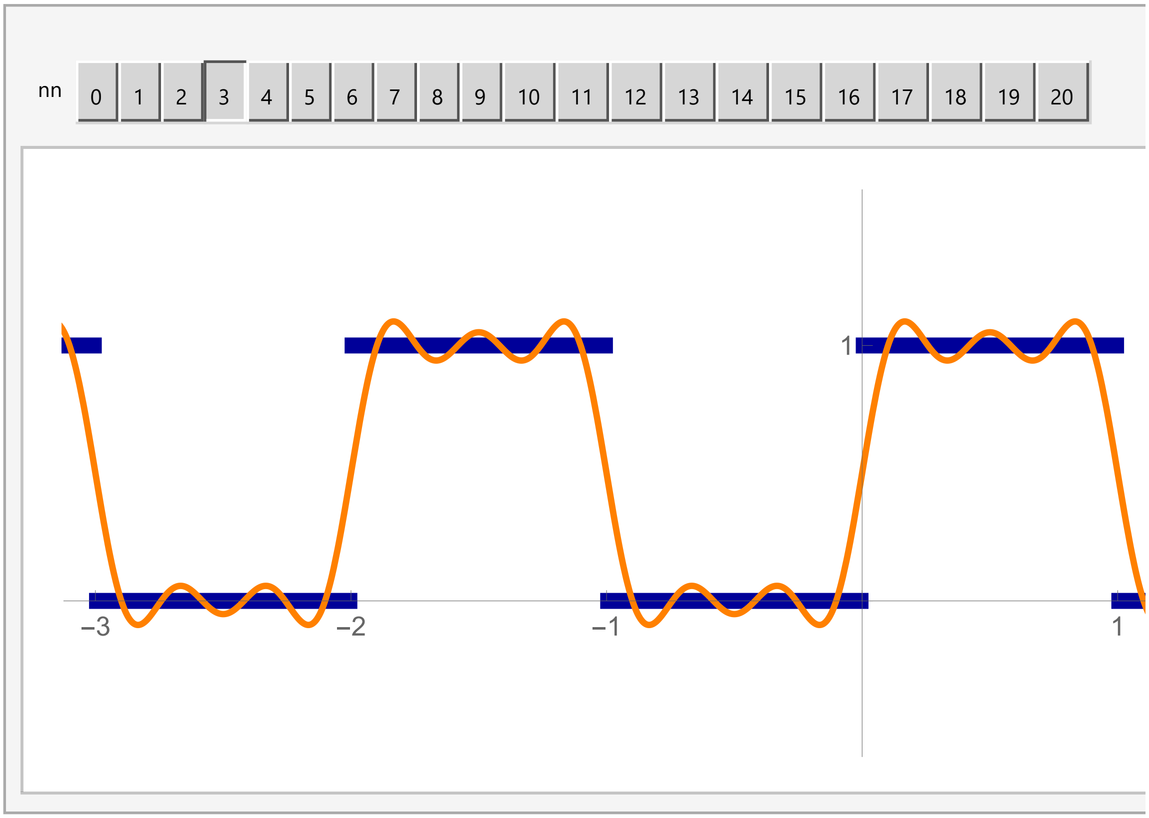
Out[130]=




It might be interesting to include different partial sums of the Fourier series:

In[131]:=
```
Manipulate[
 Show[Plot[PerExt[x, (UnitStep[#]) &, -1, 1],
   {x, -10, 10},
   PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.01]}},
   Exclusions → Range[-10, 10, 1]],
  Plot[1/2 + Sum[2/((2 k - 1) Pi) Sin[(2 k - 1) Pi x], {k, 1, nn}],
   {x, -6, 6},
   PlotStyle → {{RGBColor[1, 0.5, 0], Thickness[0.004]}}],
  PlotRange → {{-3 , 3}, {-0.5, 1.5}},
  Ticks → {Range[-6, 6, 1], Range[-2, 2, 1]},
  AspectRatio → Automatic, ImageSize → 600],
 {{nn, 3}, Range[0, 20], Setter}, ControlPlacement → Top]
```

Out[131]=

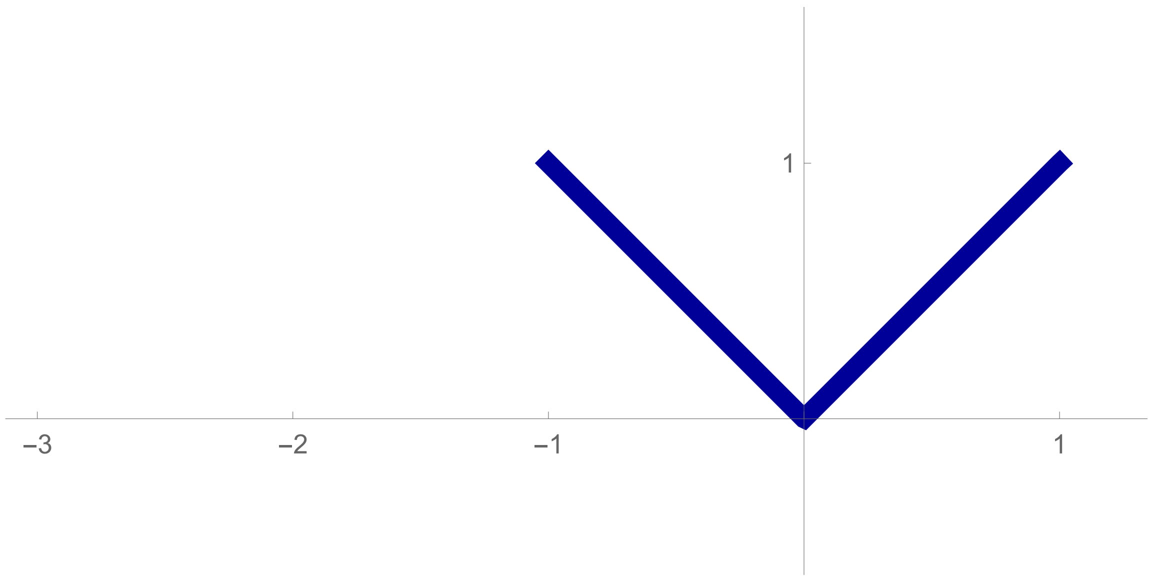



## Example 2, the absolute value function on the interval (-1,1)

Let us find the Fourier series of the function

In[132]:=
```
Show[Plot[(Abs[#]) &[x], {x, -1, 1},
   PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.012]}}],
 PlotRange → {{-3 , 3}, {-0.5, 1.5}},
 Ticks → {Range[-6, 6, 1], Range[-2, 2, 1]},
 AspectRatio → Automatic, ImageSize → 600]
```

Out[132]=




The coefficient $a_0$ is

In[133]:=
```
1/2 Integrate[Abs[x], {x, -1, 1}]
```

Out[133]= $\frac{1}{2}$

The coefficients $a_k, \; k \in \mathbb{N}$ are

In[134]:=
```
FullSimplify[1/1 Integrate[Abs[x] Cos[k Pi x], {x, -1, 1}],
 And[k ∈ Integers, k > 0]]
```

Out[134]= $\frac{2\left(-1+(-1)^{k}\right)}{k^2 \pi^2}$

The coefficients $b_k$, $k \in \mathbb{N}$ are

In[135]:=
```
FullSimplify[1/1 Integrate[Abs[x] Sin[k Pi x], {x, -1, 1}],
 And[k ∈ Integers, k > 0]]
```

Out[135]= 0

Thus the partial sum with 40 terms of the Fourier Series is

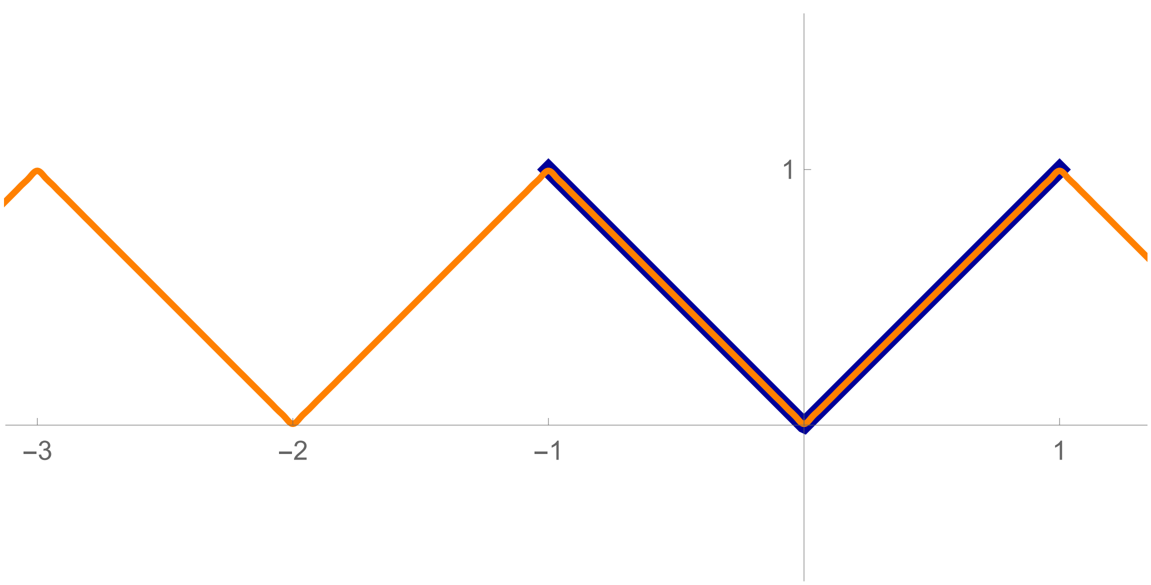
In[136]:=
```
nn = 20;
1/2 + Sum[-4/((2 k - 1)^2 Pi^2) Cos[(2 k - 1) Pi x], {k, 1, nn}]
```

Out[136]=
$$\frac{1}{2}-\frac{4\,\mathrm{Cos}[\pi x]}{\pi^2}-\frac{4\,\mathrm{Cos}[3\pi x]}{9\pi^2}-\frac{4\,\mathrm{Cos}[5\pi x]}{25\pi^2}-\frac{4\,\mathrm{Cos}[7\pi x]}{49\pi^2}-\frac{4\,\mathrm{Cos}[9\pi x]}{81\pi^2}-\frac{4\,\mathrm{Cos}[11\pi x]}{121\pi^2}-\frac{4\,\mathrm{Cos}[13\pi x]}{169\pi^2}-\frac{4\,\mathrm{Cos}[15\pi x]}{225\pi^2}-\frac{4\,\mathrm{Cos}[17\pi x]}{289\pi^2}-\frac{4\,\mathrm{Cos}[19\pi x]}{361\pi^2}-\frac{4\,\mathrm{Cos}[21\pi x]}{441\pi^2}-\frac{4\,\mathrm{Cos}[23\pi x]}{529\pi^2}-\frac{4\,\mathrm{Cos}[25\pi x]}{625\pi^2}-\frac{4\,\mathrm{Cos}[27\pi x]}{729\pi^2}-\frac{4\,\mathrm{Cos}[29\pi x]}{841\pi^2}-\frac{4\,\mathrm{Cos}[31\pi x]}{961\pi^2}-\frac{4\,\mathrm{Cos}[33\pi x]}{1089\pi^2}-\frac{4\,\mathrm{Cos}[35\pi x]}{1225\pi^2}-\frac{4\,\mathrm{Cos}[37\pi x]}{1369\pi^2}-\frac{4\,\mathrm{Cos}[39\pi x]}{1521\pi^2}$$

Verify this with graphs

In[137]:=
```
Show[Plot[(Abs[#]) &[x], {x, -1, 1},
  PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.01]}}],
 Plot[1/2 - 4 Cos[π x]/π^2 - 4 Cos[3 π x]/(9 π^2) - 4 Cos[5 π x]/(25 π^2) -
   4 Cos[7 π x]/(49 π^2) - 4 Cos[9 π x]/(81 π^2) - 4 Cos[11 π x]/(121 π^2) - 4 Cos[13 π x]/(169 π^2) -
   4 Cos[15 π x]/(225 π^2) - 4 Cos[17 π x]/(289 π^2) - 4 Cos[19 π x]/(361 π^2) -
   4 Cos[21 π x]/(441 π^2) - 4 Cos[23 π x]/(529 π^2) - 4 Cos[25 π x]/(625 π^2) -
   4 Cos[27 π x]/(729 π^2) - 4 Cos[29 π x]/(841 π^2) - 4 Cos[31 π x]/(961 π^2) -
   4 Cos[33 π x]/(1089 π^2) - 4 Cos[35 π x]/(1225 π^2) - 4 Cos[37 π x]/(1369 π^2) - 4 Cos[39 π x]/(1521 π^2),
  {x, -6, 6},
  PlotStyle → {{RGBColor[1, 0.5, 0], Thickness[0.004]}}],
 PlotRange → {{-3 , 3}, {-0.5, 1.5}},
 Ticks → {Range[-6, 6, 1], Range[-2, 2, 1]},
 AspectRatio → Automatic, ImageSize → 600]
```

Out[137]=

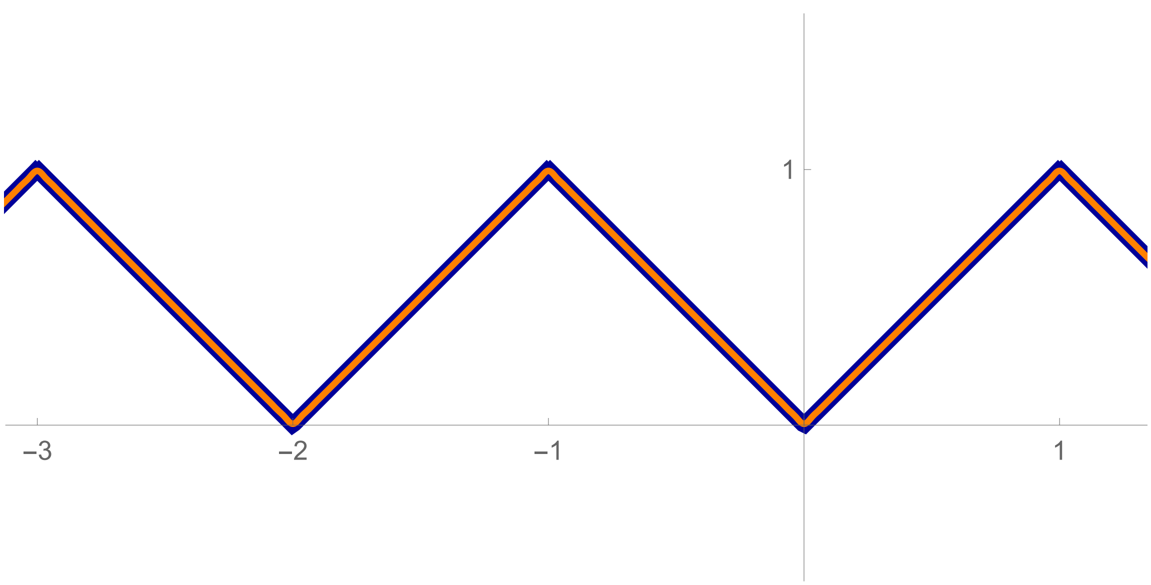


It is nice to include the periodic extension

In[138]:=

```
Show[Plot[PerExt[x, (Abs[#]) &, -1, 1], {x, -10, 10},
  PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.01]}}],
 Plot[1/2 - 4 Cos[π x]/π^2 - 4 Cos[3 π x]/(9 π^2) - 4 Cos[5 π x]/(25 π^2) -
   4 Cos[7 π x]/(49 π^2) - 4 Cos[9 π x]/(81 π^2) - 4 Cos[11 π x]/(121 π^2) - 4 Cos[13 π x]/(169 π^2) -
   4 Cos[15 π x]/(225 π^2) - 4 Cos[17 π x]/(289 π^2) - 4 Cos[19 π x]/(361 π^2) -
   4 Cos[21 π x]/(441 π^2) - 4 Cos[23 π x]/(529 π^2) - 4 Cos[25 π x]/(625 π^2) -
   4 Cos[27 π x]/(729 π^2) - 4 Cos[29 π x]/(841 π^2) - 4 Cos[31 π x]/(961 π^2) -
   4 Cos[33 π x]/(1089 π^2) - 4 Cos[35 π x]/(1225 π^2) - 4 Cos[37 π x]/(1369 π^2) - 4 Cos[39 π x]/(1521 π^2),
  {x, -6, 6},
  PlotStyle → {{RGBColor[1, 0.5, 0], Thickness[0.004]}}],
 PlotRange → {{-3 , 3}, {-0.5, 1.5}},
 Ticks → {Range[-6, 6, 1], Range[-2, 2, 1]},
 AspectRatio → Automatic, ImageSize → 600]
```

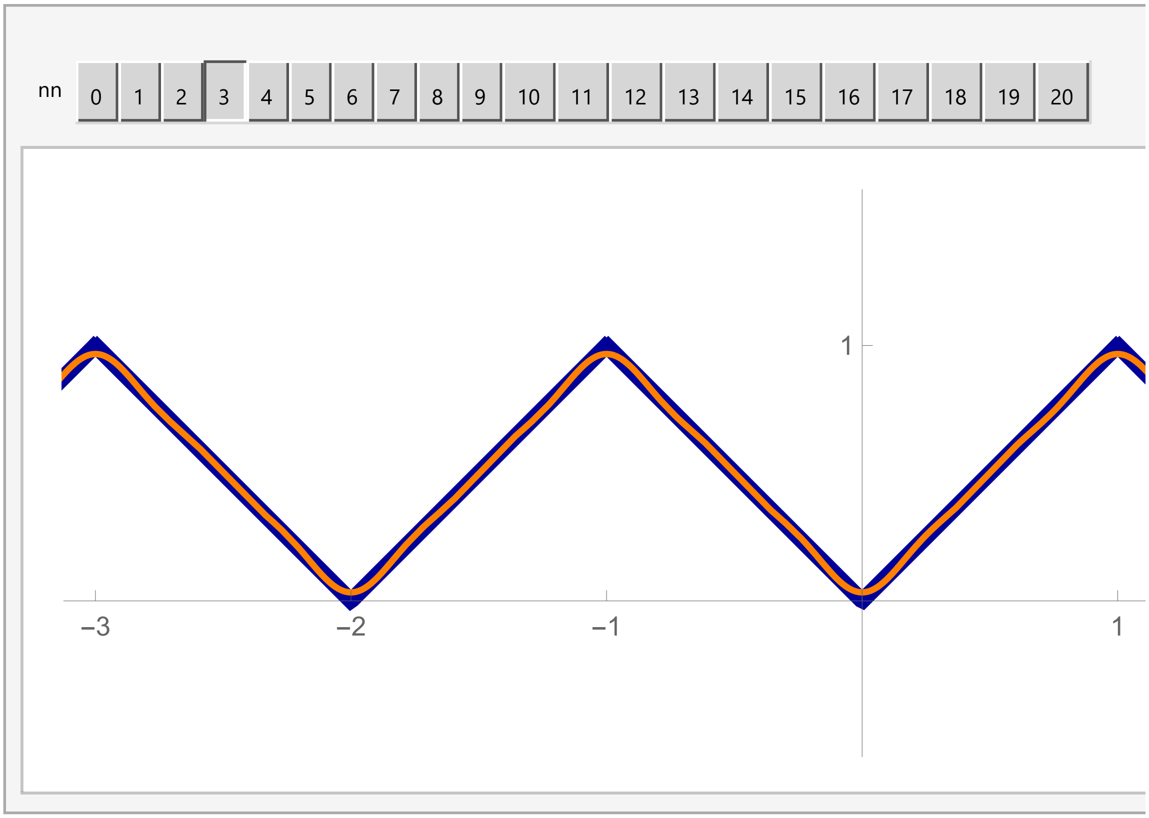
Out[138]=



It might be interesting to include different partial sums of the Fourier series:

In[139]:=
```
Manipulate[
 Show[Plot[PerExt[x, (Abs[#]) &, -1, 1], {x, -10, 10},
   PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.01]}}],
  Plot[1/2 + Sum[-4/((2 k - 1)^2 Pi^2) Cos[(2 k - 1) Pi x], {k, 1, nn}],
   {x, -6, 6},
   PlotStyle → {{RGBColor[1, 0.5, 0], Thickness[0.004]}}],
  PlotRange → {{-3 , 3}, {-0.5, 1.5}},
  Ticks → {Range[-6, 6, 1], Range[-2, 2, 1]},
  AspectRatio → Automatic, ImageSize → 600],
 {{nn, 3}, Range[0, 20], Setter}, ControlPlacement → Top]
```

Out[139]=

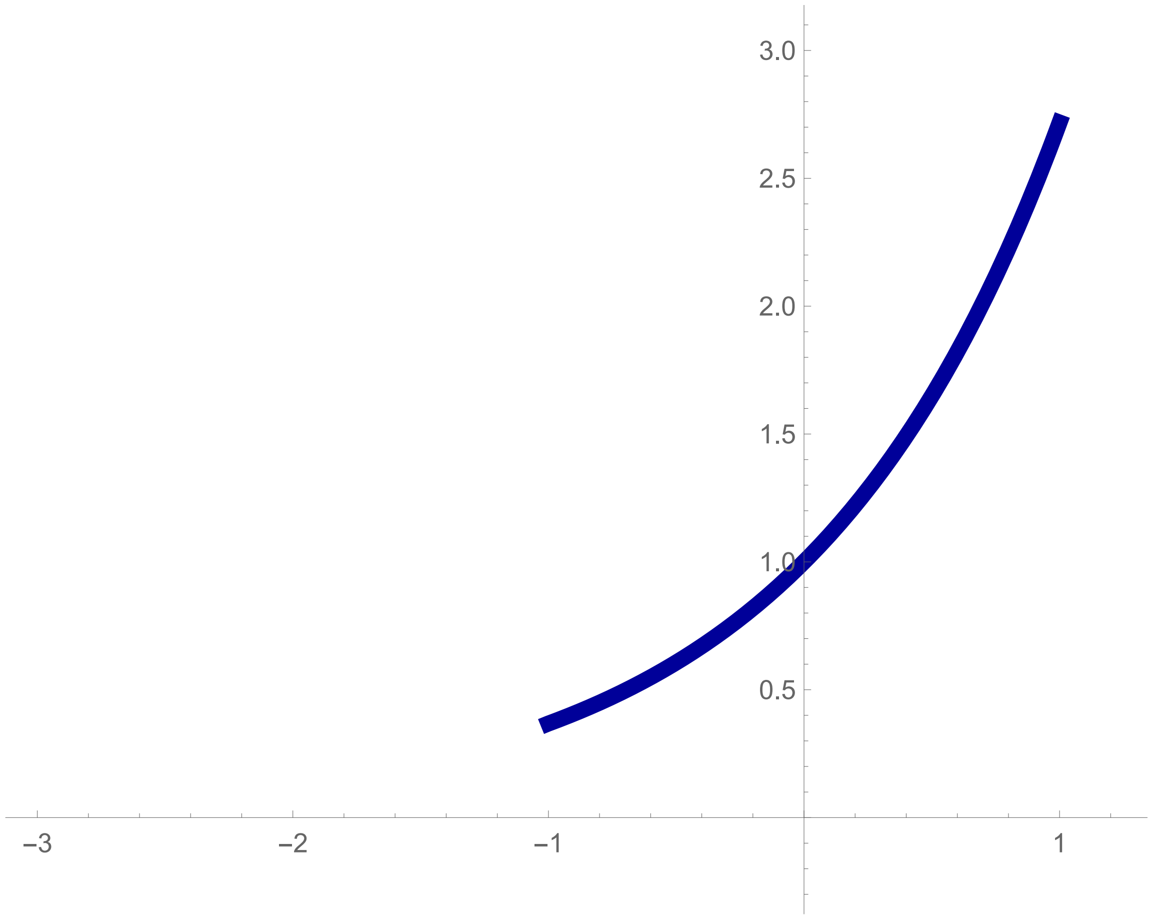



## Example 3, the exponential function on the interval (-1,1)

Let us find the Fourier series of the function

In[140]:=
```
Show[Plot[(Exp[#]) &[x], {x, -1, 1},
   PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.01]}}],
  AxesOrigin → 0, PlotRange → {{-3 , 3}, {-0.2, 3}},
  AspectRatio → Automatic, ImageSize → 600]
```

Out[140]=




The coefficient $a_0$ is

In[141]:=
```
FullSimplify[1/2 Integrate[Exp[x], {x, -1, 1}]]
```

Out[141]= $\frac{1}{2}\left(-\frac{1}{e}+e\right)$

In[142]:= 
```
FullSimplify[1/2 Integrate[Exp[x], {x, -1, 1}] == Sinh[1]]
```

Out[142]= True

The coefficients $a_k$, $k \in \mathbb{N}$ are

In[143]:= 
```
FullSimplify[1/1 Integrate[Exp[x] Cos[k Pi x], {x, -1, 1}],
  And[k ∈ Integers, k > 0]]
```

Out[143]= $$\frac{(-1)^k \left(-1 + e^2\right)}{e + e\, k^2 \pi^2}$$

In[144]:= 
```
FullSimplify[((-1)^k (-1 + e^2))/(e + e k^2 π^2) == 2 Sinh[1] * (-1)^k/(1 + (k Pi)^2)]
```

Out[144]= True

The coefficients $b_k$, $k \in \mathbb{N}$ are

In[145]:= 
```
FullSimplify[1/1 Integrate[Exp[x] Sin[k Pi x], {x, -1, 1}],
  And[k ∈ Integers, k > 0]]
```

Out[145]= $$\frac{(-1)^{1+k} \left(-1 + e^2\right) k \pi}{e + e\, k^2 \pi^2}$$

In[146]:= 
```
FullSimplify[((-1)^(1+k) (-1 + e^2) k π)/(e + e k^2 π^2) ==
  -2 Pi Sinh[1] * ((-1)^k k)/(1 + (k Pi)^2)]
```

Out[146]= True

Thus the partial sum with 40 terms of the Fourier Series is

In[147]:=
```
nn = 20;
Clear[FS3];
FS3[x_] =
 Sinh[1] + 2 Sinh[1] Sum[(-1)^k/(1 + (k Pi)^2) Cos[k Pi x], {k, 1, nn}] -
  2 Pi Sinh[1] Sum[((-1)^k k)/(1 + (k Pi)^2) Sin[k Pi x], {k, 1, nn}]
```

Out[147]=

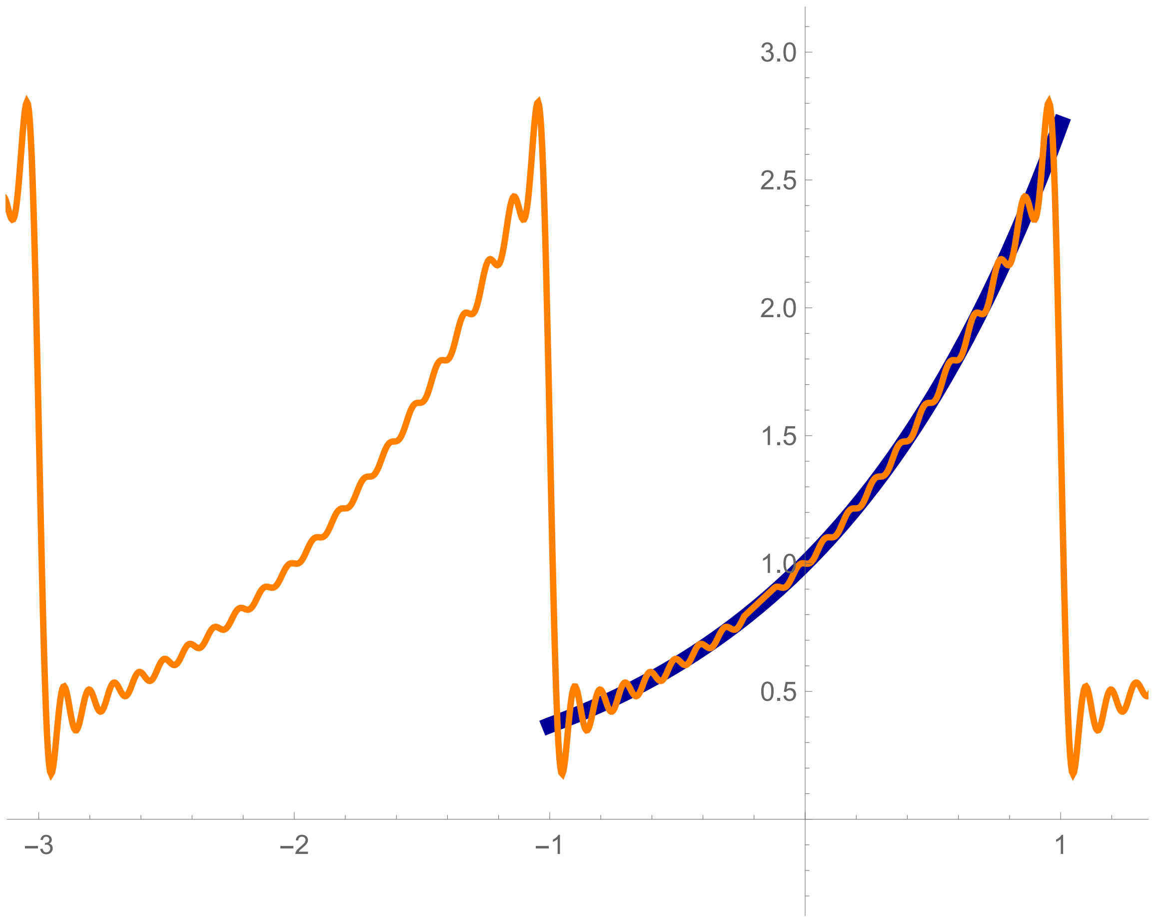
$$\mathrm{Sinh}[1] + 2\left(-\frac{\mathrm{Cos}[\pi x]}{1+\pi^2} + \frac{\mathrm{Cos}[2\pi x]}{1+4\pi^2} - \frac{\mathrm{Cos}[3\pi x]}{1+9\pi^2} + \frac{\mathrm{Cos}[4\pi x]}{1+16\pi^2} - \frac{\mathrm{Cos}[5\pi x]}{1+25\pi^2} + \frac{\mathrm{Cos}[6\pi x]}{1+36\pi^2} - \frac{\mathrm{Cos}[7\pi x]}{1+49\pi^2} + \frac{\mathrm{Cos}[8\pi x]}{1+64\pi^2} - \frac{\mathrm{Cos}[9\pi x]}{1+81\pi^2} + \frac{\mathrm{Cos}[10\pi x]}{1+100\pi^2} - \frac{\mathrm{Cos}[11\pi x]}{1+121\pi^2} + \frac{\mathrm{Cos}[12\pi x]}{1+144\pi^2} - \frac{\mathrm{Cos}[13\pi x]}{1+169\pi^2} + \frac{\mathrm{Cos}[14\pi x]}{1+196\pi^2} - \frac{\mathrm{Cos}[15\pi x]}{1+225\pi^2} + \frac{\mathrm{Cos}[16\pi x]}{1+256\pi^2} - \frac{\mathrm{Cos}[17\pi x]}{1+289\pi^2} + \frac{\mathrm{Cos}[18\pi x]}{1+324\pi^2} - \frac{\mathrm{Cos}[19\pi x]}{1+361\pi^2} + \frac{\mathrm{Cos}[20\pi x]}{1+400\pi^2}\right)\mathrm{Sinh}[1] -$$

$$2\pi\left(-\frac{\mathrm{Sin}[\pi x]}{1+\pi^2} + \frac{2\,\mathrm{Sin}[2\pi x]}{1+4\pi^2} - \frac{3\,\mathrm{Sin}[3\pi x]}{1+9\pi^2} + \frac{4\,\mathrm{Sin}[4\pi x]}{1+16\pi^2} - \frac{5\,\mathrm{Sin}[5\pi x]}{1+25\pi^2} + \frac{6\,\mathrm{Sin}[6\pi x]}{1+36\pi^2} - \frac{7\,\mathrm{Sin}[7\pi x]}{1+49\pi^2} + \frac{8\,\mathrm{Sin}[8\pi x]}{1+64\pi^2} - \frac{9\,\mathrm{Sin}[9\pi x]}{1+81\pi^2} + \frac{10\,\mathrm{Sin}[10\pi x]}{1+100\pi^2} - \frac{11\,\mathrm{Sin}[11\pi x]}{1+121\pi^2} + \frac{12\,\mathrm{Sin}[12\pi x]}{1+144\pi^2} - \frac{13\,\mathrm{Sin}[13\pi x]}{1+169\pi^2} + \frac{14\,\mathrm{Sin}[14\pi x]}{1+196\pi^2} - \frac{15\,\mathrm{Sin}[15\pi x]}{1+225\pi^2} + \frac{16\,\mathrm{Sin}[16\pi x]}{1+256\pi^2} - \frac{17\,\mathrm{Sin}[17\pi x]}{1+289\pi^2} + \frac{18\,\mathrm{Sin}[18\pi x]}{1+324\pi^2} - \frac{19\,\mathrm{Sin}[19\pi x]}{1+361\pi^2} + \frac{20\,\mathrm{Sin}[20\pi x]}{1+400\pi^2}\right)\mathrm{Sinh}[1]$$

Verify this with graphs

In[148]:=
```
Show[Plot[(Exp[#]) &[x], {x, -1, 1},
   PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.01]}}],
  Plot[FS3[x], {x, -6, 6},
   PlotStyle → {{RGBColor[1, 0.5, 0], Thickness[0.004]}}],
  AxesOrigin → 0, PlotRange → {{-3 , 3}, {-0.2, 3}},
  AspectRatio → Automatic, ImageSize → 600]
```

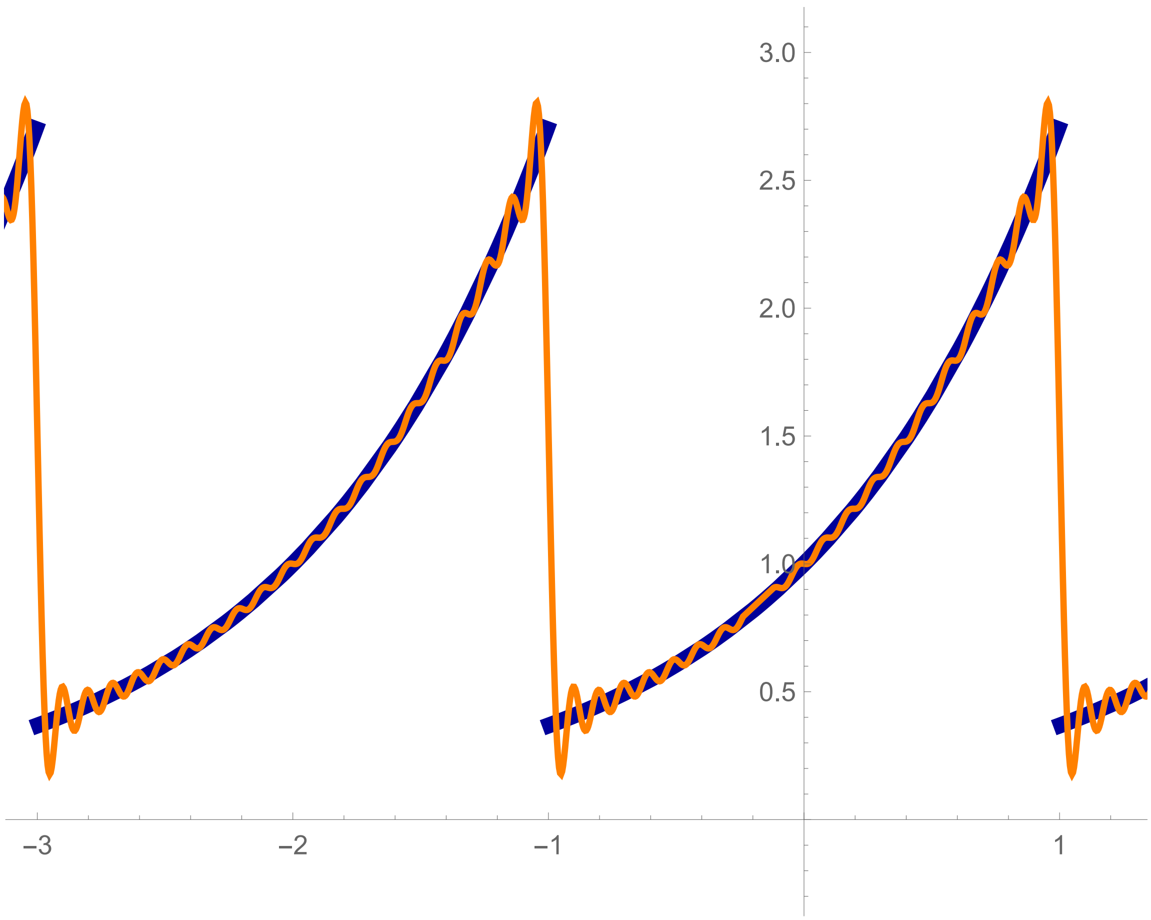
Out[148]=




It is nice to include the periodic extension

In[149]:=
```
Show[Plot[PerExt[x, (Exp[#]) &, -1, 1], {x, -10, 10},
   PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.01]}}],
 Plot[FS3[x], {x, -6, 6},
   PlotStyle → {{RGBColor[1, 0.5, 0], Thickness[0.004]}}],
 AxesOrigin → 0, PlotRange → {{-3 , 3}, {-0.2, 3}},
 AspectRatio → Automatic, ImageSize → 600]
```

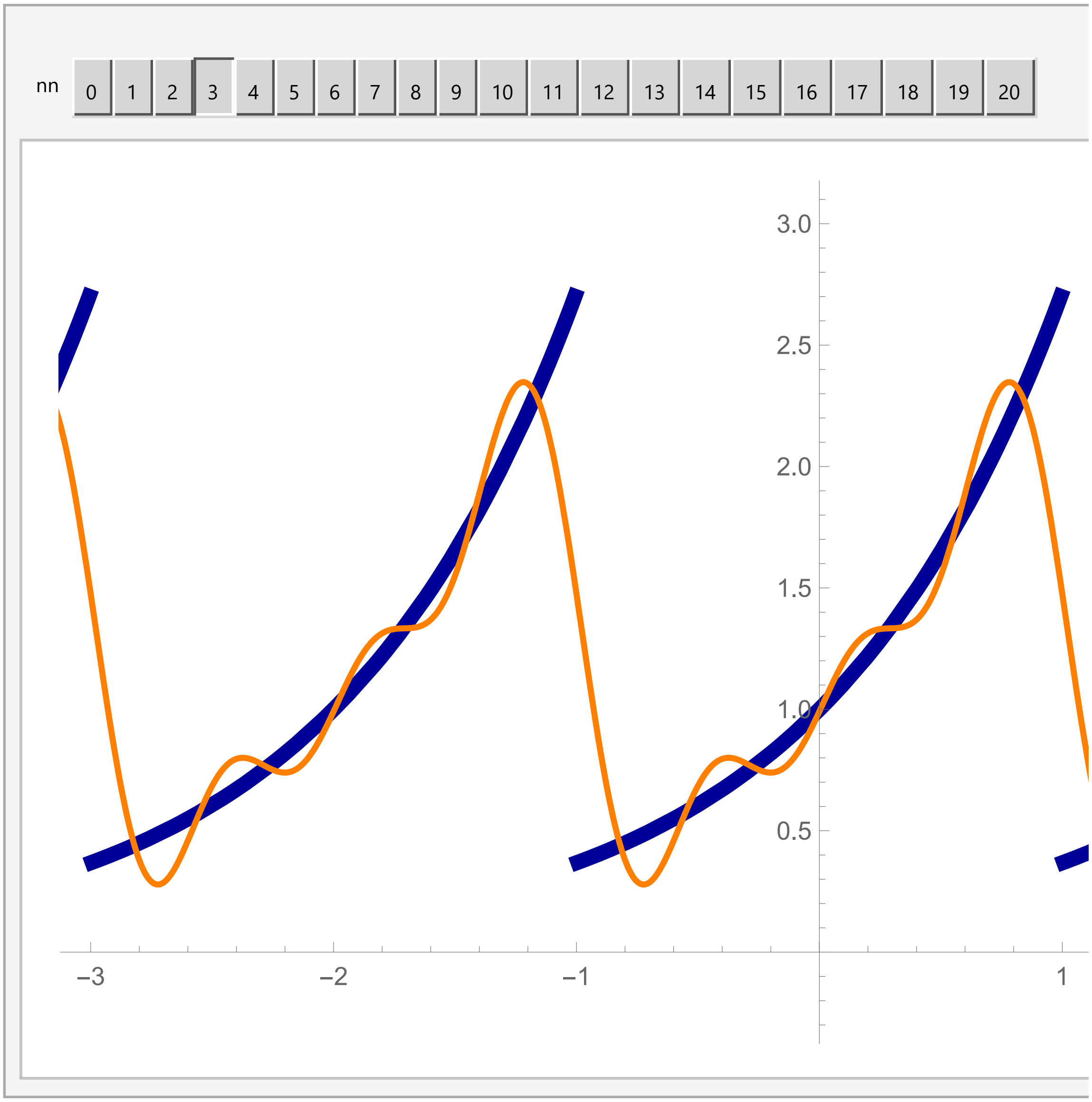
Out[149]=




It might be interesting to include different partial sums of the Fourier series:

In[150]:=
```
Manipulate[
  Show[Plot[PerExt[x, (Exp[#]) &, -1, 1], {x, -10, 10},
    PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.01]}}],
   Plot[
    Sinh[1] +
      2 Sinh[1] Sum[(-1)^k/(1 + (k Pi)^2) Cos[k Pi x], {k, 1, nn}] -
      2 Pi Sinh[1] Sum[((-1)^k k)/(1 + (k Pi)^2) Sin[k Pi x], {k, 1, nn}],
    {x, -6, 6},
    PlotStyle → {{RGBColor[1, 0.5, 0], Thickness[0.004]}}],
   AxesOrigin → 0, PlotRange → {{-3 , 3}, {-0.2, 3}},
   AspectRatio → Automatic, ImageSize → 600],
  {{nn, 3}, Range[0, 20], Setter}, ControlPlacement → Top]
```

Out[150]=

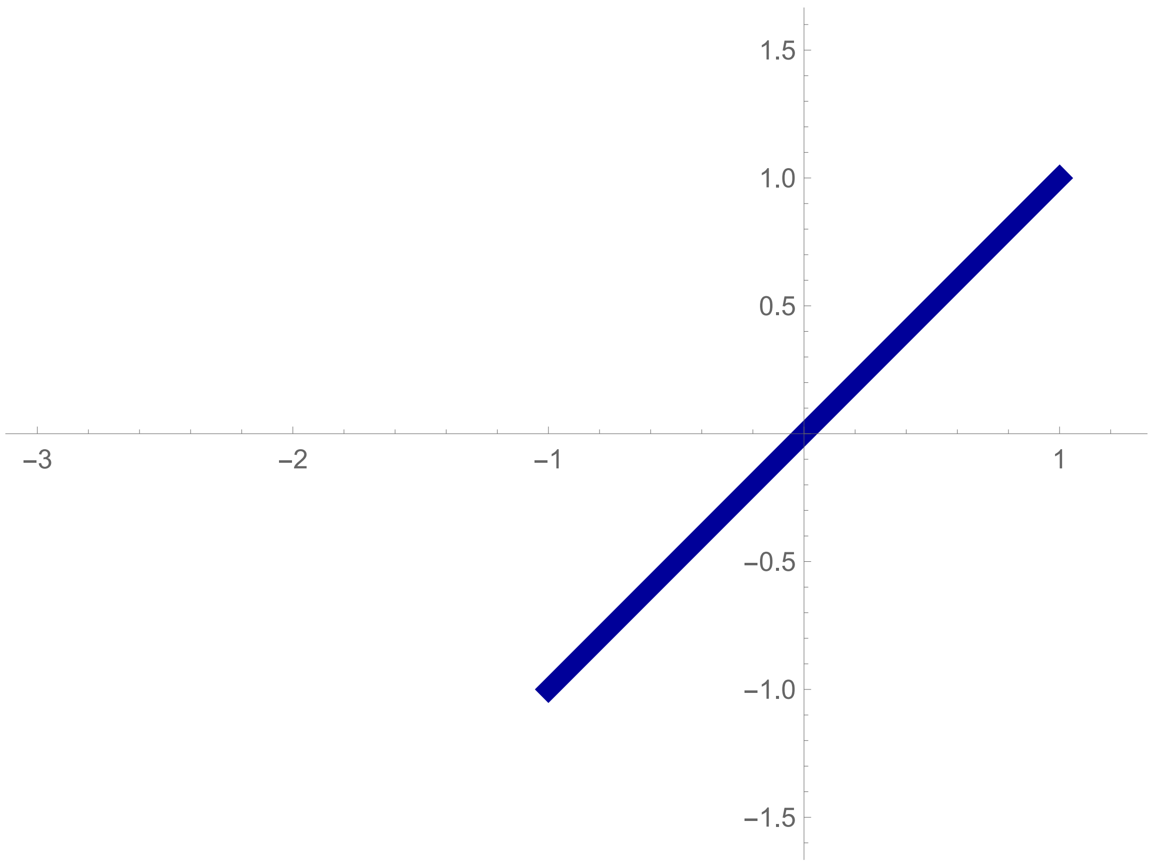



## Example 4, the identity function on the interval (-1,1)

Let us find the Fourier series of the function

In[151]:=
```
Show[Plot[(#) &[x], {x, -1, 1},
   PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.012]}}],
  PlotRange → {{-3 , 3}, {-1.5, 1.5}},
  AspectRatio → Automatic, ImageSize → 600]
```

Out[151]=




The coefficient $a_0$ is

In[152]:=
```
1/2 Integrate[x, {x, -1, 1}]
```

Out[152]= 0

The coefficients $a_k,\ k \in \mathbb{N}$ are

In[153]:=
```
FullSimplify[1/1 Integrate[x Cos[k Pi x], {x, -1, 1}],
  And[k ∈ Integers, k > 0]]
```

Out[153]= 0

The coefficients $b_k$, $k \in \mathbb{N}$ are

In[154]:=
```
FullSimplify[1/1 Integrate[x Sin[k Pi x], {x, -1, 1}],
  And[k ∈ Integers, k > 0]]
```

Out[154]= $-\frac{2\,(-1)^k}{k\,\pi}$

Thus the partial sum with 40 terms of the Fourier Series is

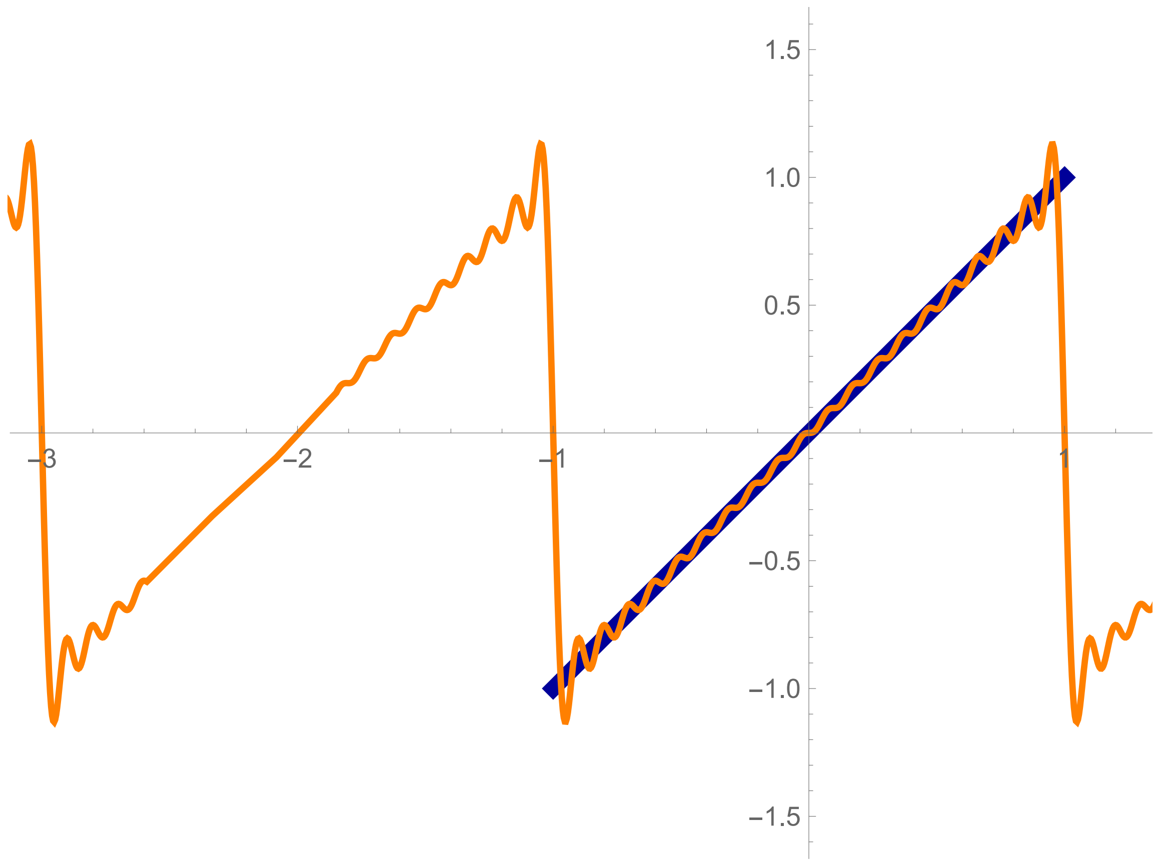
In[155]:=
```
nn = 20;
Clear[FS4]; FS4[x_] = Sum[-(2 (-1)^k)/(k π) Sin[k Pi x], {k, 1, nn}]
```

Out[155]=
$$\frac{2\,\text{Sin}[\pi\,x]}{\pi}-\frac{\text{Sin}[2\,\pi\,x]}{\pi}+\frac{2\,\text{Sin}[3\,\pi\,x]}{3\,\pi}-\frac{\text{Sin}[4\,\pi\,x]}{2\,\pi}+\frac{2\,\text{Sin}[5\,\pi\,x]}{5\,\pi}-\frac{\text{Sin}[6\,\pi\,x]}{3\,\pi}+\frac{2\,\text{Sin}[7\,\pi\,x]}{7\,\pi}-\frac{\text{Sin}[8\,\pi\,x]}{4\,\pi}+\frac{2\,\text{Sin}[9\,\pi\,x]}{9\,\pi}-\frac{\text{Sin}[10\,\pi\,x]}{5\,\pi}+\frac{2\,\text{Sin}[11\,\pi\,x]}{11\,\pi}-\frac{\text{Sin}[12\,\pi\,x]}{6\,\pi}+\frac{2\,\text{Sin}[13\,\pi\,x]}{13\,\pi}-\frac{\text{Sin}[14\,\pi\,x]}{7\,\pi}+\frac{2\,\text{Sin}[15\,\pi\,x]}{15\,\pi}-\frac{\text{Sin}[16\,\pi\,x]}{8\,\pi}+\frac{2\,\text{Sin}[17\,\pi\,x]}{17\,\pi}-\frac{\text{Sin}[18\,\pi\,x]}{9\,\pi}+\frac{2\,\text{Sin}[19\,\pi\,x]}{19\,\pi}-\frac{\text{Sin}[20\,\pi\,x]}{10\,\pi}$$

Verify this with graphs

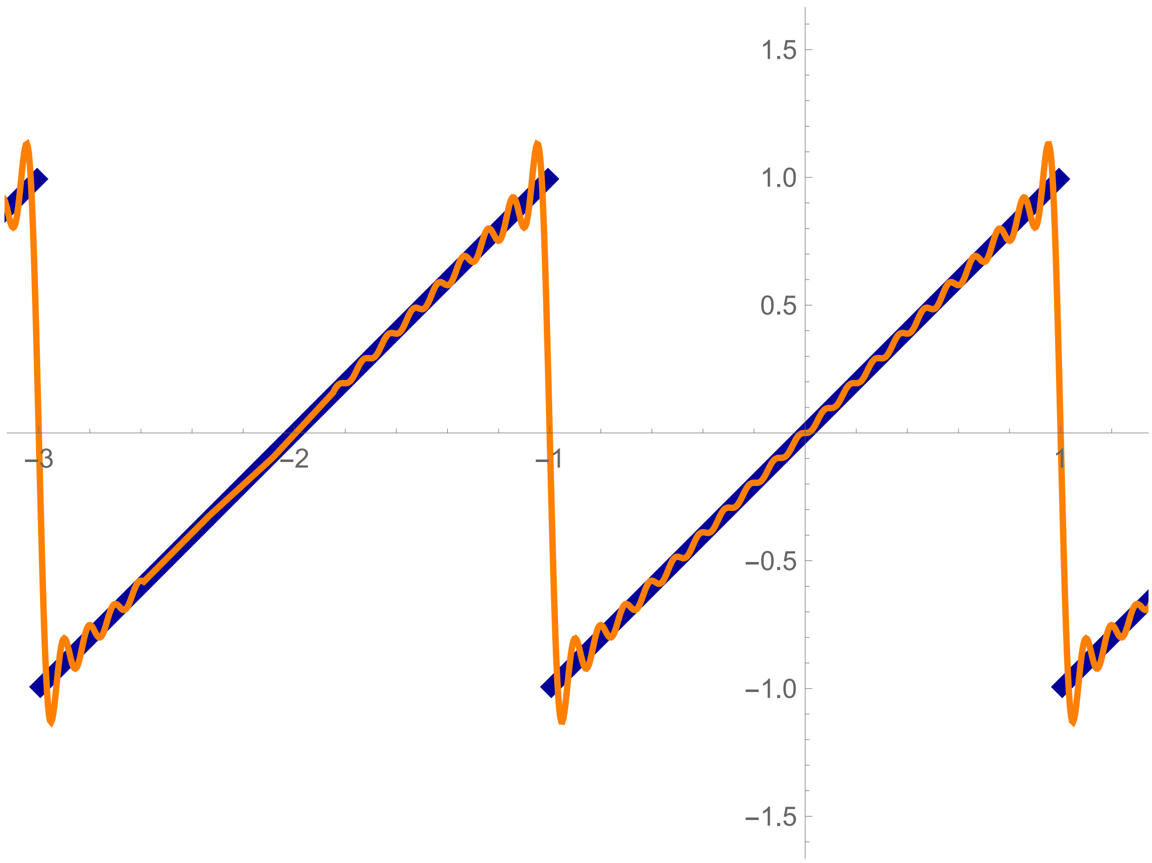
In[156]:=
```
Show[Plot[(#) &[x], {x, -1, 1},
   PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.01]}}],
  Plot[FS4[x], {x, -6, 6},
   PlotStyle → {{RGBColor[1, 0.5, 0], Thickness[0.004]}}],
  PlotRange → {{-3 , 3}, {-1.5, 1.5}},
  AspectRatio → Automatic, ImageSize → 600]
```

Out[156]=




It is nice to include the periodic extension

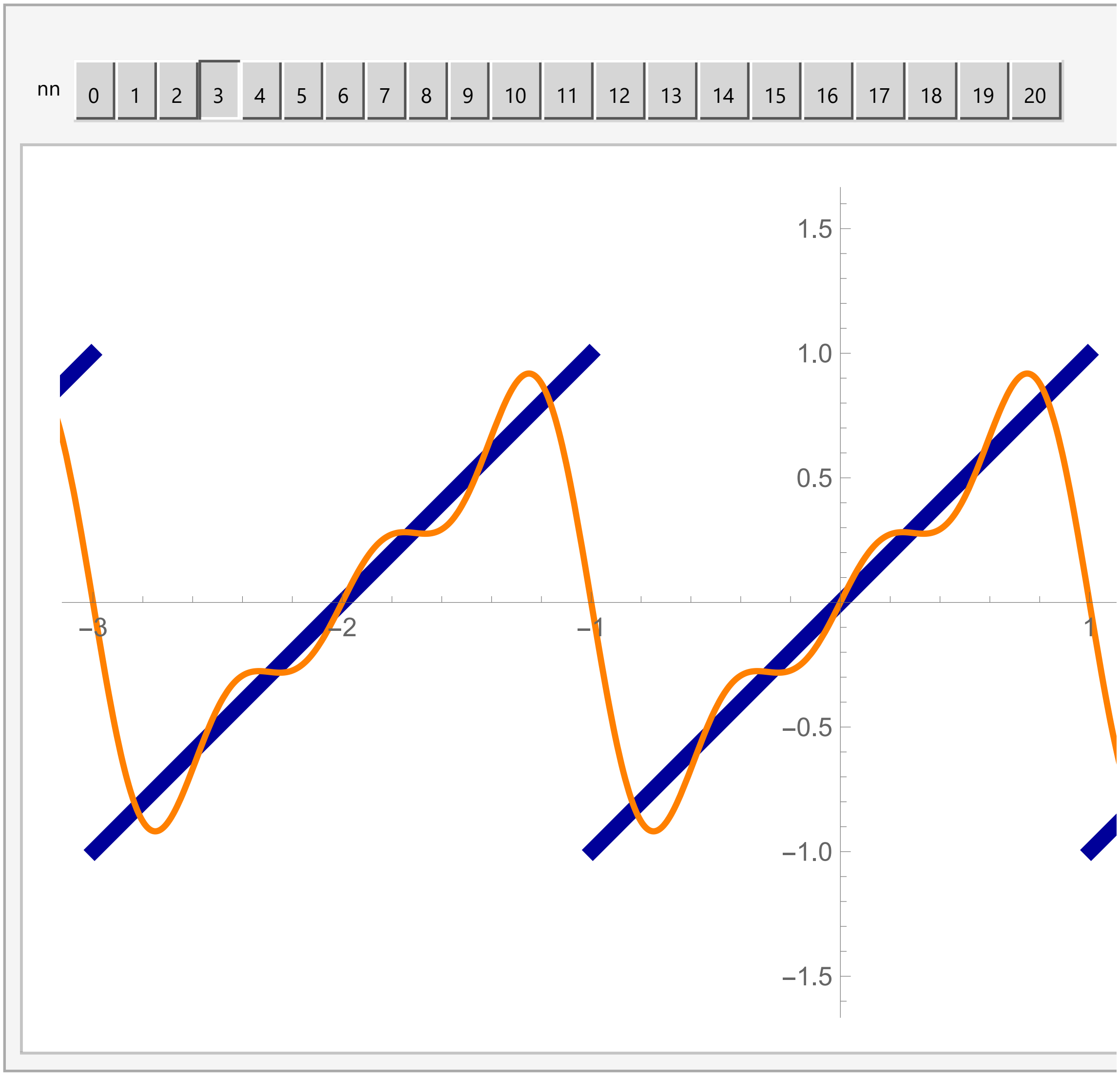
In[157]:=
```
Show[Plot[PerExt[x, ( #) &, -1, 1], {x, -10, 10},
   PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.01]}}],
  Plot[FS4[x], {x, -6, 6},
   PlotStyle → {{RGBColor[1, 0.5, 0], Thickness[0.004]}}],
  PlotRange → {{-3 , 3}, {-1.5, 1.5}},
  AspectRatio → Automatic, ImageSize → 600]
```

Out[157]=



It might be interesting to include different partial sums of the Fourier series:

```
In[158]:= Manipulate[
   Show[Plot[PerExt[x, (#) &, -1, 1], {x, -10, 10},
     PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.01]}}],
    Plot[Sum[-(2 (-1)^k)/(k π) Sin[k Pi x], {k, 1, nn}], {x, -6, 6},
     PlotStyle → {{RGBColor[1, 0.5, 0], Thickness[0.004]}}],
    PlotRange → {{-3 , 3}, {-1.5, 1.5}},
    AspectRatio → Automatic, ImageSize → 600],
   {{nn, 3}, Range[0, 20], Setter}, ControlPlacement → Top]
```

Out[158]=

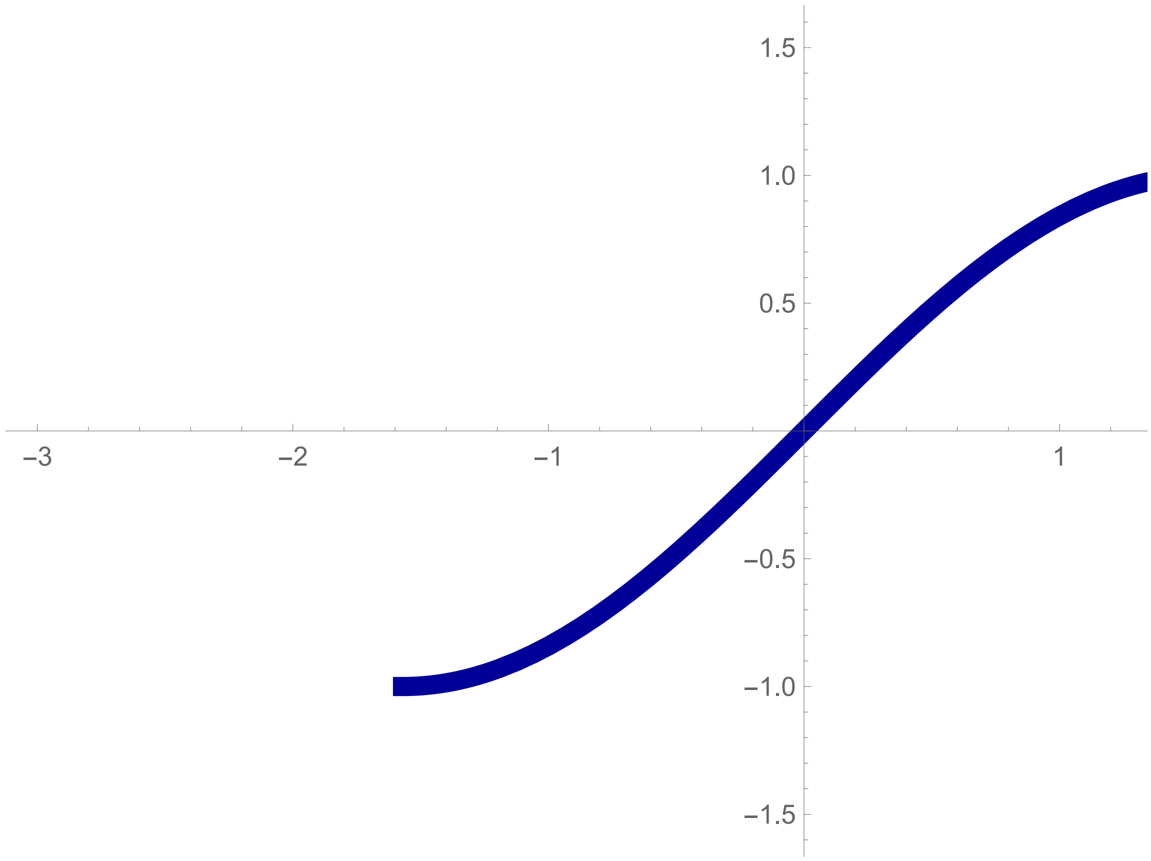



## Example 5, the sine function on the interval (-$\pi$/2,$\pi$/2)

Let us find the Fourier series of the function

In[159]:=
```
Show[Plot[(Sin[#]) &[x], {x, -Pi / 2, Pi / 2},
  PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.012]}}],
 PlotRange → {{-3 , 3}, {-1.5, 1.5}},
 AspectRatio → Automatic, ImageSize → 600]
```

Out[159]=




The coefficient $a_0$ is

In[160]:=
$$\frac{1}{2\,\text{Pi}/2}\ \texttt{Integrate[Sin[x], \{x, -Pi / 2, Pi / 2\}]}$$

Out[160]= 0

The coefficients $a_k,\ k \in \mathbb{N}$ are

In[161]:=
```
FullSimplify[
 1/(Pi/2) Integrate[Sin[x] Cos[2 k x], {x, -Pi/2, Pi/2}],
 And[k ∈ Integers, k > 0]]
```

Out[161]= 0

The coefficients $b_k$, $k \in \mathbb{N}$ are

In[162]:=
```
FullSimplify[
 1/(Pi/2) Integrate[Sin[x] Sin[2 k x], {x, -Pi/2, Pi/2}],
 And[k ∈ Integers, k > 0]]
```

Out[162]= $\frac{8\,(-1)^k\,k}{\pi - 4\,k^2\,\pi}$

Thus the partial sum with 40 terms of the Fourier Series is

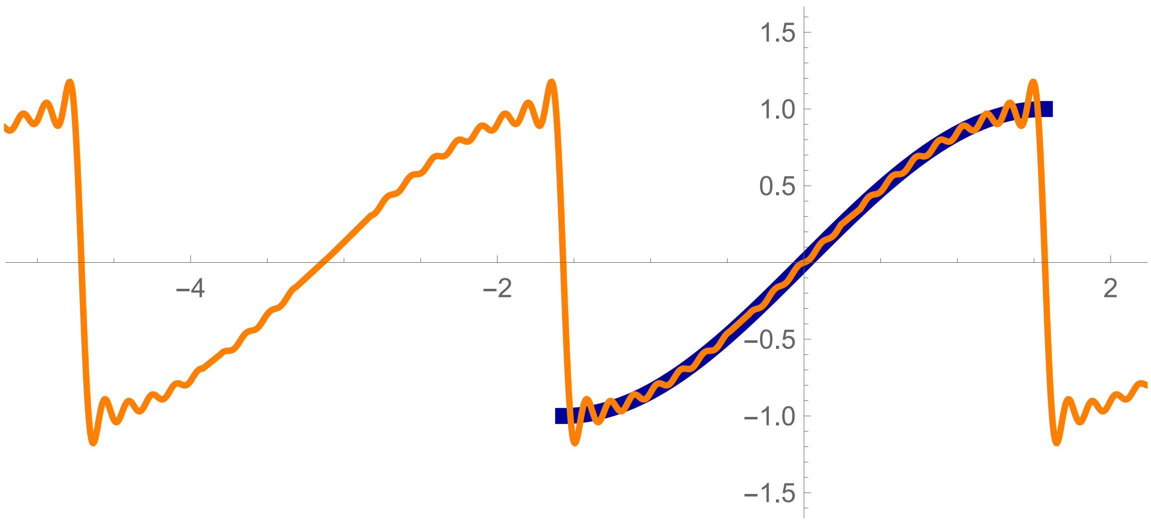
In[163]:=
```
nn = 20;
Clear[FS5]; FS5[x_] = Sum[(8 (-1)^k k)/(π - 4 k^2 π) Sin[2 k x], {k, 1, nn}]
```

Out[163]=
$$\frac{8\,\text{Sin}[2\,x]}{3\,\pi} - \frac{16\,\text{Sin}[4\,x]}{15\,\pi} + \frac{24\,\text{Sin}[6\,x]}{35\,\pi} - \frac{32\,\text{Sin}[8\,x]}{63\,\pi} + \frac{40\,\text{Sin}[10\,x]}{99\,\pi} - \frac{48\,\text{Sin}[12\,x]}{143\,\pi} + \frac{56\,\text{Sin}[14\,x]}{195\,\pi} - \frac{64\,\text{Sin}[16\,x]}{255\,\pi} + \frac{72\,\text{Sin}[18\,x]}{323\,\pi} - \frac{80\,\text{Sin}[20\,x]}{399\,\pi} + \frac{88\,\text{Sin}[22\,x]}{483\,\pi} - \frac{96\,\text{Sin}[24\,x]}{575\,\pi} + \frac{104\,\text{Sin}[26\,x]}{675\,\pi} - \frac{112\,\text{Sin}[28\,x]}{783\,\pi} + \frac{120\,\text{Sin}[30\,x]}{899\,\pi} - \frac{128\,\text{Sin}[32\,x]}{1023\,\pi} + \frac{136\,\text{Sin}[34\,x]}{1155\,\pi} - \frac{144\,\text{Sin}[36\,x]}{1295\,\pi} + \frac{152\,\text{Sin}[38\,x]}{1443\,\pi} - \frac{160\,\text{Sin}[40\,x]}{1599\,\pi}$$

Verify this with graphs

In[164]:=
```
Show[Plot[(Sin[#]) &[x], {x, -Pi / 2, Pi / 2},
   PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.01]}}],
  Plot[FS5[x], {x, -6, 6},
   PlotStyle → {{RGBColor[1, 0.5, 0], Thickness[0.004]}}],
  PlotRange → {{-5, 5}, {-1.5, 1.5}}, AspectRatio → Automatic,
  ImageSize → 600]
```

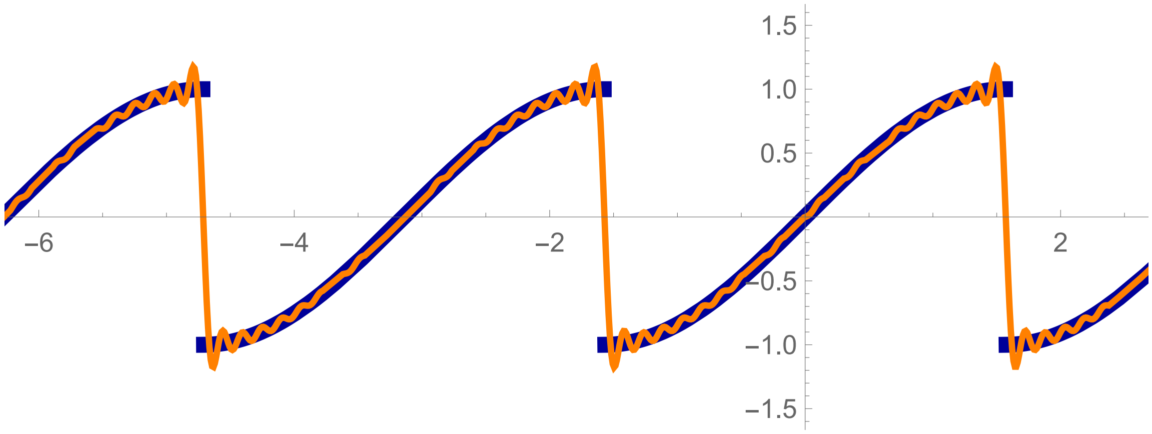
Out[164]=




It is nice to include the periodic extension

In[165]:=
```
Show[Plot[PerExt[x, (Sin[#]) &, -Pi / 2, Pi / 2],
   {x, -10, 10},
   PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.01]}}],
  Plot[FS5[x], {x, -16, 16},
   PlotStyle → {{RGBColor[1, 0.5, 0], Thickness[0.004]}}],
  PlotRange → {{-6, 6}, {-1.5, 1.5}}, AspectRatio → Automatic,
  ImageSize → 600]
```

Out[165]=




It might be interesting to include different partial sums of the Fourier series:

In[166]:=
```
Manipulate[
 Show[Plot[PerExt[x, (Sin[#]) &, -Pi / 2, Pi / 2],
   {x, -10, 10},
   PlotStyle → {{RGBColor[0, 0, 0.6], Thickness[0.01]}}],
  Plot[Sum[(8 (-1)^k k)/(π - 4 k^2 π) Sin[2 k x], {k, 1, nn}], {x, -6, 6},
   PlotStyle → {{RGBColor[1, 0.5, 0], Thickness[0.004]}}],
  PlotRange → {{-6 , 6}, {-1.5, 1.5}},
  AspectRatio → Automatic, ImageSize → 600],
 {{nn, 3}, Range[0, 20], Setter}, ControlPlacement → Top]
```

Out[166]=