In[383]:= `NotebookDirectory[]`

Out[383]= `C:\Dropbox\Work\myweb\Courses\Math_pages\Math_430\`

A collection of Fourier series

Preliminaries

Below is the definition of a periodic extension of a function defined on $(-L, L]$. This definition takes a function as a variable. The function has to be inputted as a so called pure function (that is instead of the variable we put # and the formula ends with &).

In[384]:= `Clear[ff, x, lL];`

```
fft[ff_, x_, lL_] := ff[x - (Ceiling[(x - (-lL))/(2 lL)] - 1) (2 lL)]
```

In[386]:= `(#^2) &[2]`

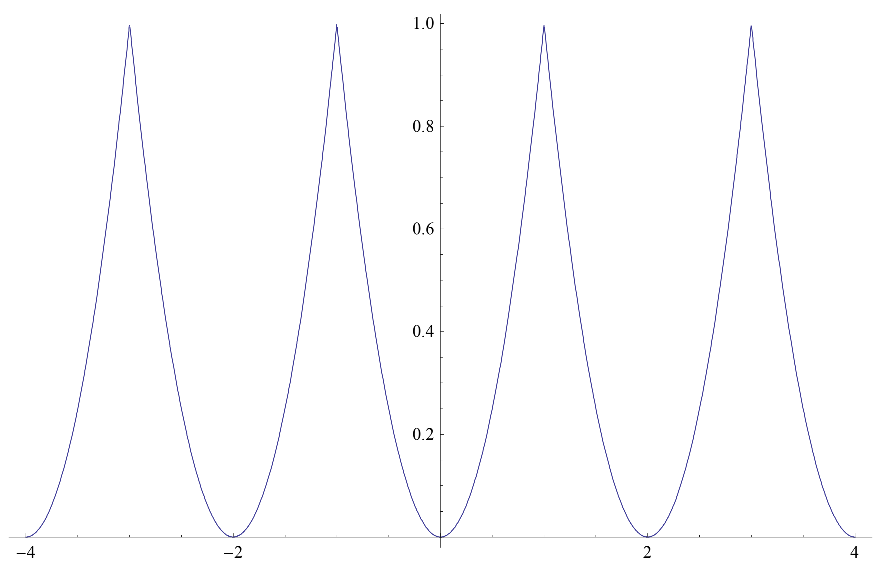
Out[386]= 4

In[387]:= `fft[#^2 &, x, 1]`

Out[387]= $\left(x - 2\left(-1 + \text{Ceiling}\left[\frac{1+x}{2}\right]\right)\right)^2$

In[388]:= `Plot[fft[#^2 &, x, 1], {x, -4, 4}, ImageSize → 450]`

Out[388]=



In[389]:= `is = 500`

Out[389]= 500

Example 1: Sign[x] on $-\pi < x \leq \pi$

In[390]:=
```
Clear[f1];

f1[x_] = Sign[x];
```

on the interval $(-\pi,\pi]$

The coefficient a_0

In[392]:=
```
FullSimplify[1/(2 Pi) Integrate[f1[x], {x, -Pi, Pi}]]
```

Out[392]= 0

The coefficients a_n

In[393]:=
```
FullSimplify[1/Pi Integrate[f1[x] Cos[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[393]= 0

The coefficients b_n

In[394]:=
```
FullSimplify[1/Pi Integrate[f1[x] Sin[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[394]= $-\frac{2\left(-1+(-1)^{n}\right)}{n\,\pi}$

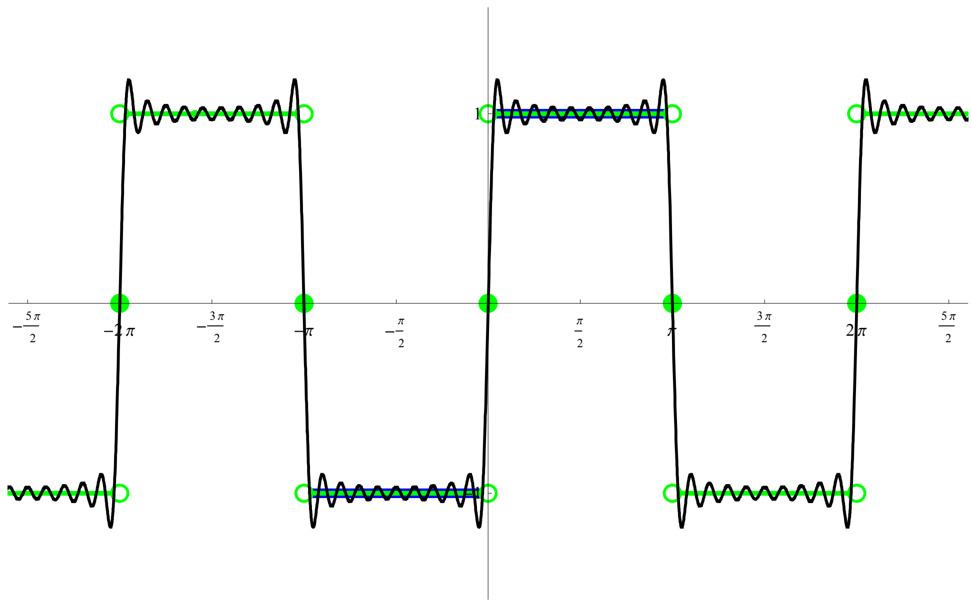
This formula simplifies; for even n to 0 and for odd n to $\frac{4}{\pi\, n}$. Thus the Fourier series of the given function is

$$\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2\,k-1}\,\mathrm{Sin}[(2\,k-1)\,x]$$

This series converges pointwise to the Fourier 2π-periodic extension of Sign[x], as illustrated in the following graph and manipulation

In[395]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, nn = 10;
 pic1 = Plot[{f1[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}, Exclusions → {0}];
 pic2 = Plot[{fft[f1[#] &, x, Pi]}, {x, -5, 10},
   PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

 pic2a = Graphics[{
    {PointSize[0.02], Green,
     {Point[{# Pi, -1}], Point[{# Pi, 1}], Point[{# Pi, 0}]} & /@ Range[-10, 13, 1]},
    {PointSize[0.014], White, {Point[{# Pi, -1}], Point[{# Pi, 1}]} & /@ Range[-10, 13, 1]}
   }];

 pic3 = Plot[Evaluate[{4/π Sum[1/(2 k - 1) Sin[(2 k - 1) x], {k, 1, nn}]}],
   {x, -12, 14}, PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200];
 Show[pic1, pic2, pic2a, pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {-1.5, 1.5}},
  Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-2, 2, 1]}, ImageSize → is]]
```

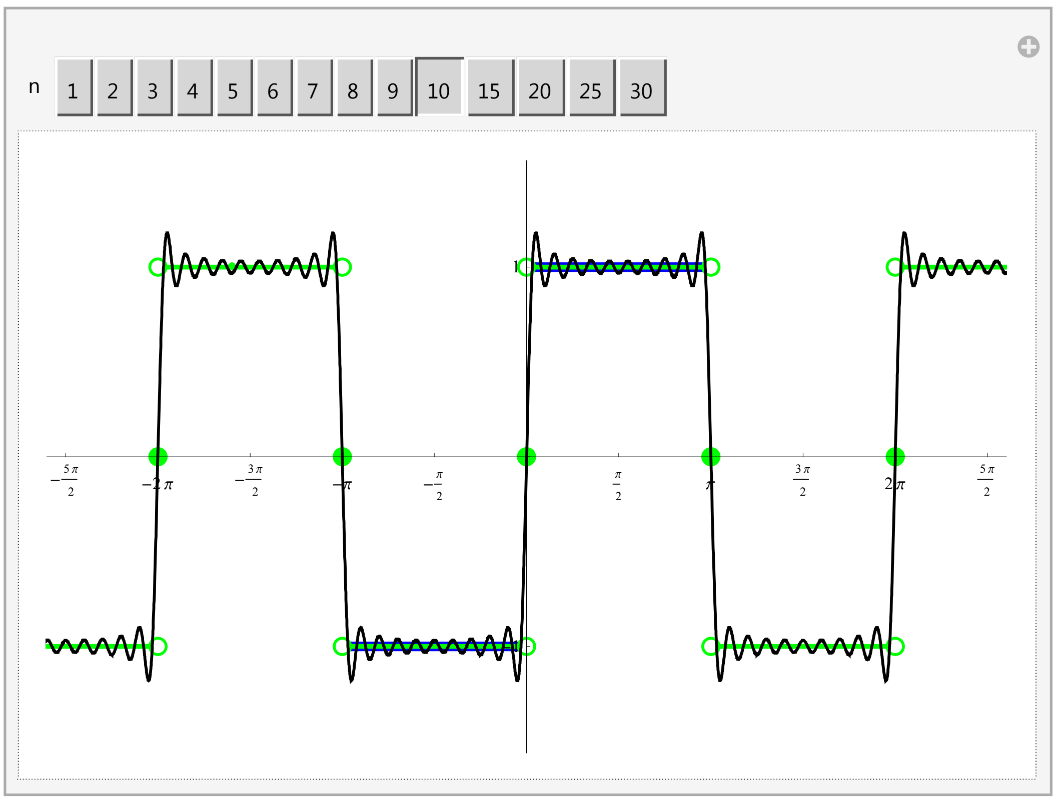
Out[395]=



Or, the same picture with Manipulate

In[396]:=
```
Module[{pic1, pic2, pic2a, pic3, nn},
 Manipulate[pic1 = Plot[{f1[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}},
   Exclusions → {0}]; pic2 = Plot[{fft[f1[#] &, x, Pi]}, {x, -5, 10},
   PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

  pic2a = Graphics[{
     {PointSize[0.02], Green,
      {Point[{# Pi, -1}], Point[{# Pi, 1}], Point[{# Pi, 0}]} & /@ Range[-10, 13, 1]},
     {PointSize[0.014], White, {Point[{# Pi, -1}], Point[{# Pi, 1}]} & /@ Range[-10, 13, 1]}
    }];

  pic3 = Plot[Evaluate[{(4/π) Sum[1/(2 k - 1) Sin[(2 k - 1) x], {k, 1, nn}]}],
    {x, -12, 14}, PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200];
  Show[pic1, pic2, pic2a, pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {-1.5, 1.5}},
   Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-2, 2, 1]}, ImageSize → is],
  {{nn, 10, "n"}, Join[Range[10], Range[15, 30, 5]]}, ControlType → Setter]]
```

Out[396]=



- **Numerical series at special values of *x***
- $x = \pi/2$

Notice that the convergence theorem implies that for a specific $x = \frac{\pi}{2}$ the following numerical series

$$\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\,\text{Sin}\left[(2k-1)\frac{\pi}{2}\right]$$

which equals

$$\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}$$

converges to 1.

Mathematica knows this

In[397]:=
$$\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}$$

Out[397]= 1

In other words, from the convergence theorem for Fourier series we deduce that

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}=\frac{\pi}{4}$$

which is the famous Leibniz formula for π.

- $x = \pi/4$

In[398]:= **Table[{k, Sin[(2 k - 1) * Pi/4]}, {k, 1, 20}]**

Out[398]=
$$\left\{\left\{1,\frac{1}{\sqrt{2}}\right\},\left\{2,\frac{1}{\sqrt{2}}\right\},\left\{3,-\frac{1}{\sqrt{2}}\right\},\left\{4,-\frac{1}{\sqrt{2}}\right\},\left\{5,\frac{1}{\sqrt{2}}\right\},\left\{6,\frac{1}{\sqrt{2}}\right\},\left\{7,-\frac{1}{\sqrt{2}}\right\},\left\{8,-\frac{1}{\sqrt{2}}\right\},\left\{9,\frac{1}{\sqrt{2}}\right\},\left\{10,\frac{1}{\sqrt{2}}\right\},\left\{11,-\frac{1}{\sqrt{2}}\right\},\left\{12,-\frac{1}{\sqrt{2}}\right\},\left\{13,\frac{1}{\sqrt{2}}\right\},\left\{14,\frac{1}{\sqrt{2}}\right\},\left\{15,-\frac{1}{\sqrt{2}}\right\},\left\{16,-\frac{1}{\sqrt{2}}\right\},\left\{17,\frac{1}{\sqrt{2}}\right\},\left\{18,\frac{1}{\sqrt{2}}\right\},\left\{19,-\frac{1}{\sqrt{2}}\right\},\left\{20,-\frac{1}{\sqrt{2}}\right\}\right\}$$

In[399]:= **Table[{k, (-1)^(Ceiling[k/2]-1)}, {k, 1, 20}]**

Out[399]= {{1, 1}, {2, 1}, {3, -1}, {4, -1}, {5, 1}, {6, 1}, {7, -1}, {8, -1}, {9, 1}, {10, 1}, {11, -1}, {12, -1}, {13, 1}, {14, 1}, {15, -1}, {16, -1}, {17, 1}, {18, 1}, {19, -1}, {20, -1}}

The convergence theorem implies that for a specific $x=\frac{\pi}{4}$ the following numerical series

$$\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\,\text{Sin}\left[(2k-1)\frac{\pi}{4}\right]$$

which equals

$$\frac{2\sqrt{2}}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{\text{Ceiling}[k/2]-1}}{2k-1}$$

converges to 1.

Does Mathematica knows this

In[400]:= $\frac{2\sqrt{2}}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{\text{Ceiling[k/2]-1}}}{2k-1}$

Sum::div : Sum does not converge. ≫

Out[400]= $\frac{2\sqrt{2}\sum_{k=1}^{\infty}\frac{(-1)^{-1+\text{Ceiling}\left[\frac{k}{2}\right]}}{-1+2k}}{\pi}$

Check numerically

In[401]:= $N\left[\frac{2\sqrt{2}}{\pi}\sum_{k=1}^{10000}\frac{(-1)^{\text{Ceiling[k/2]-1}}}{2k-1}\right]$

Out[401]= 0.999955

- $x = \pi/3$

But we can get more numerical series sums from the above Fourier series

In[402]:= $\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\text{Sin}\left[(2k-1)*\frac{\text{Pi}}{3}\right]$

Out[402]= $\frac{2\left(\text{ArcTan}\left[(-1)^{1/6}\right]-i\,\text{ArcTanh}\left[(-1)^{1/3}\right]\right)}{\pi}$

In[403]:= $\text{FullSimplify}\left[\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\text{Sin}\left[(2k-1)*\frac{\text{Pi}}{3}\right]\right]$

Out[403]= 1

In[404]:= $\text{Table}\left[\left\{k,\text{Sin}\left[(2k-1)*\frac{\text{Pi}}{3}\right]\right\},\{k,1,20\}\right]$

Out[404]= $\left\{\left\{1,\frac{\sqrt{3}}{2}\right\},\{2,0\},\left\{3,-\frac{\sqrt{3}}{2}\right\},\left\{4,\frac{\sqrt{3}}{2}\right\},\{5,0\},\left\{6,-\frac{\sqrt{3}}{2}\right\},\left\{7,\frac{\sqrt{3}}{2}\right\},\{8,0\},\left\{9,-\frac{\sqrt{3}}{2}\right\},\left\{10,\frac{\sqrt{3}}{2}\right\},\{11,0\},\left\{12,-\frac{\sqrt{3}}{2}\right\},\left\{13,\frac{\sqrt{3}}{2}\right\},\{14,0\},\left\{15,-\frac{\sqrt{3}}{2}\right\},\left\{16,\frac{\sqrt{3}}{2}\right\},\{17,0\},\left\{18,-\frac{\sqrt{3}}{2}\right\},\left\{19,\frac{\sqrt{3}}{2}\right\},\{20,0\}\right\}$

In[405]:= $\text{Table}\left[3\,\text{Floor}[j/2]+\frac{1+(-1)^{j-1}}{2},\{j,1,10\}\right]$

Out[405]= {1, 3, 4, 6, 7, 9, 10, 12, 13, 15}

In[406]:= $\text{Table}\left[2\left(3\,\text{Floor}[j/2]+\frac{1+(-1)^{j-1}}{2}\right)-1,\{j,1,10\}\right]$

Out[406]= {1, 5, 7, 11, 13, 17, 19, 23, 25, 29}

In[407]:= $\text{Table}\left[\left\{j,3\,\text{Floor}[j/2]+\frac{1+(-1)^{j-1}}{2},(-1)^{j-1}\right\},\{j,1,10\}\right]$

Out[407]= {{1, 1, 1}, {2, 3, -1}, {3, 4, 1}, {4, 6, -1}, {5, 7, 1}, {6, 9, -1}, {7, 10, 1}, {8, 12, -1}, {9, 13, 1}, {10, 15, -1}}

In[408]:= Table$\left[\left\{j, 2\left(3\,\text{Floor}[j/2] + \frac{1+(-1)^{j-1}}{2}\right) - 1\right\}, \{j, 1, 10\}\right]$

```
Out[408]= {{1, 1}, {2, 5}, {3, 7}, {4, 11}, {5, 13}, {6, 17}, {7, 19}, {8, 23}, {9, 25}, {10, 29}}
```

In[409]:= Table$\left[\left\{j, 3\left(j - \frac{1+(-1)^{j-1}}{2}\right) + (-1)^{j-1}\right\}, \{j, 1, 10\}\right]$

```
Out[409]= {{1, 1}, {2, 5}, {3, 7}, {4, 11}, {5, 13}, {6, 17}, {7, 19}, {8, 23}, {9, 25}, {10, 29}}
```

In[410]:= Table$\left[\left\{j, \frac{6j - 3 - (-1)^{j-1}}{2}\right\}, \{j, 1, 10\}\right]$

```
Out[410]= {{1, 1}, {2, 5}, {3, 7}, {4, 11}, {5, 13}, {6, 17}, {7, 19}, {8, 23}, {9, 25}, {10, 29}}
```

In[411]:= $\frac{4\sqrt{3}}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{6j-3-(-1)^{j-1}}$

Out[411]= $\frac{4\sqrt{3}\sum_{j=1}^{\infty}\frac{(-1)^{-1+j}}{-3-(-1)^{-1+j}+6j}}{\pi}$

In[412]:= N$\left[\frac{4\sqrt{3}}{\pi}\sum_{j=1}^{1200}\frac{(-1)^{j-1}}{6j-3-(-1)^{j-1}}\right]$

Out[412]= 0.999796

Example 2: UnitStep[x] on $-\pi < x \le \pi$

```
In[413]:= Clear[f2];

f2[x_] = UnitStep[x];
```

on the interval $(-\pi,\pi]$

The coefficient a_0

In[415]:= FullSimplify$\left[\frac{1}{2\,\text{Pi}}\text{Integrate}[f2[x], \{x, -\text{Pi}, \text{Pi}\}]\right]$

Out[415]= $\frac{1}{2}$

The coefficients a_n

In[416]:= FullSimplify$\left[\frac{1}{\text{Pi}}\text{Integrate}[f2[x]\,\text{Cos}[n\,x], \{x, -\text{Pi}, \text{Pi}\}], \text{And}[n \in \text{Integers}, n > 0]\right]$

Out[416]= 0

The coefficients b_n

In[417]:= FullSimplify$\left[\frac{1}{\text{Pi}}\text{Integrate}[f2[x]\,\text{Sin}[n\,x], \{x, -\text{Pi}, \text{Pi}\}], \text{And}[n \in \text{Integers}, n > 0]\right]$

Out[417]= $-\frac{-1+(-1)^n}{n\,\pi}$

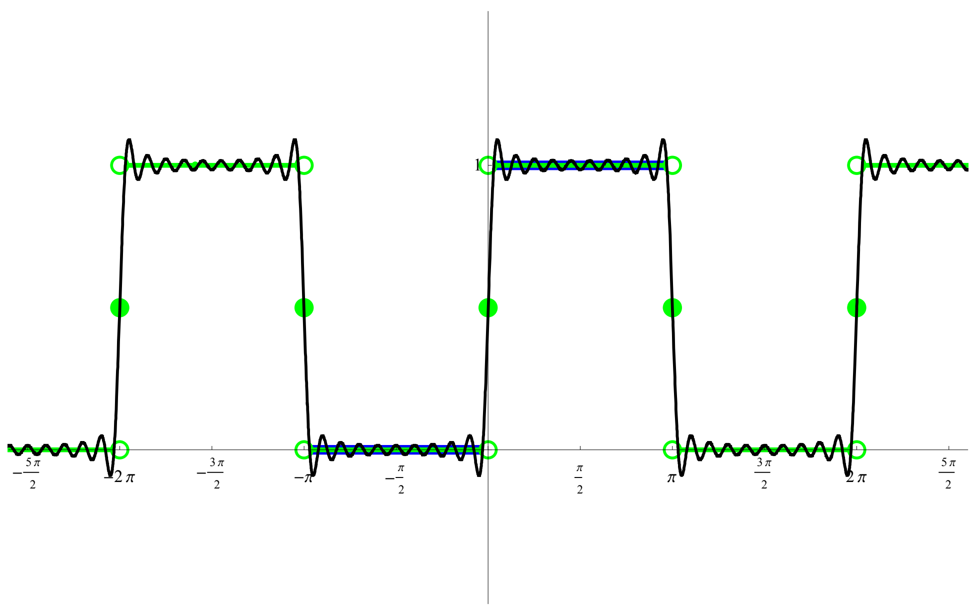
This formula simplifies; for even n to 0 and for odd n to $\frac{2}{\pi n}$. Thus the Fourier series of the given function is

$$\frac{1}{2}+\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\text{Sin}[(2k-1)x]$$

This series converges pointwise to the Fourier 2π-periodic extension of UnitStep[x], as illustrated in the following graph and manipulation

In[418]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, nn = 10;
 pic1 = Plot[{f2[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}, Exclusions → {0}];
 pic2 = Plot[{fft[f2[#] &, x, Pi]}, {x, -5, 10},
   PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

 pic2a = Graphics[{
    {PointSize[0.02], Green,
     {Point[{# Pi, 1}], Point[{# Pi, 1 / 2}], Point[{# Pi, 0}]} & /@ Range[-10, 13, 1]},
    {PointSize[0.014], White, {Point[{# Pi, 0}], Point[{# Pi, 1}]} & /@ Range[-10, 13, 1]}
   }];

 pic3 = Plot[Evaluate[{1/2 + 2/π Sum[1/(2 k - 1) Sin[(2 k - 1) x], {k, 1, nn}]}],
   {x, -12, 14}, PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200];
 Show[pic1, pic2, pic2a, pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {-.5, 1.5}},
  Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-2, 2, 1]}, ImageSize → is]]
```

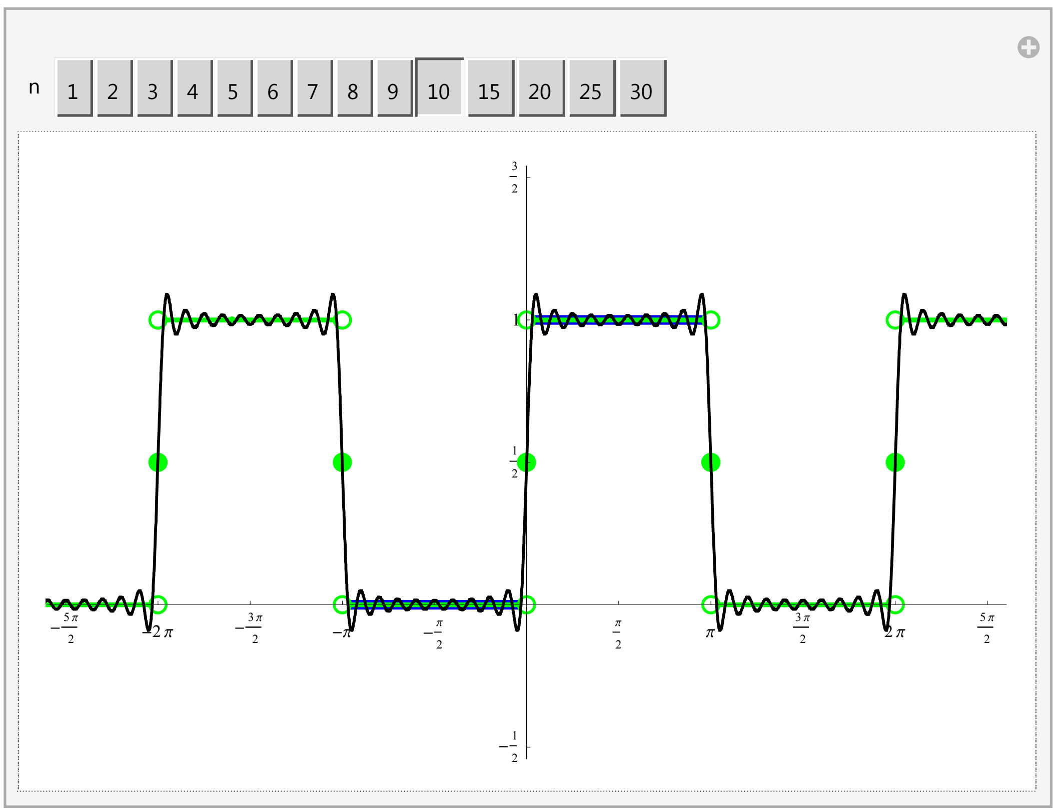
Out[418]=



Or, the same picture with Manipulate

In[419]:=
```
Module[{pic1, pic2, pic2a, pic3, nn},
 Manipulate[pic1 = Plot[{f2[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}},
   Exclusions → {0}]; pic2 = Plot[{fft[f2[#] &, x, Pi]}, {x, -5, 10},
   PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

  pic2a = Graphics[{
     {PointSize[0.02], Green,
      {Point[{# Pi, 1}], Point[{# Pi, 1 / 2}], Point[{# Pi, 0}]} & /@ Range[-10, 13, 1]},
     {PointSize[0.014], White, {Point[{# Pi, 0}], Point[{# Pi, 1}]} & /@ Range[-10, 13, 1]}
    }];

  pic3 = Plot[Evaluate[{1/2 + 2/π Sum[1/(2 k - 1) Sin[(2 k - 1) x], {k, 1, nn}]}],
    {x, -12, 14}, PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200];
  Show[pic1, pic2, pic2a, pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {-.5, 1.5}},
   Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-2, 2, 1 / 2]}, ImageSize → is],
  {{nn, 10, "n"}, Join[Range[10], Range[15, 30, 5]]}, ControlType → Setter]]
```

Out[419]=



With the exception of the points of discontinuity, there is a simple connection between the functions f1 and f2: $f2[x] = \frac{1}{2}(1 + f1[x])$. Therefore the Fourier series studied in this section does not yield any new information in pointwise convergence.

Example 3: x on $-\pi < x \leq \pi$

In[420]:=
```
Clear[f3];

f3[x_] = x;
```

on the interval $(-\pi,\pi]$

The coefficient a_0

In[422]:=
```
FullSimplify[1/(2 Pi) Integrate[f3[x], {x, -Pi, Pi}]]
```

Out[422]= 0

The coefficients a_n

In[423]:=
```
FullSimplify[1/Pi Integrate[f3[x] Cos[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[423]= 0

The coefficients b_n

In[424]:=
```
FullSimplify[1/Pi Integrate[f3[x] Sin[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[424]= $-\frac{2 (-1)^n}{n}$

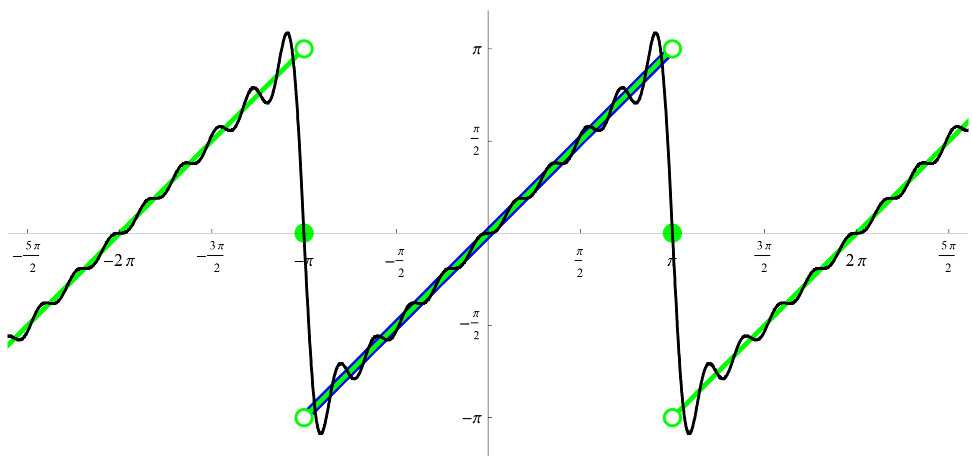
Thus the Fourier series of the given function is

$$2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \mathrm{Sin}[n\, x]$$

This series converges pointwise to the Fourier 2π-periodic extension of the function x, restricted to $(-\pi,\pi]$, as illustrated in the following graph and manipulation

In[425]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, nn = 10;
 pic1 = Plot[{f3[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}];

 pic2 = Plot[{fft[f3[#] &, x, Pi]}, {x, -15, 10},
   PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

 pic2a = Graphics[{
    {PointSize[0.02], Green,
     {Point[{# Pi, -Pi}], Point[{# Pi, 0}], Point[{# Pi, Pi}]} & /@ Range[-11, 13, 2]},
    {PointSize[0.014], White, {Point[{# Pi, -Pi}], Point[{# Pi, Pi}]} & /@ Range[-11, 13, 2]}
   }];

 pic3 = Plot[Evaluate[{2 Sum[(-1)^(n+1)/n Sin[(n) x], {n, 1, nn}]}], {x, -12, 14},
   PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200]; Show[pic1, pic2, pic2a,
  pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {-Pi - .5, Pi + .5}}, AspectRatio → Automatic,
  Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-Pi, Pi, Pi/2]}, ImageSize → is]]
```

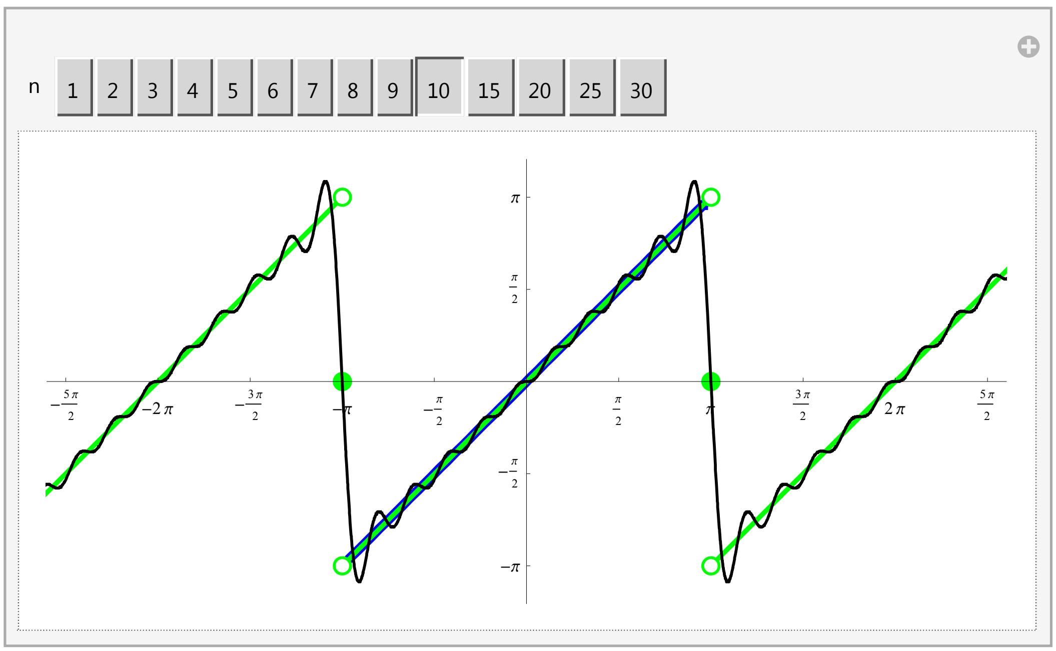
Out[425]=



Or, the same picture with Manipulate

In[426]:=
```
Module[{pic1, pic2, pic2a, pic3, nn},
 Manipulate[pic1 = Plot[{f3[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}];

  pic2 = Plot[{fft[f3[#] &, x, Pi]}, {x, -15, 10},
    PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

  pic2a = Graphics[{
     {PointSize[0.02], Green, {Point[{# Pi, -Pi}], Point[{# Pi, 0}], Point[{# Pi, Pi}]} & /@
       Range[-11, 13, 2]}, {PointSize[0.014], White,
      {Point[{# Pi, -Pi}], Point[{# Pi, Pi}]} & /@ Range[-11, 13, 2]}
    }];

  pic3 = Plot[Evaluate[{2 Sum[(-1)^(n+1)/n Sin[(n) x], {n, 1, nn}]}], {x, -12, 14},
    PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200]; Show[pic1, pic2, pic2a,
   pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {-Pi - .5, Pi + .5}}, AspectRatio → Automatic,
   Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-Pi, Pi, Pi/2]}, ImageSize → is],
  {{nn, 10, "n"}, Join[Range[10], Range[15, 30, 5]]}, ControlType → Setter]]
```

Out[426]=



- **Numerical series at special values of *x***
- ***x* = π / 2**

Notice that the convergence theorem implies that for a specific $x = \frac{\pi}{2}$ the following numerical series

$$2 \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Sin}\left[n \frac{\pi}{2}\right]$$

which equals

$$2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1}$$

converges to $\pi/2$.

This is again the Leibniz series.

- **$x = \pi/3$**

```
In[427]:= Table[{n, Sin[n * Pi/3]}, {n, 1, 20}]
```

Out[427]= $\left\{\left\{1, \frac{\sqrt{3}}{2}\right\}, \left\{2, \frac{\sqrt{3}}{2}\right\}, \{3, 0\}, \left\{4, -\frac{\sqrt{3}}{2}\right\}, \left\{5, -\frac{\sqrt{3}}{2}\right\}, \{6, 0\}, \left\{7, \frac{\sqrt{3}}{2}\right\}, \left\{8, \frac{\sqrt{3}}{2}\right\}, \{9, 0\}, \left\{10, -\frac{\sqrt{3}}{2}\right\}, \left\{11, -\frac{\sqrt{3}}{2}\right\}, \{12, 0\}, \left\{13, \frac{\sqrt{3}}{2}\right\}, \left\{14, \frac{\sqrt{3}}{2}\right\}, \{15, 0\}, \left\{16, -\frac{\sqrt{3}}{2}\right\}, \left\{17, -\frac{\sqrt{3}}{2}\right\}, \{18, 0\}, \left\{19, \frac{\sqrt{3}}{2}\right\}, \left\{20, \frac{\sqrt{3}}{2}\right\}\right\}$

Figuring out the pattern here is more complicated

```
In[428]:= Table[(-1)^Floor[k/2] Floor[3/2 k + 1], {k, 0, 20}]
```

```
Out[428]= {1, 2, -4, -5, 7, 8, -10, -11, 13, 14, -16, -17, 19, 20, -22, -23, 25, 26, -28, -29, 31}
```

```
In[429]:= Table[2 (-1)^(n+1)/n Sin[n π/3], {n, 1, 23}]
```

Out[429]= $\left\{\sqrt{3}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{4}, -\frac{\sqrt{3}}{5}, 0, \frac{\sqrt{3}}{7}, -\frac{\sqrt{3}}{8}, 0, \frac{\sqrt{3}}{10}, -\frac{\sqrt{3}}{11}, 0, \frac{\sqrt{3}}{13}, -\frac{\sqrt{3}}{14}, 0, \frac{\sqrt{3}}{16}, -\frac{\sqrt{3}}{17}, 0, \frac{\sqrt{3}}{19}, -\frac{\sqrt{3}}{20}, 0, \frac{\sqrt{3}}{22}, -\frac{\sqrt{3}}{23}\right\}$

```
In[430]:= Table[Sqrt[3] (-1)^(Floor[k/2]+Floor[3/2 k]) / (Floor[3/2 k] + 1), {k, 0, 15}]
```

Out[430]= $\left\{\sqrt{3}, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{4}, -\frac{\sqrt{3}}{5}, \frac{\sqrt{3}}{7}, -\frac{\sqrt{3}}{8}, \frac{\sqrt{3}}{10}, -\frac{\sqrt{3}}{11}, \frac{\sqrt{3}}{13}, -\frac{\sqrt{3}}{14}, \frac{\sqrt{3}}{16}, -\frac{\sqrt{3}}{17}, \frac{\sqrt{3}}{19}, -\frac{\sqrt{3}}{20}, \frac{\sqrt{3}}{22}, -\frac{\sqrt{3}}{23}\right\}$

The convergence theorem implies that for a specific $x = \frac{\pi}{3}$ the following numerical series

$$2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \operatorname{Sin}\left[n \frac{\pi}{3}\right]$$

which equals

$$\sqrt{3} \sum_{k=0}^{\infty} \frac{(-1)^{\operatorname{Floor}[k/2]+\operatorname{Floor}\left[\frac{3}{2} k\right]}}{\operatorname{Floor}\left[\frac{3}{2} k\right]+1}$$

converges to $\pi/3$.

Does Mathematica knows this

In[431]:= $$\frac{3\sqrt{3}}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{\text{Floor}[k/2]+\text{Floor}\left[\frac{3}{2}k\right]}}{\text{Floor}\left[\frac{3}{2}k\right]+1}$$

Sum::div : Sum does not converge. ≫

Out[431]= $$\frac{3\sqrt{3}\sum_{k=0}^{\infty}\frac{(-1)^{\text{Floor}\left[\frac{k}{2}\right]+\text{Floor}\left[\frac{3k}{2}\right]}}{1+\text{Floor}\left[\frac{3k}{2}\right]}}{\pi}$$

Check numerically

In[432]:= $$\text{N}\left[\frac{3\sqrt{3}}{\pi}\sum_{k=0}^{10000}\frac{(-1)^{\text{Floor}[k/2]+\text{Floor}\left[\frac{3}{2}k\right]}}{\text{Floor}\left[\frac{3}{2}k\right]+1}\right]$$

Out[432]= 1.00007

So, we established that the series with terms

In[433]:= $$\text{Table}\left[\frac{(-1)^{\text{Floor}[k/2]+\text{Floor}\left[\frac{3}{2}k\right]}}{\text{Floor}\left[\frac{3}{2}k\right]+1}, \{k, 0, 30\}\right]$$

Out[433]= $$\left\{1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{7}, -\frac{1}{8}, \frac{1}{10}, -\frac{1}{11}, \frac{1}{13}, -\frac{1}{14}, \frac{1}{16}, -\frac{1}{17}, \frac{1}{19}, -\frac{1}{20}, \frac{1}{22}, -\frac{1}{23}, \frac{1}{25}, -\frac{1}{26}, \frac{1}{28}, -\frac{1}{29}, \frac{1}{31}, -\frac{1}{32}, \frac{1}{34}, -\frac{1}{35}, \frac{1}{37}, -\frac{1}{38}, \frac{1}{40}, -\frac{1}{41}, \frac{1}{43}, -\frac{1}{44}, \frac{1}{46}\right\}$$

and so on

converges to $\frac{\pi}{3\sqrt{3}}$

Adding the consecutive terms of the above series we get the series, for which *Mathematica* knows the sum

In[434]:= $$\sum_{k=0}^{\infty}\frac{1}{(3k+1)(3k+2)}$$

Out[434]= $$\frac{\pi}{3\sqrt{3}}$$

Example 3a: π Sign[x] - x on $-\pi < x \leq \pi$

In[435]:=
```
Clear[f3a];

f3a[x_] = π Sign[x] - x;
```

on the interval $(-\pi,\pi]$

The coefficient a_0

In[437]:= $$\text{FullSimplify}\left[\frac{1}{2\,\text{Pi}}\,\text{Integrate}[\text{f3a}[x], \{x, -\text{Pi}, \text{Pi}\}]\right]$$

Out[437]= 0

The coefficients a_n

In[438]:=
```
FullSimplify[1/Pi Integrate[f3a[x] Cos[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[438]= 0

The coefficients b_n

In[439]:=
```
FullSimplify[1/Pi Integrate[f3a[x] Sin[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

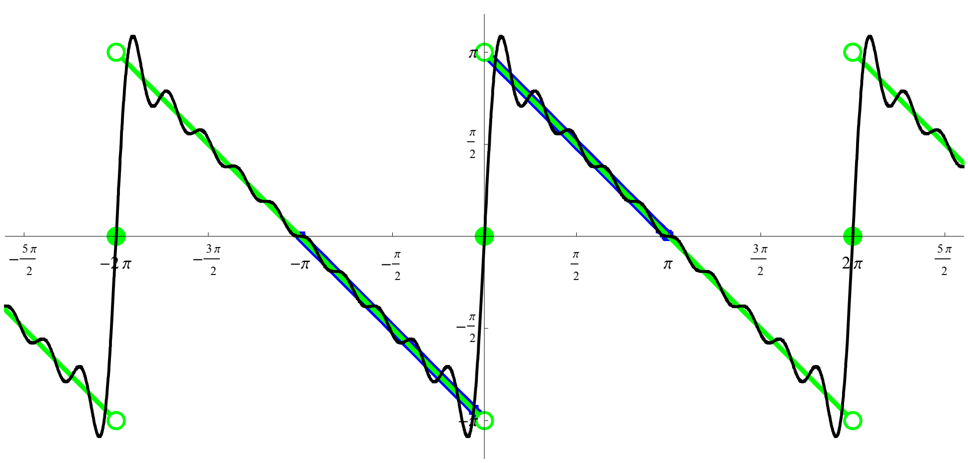
Out[439]= $\frac{2}{n}$

Thus the Fourier series of the given function is

$$2 \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Sin}[n\, x]$$

This series converges pointwise to the Fourier 2π-periodic extension of the function x, restricted to $(-\pi,\pi]$, as illustrated in the following graph and manipulation

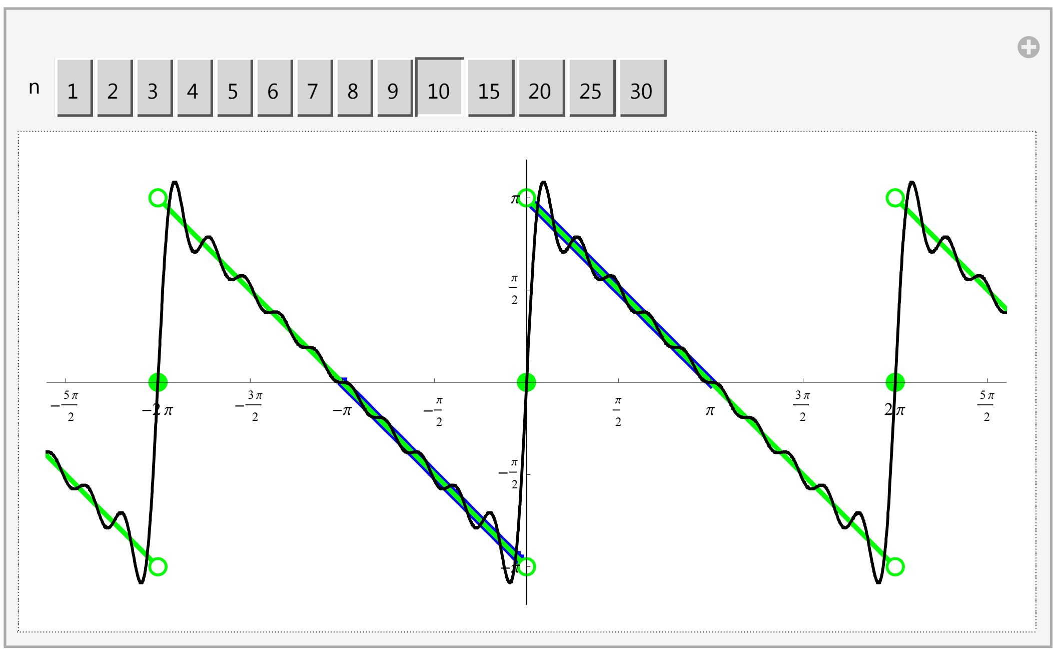
In[440]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, nn = 10;
 pic1 = Plot[{f3a[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}];

 pic2 = Plot[{fft[f3a[#] &, x, Pi]}, {x, -15, 10},
   PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

 pic2a = Graphics[{
    {PointSize[0.02], Green,
     {Point[{# Pi, -Pi}], Point[{# Pi, 0}], Point[{# Pi, Pi}]} & /@ Range[-10, 13, 2]},
    {PointSize[0.014], White, {Point[{# Pi, -Pi}], Point[{# Pi, Pi}]} & /@ Range[-10, 13, 2]}
   }];

 pic3 = Plot[Evaluate[{2 Sum[1/n Sin[(n) x], {n, 1, nn}]}], {x, -12, 14},
   PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200]; Show[pic1, pic2, pic2a,
  pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {-Pi - .5, Pi + .5}}, AspectRatio → Automatic,
  Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-Pi, Pi, Pi/2]}, ImageSize → is]]
```

Out[440]=


Or, the same picture with Manipulate

In[441]:=
```
Module[{pic1, pic2, pic2a, pic3, nn},
 Manipulate[pic1 = Plot[{f3a[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}];

  pic2 = Plot[{fft[f3a[#] &, x, Pi]}, {x, -15, 10},
    PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

  pic2a = Graphics[{
     {PointSize[0.02], Green, {Point[{# Pi, -Pi}], Point[{# Pi, 0}], Point[{# Pi, Pi}]} & /@
       Range[-10, 13, 2]}, {PointSize[0.014], White,
      {Point[{# Pi, -Pi}], Point[{# Pi, Pi}]} & /@ Range[-10, 13, 2]}
    }];

  pic3 = Plot[Evaluate[{2 Sum[1/n Sin[(n) x], {n, 1, nn}]}], {x, -12, 14},
    PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200]; Show[pic1, pic2, pic2a,
   pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {-Pi - .5, Pi + .5}}, AspectRatio → Automatic,
   Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-Pi, Pi, Pi/2]}, ImageSize → is],
  {{nn, 10, "n"}, Join[Range[10], Range[15, 30, 5]]}, ControlType → Setter]]
```

Out[441]=



- **Numerical series at special values of *x***
- ***x* = *π* / 2**

Notice that the convergence theorem implies that for a specific $x = \frac{\pi}{2}$ the following numerical series

$$2 \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Sin}\left[n \frac{\pi}{2}\right]$$

which equals

$$2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}$$

converges to $\pi/2$.

This is again the Leibniz series.

- $x = \pi/3$

In[442]:= `Table[{n, Sin[n * Pi/3]}, {n, 1, 20}]`

Out[442]= $\left\{\left\{1, \frac{\sqrt{3}}{2}\right\}, \left\{2, \frac{\sqrt{3}}{2}\right\}, \{3, 0\}, \left\{4, -\frac{\sqrt{3}}{2}\right\}, \left\{5, -\frac{\sqrt{3}}{2}\right\}, \{6, 0\}, \left\{7, \frac{\sqrt{3}}{2}\right\}, \left\{8, \frac{\sqrt{3}}{2}\right\}, \{9, 0\}, \left\{10, -\frac{\sqrt{3}}{2}\right\}, \left\{11, -\frac{\sqrt{3}}{2}\right\}, \{12, 0\}, \left\{13, \frac{\sqrt{3}}{2}\right\}, \left\{14, \frac{\sqrt{3}}{2}\right\}, \{15, 0\}, \left\{16, -\frac{\sqrt{3}}{2}\right\}, \left\{17, -\frac{\sqrt{3}}{2}\right\}, \{18, 0\}, \left\{19, \frac{\sqrt{3}}{2}\right\}, \left\{20, \frac{\sqrt{3}}{2}\right\}\right\}$

Figuring out the pattern here is more complicated

In[443]:= `Table[(-1)^Floor[k/2] Floor[3/2 k + 1], {k, 0, 20}]`

Out[443]= {1, 2, -4, -5, 7, 8, -10, -11, 13, 14, -16, -17, 19, 20, -22, -23, 25, 26, -28, -29, 31}

In[444]:= `Table[2 (-1)^(n+1)/n Sin[n π/3], {n, 1, 23}]`

Out[444]= $\left\{\sqrt{3}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{4}, -\frac{\sqrt{3}}{5}, 0, \frac{\sqrt{3}}{7}, -\frac{\sqrt{3}}{8}, 0, \frac{\sqrt{3}}{10}, -\frac{\sqrt{3}}{11}, 0, \frac{\sqrt{3}}{13}, -\frac{\sqrt{3}}{14}, 0, \frac{\sqrt{3}}{16}, -\frac{\sqrt{3}}{17}, 0, \frac{\sqrt{3}}{19}, -\frac{\sqrt{3}}{20}, 0, \frac{\sqrt{3}}{22}, -\frac{\sqrt{3}}{23}\right\}$

In[445]:= `Table[Sqrt[3] (-1)^(Floor[k/2]+Floor[3/2 k])/(Floor[3/2 k] + 1), {k, 0, 15}]`

Out[445]= $\left\{\sqrt{3}, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{4}, -\frac{\sqrt{3}}{5}, \frac{\sqrt{3}}{7}, -\frac{\sqrt{3}}{8}, \frac{\sqrt{3}}{10}, -\frac{\sqrt{3}}{11}, \frac{\sqrt{3}}{13}, -\frac{\sqrt{3}}{14}, \frac{\sqrt{3}}{16}, -\frac{\sqrt{3}}{17}, \frac{\sqrt{3}}{19}, -\frac{\sqrt{3}}{20}, \frac{\sqrt{3}}{22}, -\frac{\sqrt{3}}{23}\right\}$

The convergence theorem implies that for a specific $x = \frac{\pi}{3}$ the following numerical series

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\,\mathrm{Sin}\left[n\,\frac{\pi}{3}\right]$$

which equals

$$\sqrt{3}\sum_{k=0}^{\infty}\frac{(-1)^{\mathrm{Floor}[k/2]+\mathrm{Floor}\left[\frac{3}{2}k\right]}}{\mathrm{Floor}\left[\frac{3}{2}k\right]+1}$$

converges to $\pi/3$.

Does Mathematica knows this

In[446]:= $$\frac{3\sqrt{3}}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{\text{Floor}[k/2]+\text{Floor}\left[\frac{3}{2}k\right]}}{\text{Floor}\left[\frac{3}{2}k\right]+1}$$

Sum::div : Sum does not converge. ≫

Out[446]= $$\frac{3\sqrt{3}\sum_{k=0}^{\infty}\frac{(-1)^{\text{Floor}\left[\frac{k}{2}\right]+\text{Floor}\left[\frac{3k}{2}\right]}}{1+\text{Floor}\left[\frac{3k}{2}\right]}}{\pi}$$

Check numerically

In[447]:= $$\text{N}\left[\frac{3\sqrt{3}}{\pi}\sum_{k=0}^{10000}\frac{(-1)^{\text{Floor}[k/2]+\text{Floor}\left[\frac{3}{2}k\right]}}{\text{Floor}\left[\frac{3}{2}k\right]+1}\right]$$

Out[447]= 1.00007

So, we established that the series with terms

In[448]:= $$\text{Table}\left[\frac{(-1)^{\text{Floor}[k/2]+\text{Floor}\left[\frac{3}{2}k\right]}}{\text{Floor}\left[\frac{3}{2}k\right]+1}, \{k, 0, 30\}\right]$$

Out[448]= $$\left\{1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{7}, -\frac{1}{8}, \frac{1}{10}, -\frac{1}{11}, \frac{1}{13}, -\frac{1}{14}, \frac{1}{16}, -\frac{1}{17}, \frac{1}{19}, -\frac{1}{20}, \frac{1}{22}, -\frac{1}{23}, \frac{1}{25}, -\frac{1}{26}, \frac{1}{28}, -\frac{1}{29}, \frac{1}{31}, -\frac{1}{32}, \frac{1}{34}, -\frac{1}{35}, \frac{1}{37}, -\frac{1}{38}, \frac{1}{40}, -\frac{1}{41}, \frac{1}{43}, -\frac{1}{44}, \frac{1}{46}\right\}$$

and so on

converges to $\frac{\pi}{3\sqrt{3}}$

Adding the consecutive terms of the above series we get the series, for which *Mathematica* knows the sum

In[449]:= $$\sum_{k=0}^{\infty}\frac{1}{(3k+1)(3k+2)}$$

Out[449]= $$\frac{\pi}{3\sqrt{3}}$$

Example 4: Abs[x] = x Sign[x] on $-\pi < x \leq \pi$

```
In[450]:= Clear[f4];

f4[x_] = Abs[x];
```

on the interval $(-\pi,\pi]$

The coefficient a_0

In[452]:= $$\text{FullSimplify}\left[\frac{1}{2\,\text{Pi}}\text{Integrate}[f4[x], \{x, -\text{Pi}, \text{Pi}\}]\right]$$

Out[452]= $$\frac{\pi}{2}$$

The coefficients a_n

In[453]:=
```
FullSimplify[1/Pi Integrate[f4[x] Cos[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[453]= $\frac{2\,(-1+(-1)^n)}{n^2\,\pi}$

This formula simplifies; for even n to 0 and for odd n to $\frac{-4}{\pi\,n^2}$.

The coefficients b_n

In[454]:=
```
FullSimplify[1/Pi Integrate[f4[x] Sin[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[454]= 0

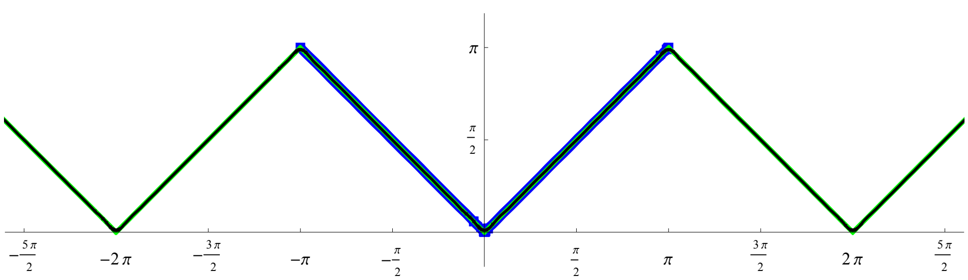
Thus the Fourier series of the given function is

$$\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2\,k-1)^2}\,\text{Cos}[(2\,k-1)\,x]$$

This series converges **uniformly** to the Fourier 2π-periodic extension of the function Abs[x], restricted to $(-\pi,\pi]$, as illustrated in the following graph and manipulation

In[455]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, nn = 10;
 pic1 = Plot[{f4[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}];

 pic2 = Plot[{fft[f4[#] &, x, Pi]}, {x, -15, 10},
   PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

 pic3 = Plot[Evaluate[{π/2 - 4/Pi Sum[1/(2 k - 1)^2 Cos[(2 k - 1) x], {k, 1, nn}]}], {x, -12, 14},
   PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200]; Show[pic1, pic2,
  pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {0 - .5, Pi + .5}}, AspectRatio → Automatic,
  Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-Pi, Pi, Pi/2]}, ImageSize → is]]
```

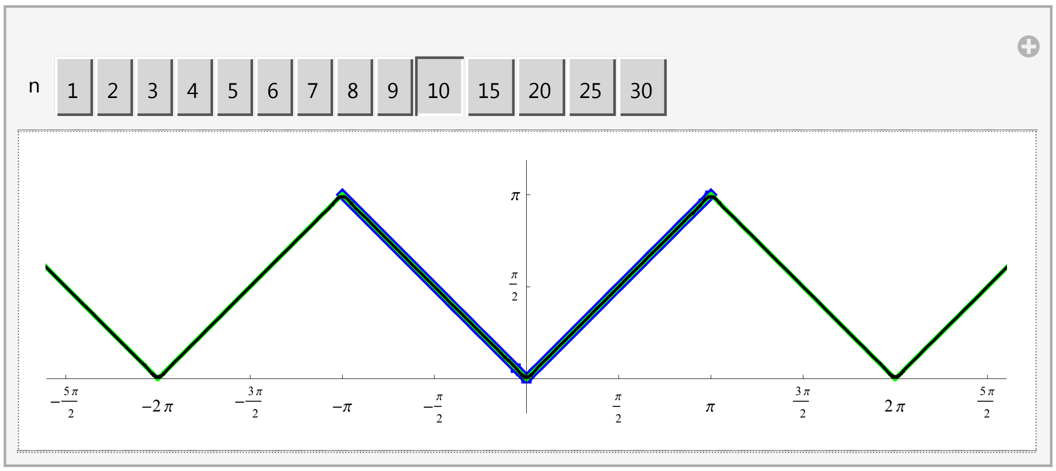
Out[455]=



Or, the same picture with Manipulate

In[456]:=
```
Module[{pic1, pic2, pic2a, pic3, nn},
 Manipulate[pic1 = Plot[{f4[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}];

  pic2 = Plot[{fft[f4[#] &, x, Pi]}, {x, -15, 10},
    PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

  pic3 = Plot[Evaluate[{π/2 - 4/Pi Sum[1/(2 k - 1)^2 Cos[(2 k - 1) x], {k, 1, nn}]}], {x, -12, 14},
    PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200]; Show[pic1, pic2,
   pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {0 - .5, Pi + .5}}, AspectRatio → Automatic,
   Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-Pi, Pi, Pi/2]}, ImageSize → is],
  {{nn, 10, "n"}, Join[Range[10], Range[15, 30, 5]]}, ControlType → Setter]]
```

Out[456]=



- **Numerical series at special values of *x***
- ***x* = 0**

Notice that the convergence theorem implies that for a specific $x = 0$ the following numerical series

$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \text{Cos}[(2k-1)\,0]$$

which equals

$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

converges to 0.

Thus

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

- $x = \pi/4$

In[457]:= `Table[Cos[(2 k - 1) * π/4], {k, 1, 20}]`

Out[457]= $\left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$

In[458]:= `Table[(-1)^Floor[k/2], {k, 1, 20}]`

Out[458]= {1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1}

Notice that the convergence theorem implies that for a specific $x = \frac{\pi}{4}$ the following numerical series

$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \text{Cos}\left[(2k-1) * \frac{\pi}{4}\right]$$

which equals

$$\frac{\pi}{2} - \frac{2\sqrt{2}}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{\text{Floor}[k/2]}}{(2k-1)^2}$$

converges to $\pi/4$.

Thus

$$\sum_{k=1}^{\infty} \frac{(-1)^{\text{Floor}[k/2]}}{(2k-1)^2} = \frac{\pi^2}{8\sqrt{2}}$$

In[459]:= $\sum_{k=1}^{\infty} \frac{(-1)^{\text{Floor}[k/2]}}{(2k-1)^2}$

Out[459]= $\sum_{k=1}^{\infty} \frac{(-1)^{\text{Floor}\left[\frac{k}{2}\right]}}{(-1+2k)^2}$

In[460]:= $\text{N}\left[\frac{\pi^2}{8\sqrt{2}}\right]$

Out[460]= 0.872358

In[461]:= $\text{N}\left[\sum_{k=1}^{10000} \frac{(-1)^{\text{Floor}[k/2]}}{(2k-1)^2}\right]$

Out[461]= 0.872358

Grouping four terms in the above series gives

In[462]:= $\text{Together}\left[\frac{1}{(2(4k+1)-1)^2} - \frac{1}{(2(4k+2)-1)^2} - \frac{1}{(2(4k+3)-1)^2} + \frac{1}{(2(4k+4)-1)^2}\right]$

Out[462]= $\frac{16\left(599 + 5504k + 17792k^2 + 24576k^3 + 12288k^4\right)}{(1+8k)^2(3+8k)^2(5+8k)^2(7+8k)^2}$

In[463]:=
```
FullSimplify[
 Sum[(16 (599 + 5504 k + 17792 k^2 + 24576 k^3 + 12288 k^4)) / ((1 + 8 k)^2 (3 + 8 k)^2 (5 + 8 k)^2 (7 + 8 k)^2), {k, 0, ∞}]]
```

$$\sum_{k=0}^{\infty}\left(16\left(599+5504\,k+17792\,k^2+24576\,k^3+12288\,k^4\right)\right)\Big/\left((1+8\,k)^2\,(3+8\,k)^2\,(5+8\,k)^2\,(7+8\,k)^2\right)$$

Out[463]= $\frac{\pi^2}{8\sqrt{2}}$

Mathematica knows these things.

Example 5: *x* UnitStep[*x*] on $-\pi < x \leq \pi$

In[464]:=
```
Clear[f5];

f5[x_] = x UnitStep[x];
```

on the interval $(-\pi,\pi]$

The coefficient a_0

In[466]:=
```
FullSimplify[1/(2 Pi) Integrate[f5[x], {x, -Pi, Pi}]]
```

Out[466]= $\frac{\pi}{4}$

The coefficients a_n

In[467]:=
```
FullSimplify[1/Pi Integrate[f5[x] Cos[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[467]= $\frac{-1+(-1)^n}{n^2\,\pi}$

This formula simplifies; for even n to 0 and for odd n to $\frac{-2}{\pi\, n^2}$.

The coefficients b_n

In[468]:=
```
FullSimplify[1/Pi Integrate[f5[x] Sin[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[468]= $-\frac{(-1)^n}{n}$

Thus the Fourier series of the given function is

$$\frac{\pi}{4}-\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2\,k-1)^2}\,\mathrm{Cos}[(2\,k-1)\,x]+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\,\mathrm{Sin}[n\,x]$$

This series converges **pointwise** to the Fourier 2π-periodic extension of the function x UnitStep[x], restricted to $(-\pi,\pi]$, as illustrated in the following graph and manipulation

In[469]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, nn = 10;
 pic1 = Plot[{f5[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}];

 pic2 = Plot[{fft[f5[#] &, x, Pi]}, {x, -15, 10},
   PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

 pic2a = Graphics[{
    {PointSize[0.02], Green,
     {Point[{# Pi, 0}], Point[{# Pi, Pi / 2}], Point[{# Pi, Pi}]} & /@ Range[-11, 13, 2]},
    {PointSize[0.014], White, {Point[{# Pi, 0}], Point[{# Pi, Pi}]} & /@ Range[-11, 13, 2]}
   }];
```

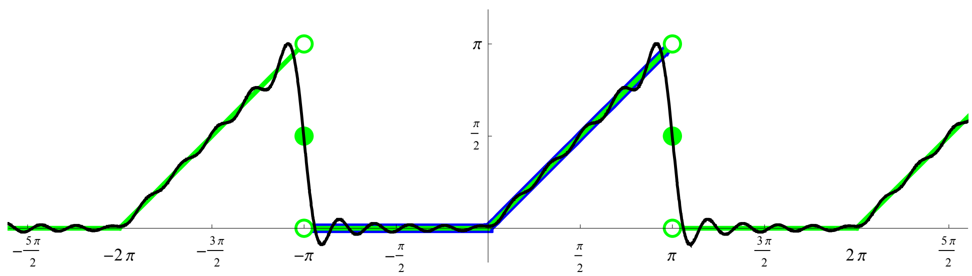
$$\text{pic3 = Plot}\left[\text{Evaluate}\left[\left\{\frac{\pi}{4} - \frac{2}{\text{Pi}} \sum_{k=1}^{\text{Floor[nn/2]}} \frac{1}{(2\,k-1)^2} \text{Cos}[(2\,k-1)\,x] + \sum_{n=1}^{nn} \frac{(-1)^{n+1}}{n} \text{Sin}[n\,x]\right\}\right],\right.$$

$$\{x, -12, 14\}, \text{PlotStyle} \to \{\{\text{Thickness}[0.003], \text{Black}\}\}, \text{PlotPoints} \to 200\Big]; \text{Show}\Big[\text{pic1, pic2,}$$

$$\text{pic2a, pic3, PlotRange} \to \{\{-2.5\,\text{Pi}, 2.5\,\text{Pi}\}, \{0 - .5, \text{Pi} + .5\}\}, \text{AspectRatio} \to \text{Automatic},$$

$$\text{Ticks} \to \left\{\text{Range}\left[-10\,\text{Pi}, 10\,\text{Pi}, \frac{\text{Pi}}{2}\right], \text{Range}\left[-\text{Pi}, \text{Pi}, \frac{\text{Pi}}{2}\right]\right\}, \text{ImageSize} \to \text{is}\Big]\Big]$$

Out[469]=



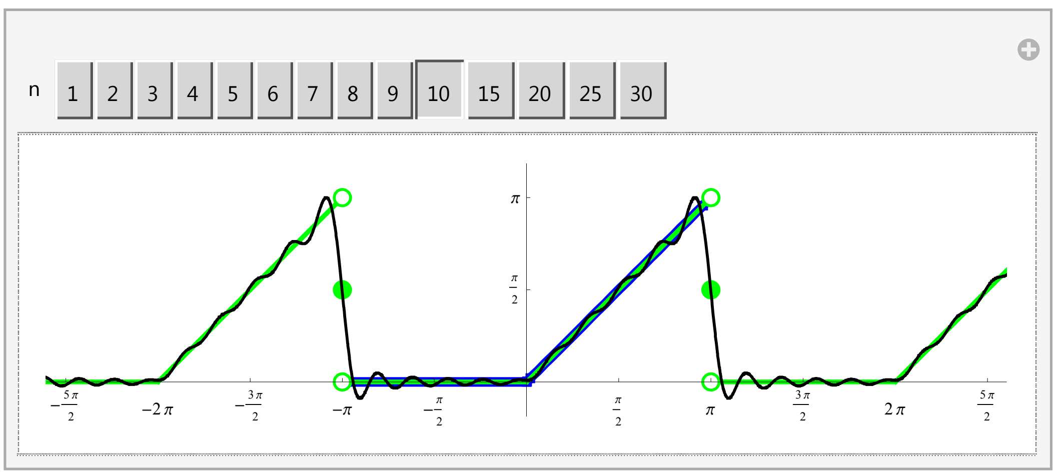
Or, the same picture with Manipulate

In[470]:=
```
Module[{pic1, pic2, pic2a, pic3, nn},
 Manipulate[pic1 = Plot[{f5[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}];

  pic2 = Plot[{fft[f5[#] &, x, Pi]}, {x, -15, 10},
    PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

  pic2a = Graphics[{
     {PointSize[0.02], Green,
      {Point[{# Pi, 0}], Point[{# Pi, Pi / 2}], Point[{# Pi, Pi}]} & /@ Range[-11, 13, 2]},
     {PointSize[0.014], White, {Point[{# Pi, 0}], Point[{# Pi, Pi}]} & /@ Range[-11, 13, 2]}
    }];
```

$$\text{pic3} = \text{Plot}\left[\text{Evaluate}\left[\left\{\frac{\pi}{4} - \frac{2}{\text{Pi}} \sum_{k=1}^{\text{Floor[nn/2]}} \frac{1}{(2\,k-1)^2} \text{Cos}[(2\,k-1)\,x] + \sum_{n=1}^{nn} \frac{(-1)^{n+1}}{n} \text{Sin}[n\,x]\right\}\right],\right.$$

```
   {x, -12, 14}, PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200];
  Show[pic1, pic2, pic2a, pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {0 - .5, Pi + .5}},
   AspectRatio → Automatic,
   Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-Pi, Pi, Pi/2]}, ImageSize → is],
  {{nn, 10, "n"}, Join[Range[10], Range[15, 30, 5]]}, ControlType → Setter]]
```

Out[470]=



- **Numerical series at special values of *x***

  I think that nothing new can be obtained here.

Example 6: x^2 on $-\pi < x \leq \pi$

In[471]:=
```
Clear[f6];

f6[x_] = x^2;
```

on the interval $(-\pi,\pi]$

The coefficient a_0

In[473]:=
```
FullSimplify[1/(2 Pi) Integrate[f6[x], {x, -Pi, Pi}]]
```

Out[473]= $\frac{\pi^2}{3}$

The coefficients a_n

In[474]:=
```
FullSimplify[1/Pi Integrate[f6[x] Cos[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[474]= $\frac{4\,(-1)^n}{n^2}$

The coefficients b_n

In[475]:=
```
FullSimplify[1/Pi Integrate[f6[x] Sin[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[475]= 0

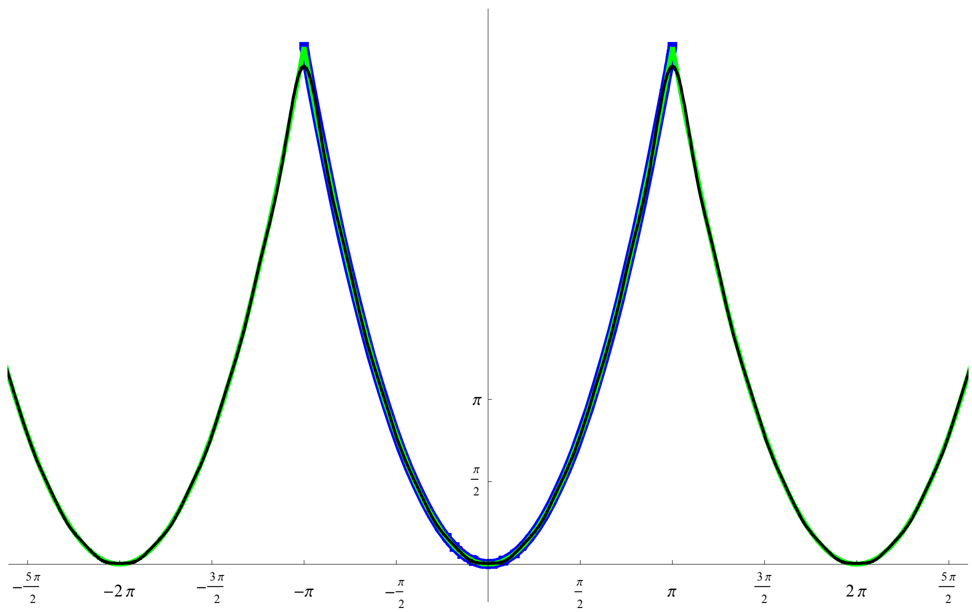
Thus the Fourier series of the given function is

$$\frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \operatorname{Cos}[n\,x]$$

This series converges **uniformly** to the Fourier 2π-periodic extension of the function x^2, restricted to $(-\pi,\pi]$, as illustrated in the following graph and manipulation

In[476]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, nn = 10;
 pic1 = Plot[{f6[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}];

 pic2 = Plot[{fft[f6[#] &, x, Pi]}, {x, -15, 10},
   PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

 pic3 = Plot[Evaluate[{π^2/3 + 4 Sum[(-1)^n/n^2 Cos[n x], {n, 1, nn}]}], {x, -12, 14},
   PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200]; Show[pic1, pic2, pic3,
  PlotRange → {{-2.5 Pi, 2.5 Pi}, {0 - .5, Pi^2 + .5}}, AspectRatio → 1 / GoldenRatio,
  Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-Pi, Pi, Pi/2]}, ImageSize → is]]
```

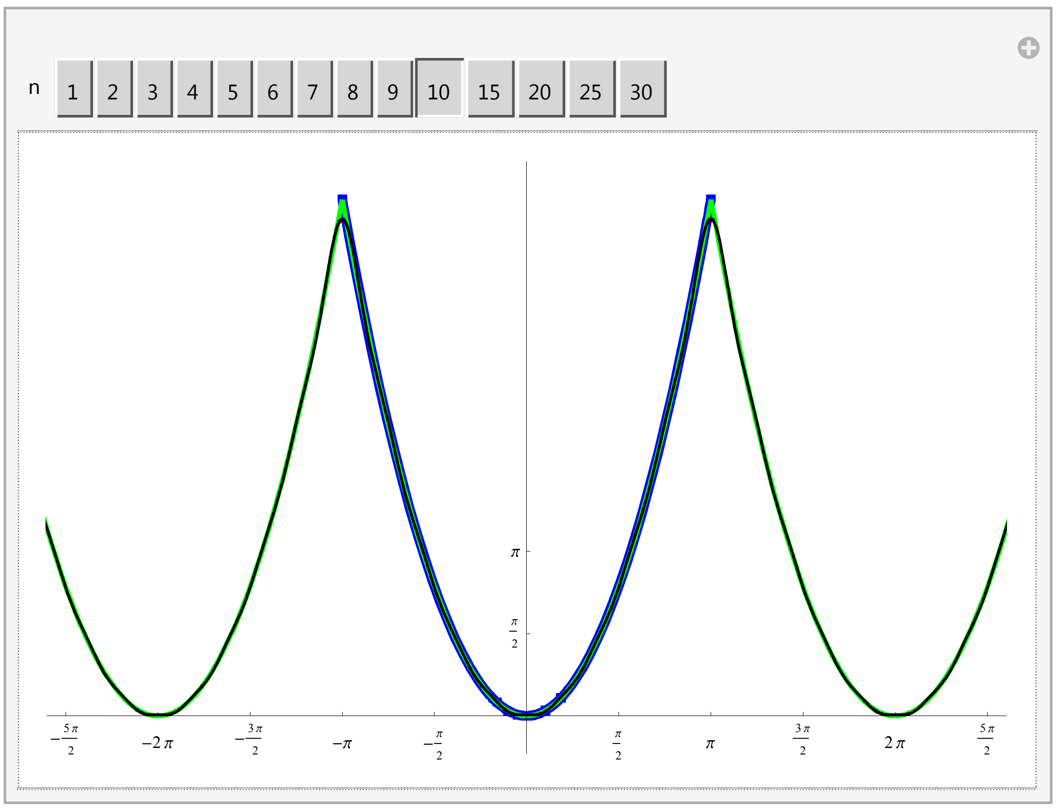
Out[476]=



Or, the same picture with Manipulate

In[477]:=
```
Module[{pic1, pic2, pic2a, pic3, nn},
 Manipulate[pic1 = Plot[{f6[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}];

  pic2 = Plot[{fft[f6[#] &, x, Pi]}, {x, -15, 10},
    PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

  pic3 = Plot[Evaluate[{π^2/3 + 4 Sum[(-1)^n/n^2 Cos[n x], {n, 1, nn}]}], {x, -12, 14},
    PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200]; Show[pic1, pic2, pic3,
   PlotRange → {{-2.5 Pi, 2.5 Pi}, {0 - .5, Pi^2 + .5}}, AspectRatio → 1 / GoldenRatio,
   Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-Pi, Pi, Pi/2]}, ImageSize → is],
  {{nn, 10, "n"}, Join[Range[10], Range[15, 30, 5]]}, ControlType → Setter]]
```

Out[477]=



- **Numerical series at special values of *x***
- ***x* = π**

Notice that the convergence theorem implies that for a specific $x = \pi$ the following numerical series

$$\frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \text{Cos}[n\,\pi]$$

which equals

$$\frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges to π^2.

Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

This is the famous formula for which direct proofs are not simple.

Example 7: x Abs[x] = x^2 Sign[x] on $-\pi < x \leq \pi$

In[478]:=
```
Clear[f7];

f7[x_] = x Abs[x];
```

on the interval $(-\pi,\pi]$

The coefficient a_0

In[480]:=
```
FullSimplify[1/(2 Pi) Integrate[f7[x], {x, -Pi, Pi}]]
```

Out[480]= 0

The coefficients a_n

In[481]:=
```
FullSimplify[1/Pi Integrate[f7[x] Cos[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[481]= 0

The coefficients b_n

In[482]:=
```
FullSimplify[1/Pi Integrate[f7[x] Sin[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[482]= $$\frac{-4 - 2\,(-1)^n\,\left(-2 + n^2\,\pi^2\right)}{n^3\,\pi}$$

In[483]:=
```
Expand[(-4 - 2 (-1)^n (-2 + n^2 π^2))/(n^3 π)]
```

Out[483]= $$-\frac{4}{n^3\,\pi} + \frac{4\,(-1)^n}{n^3\,\pi} - \frac{2\,(-1)^n\,\pi}{n}$$

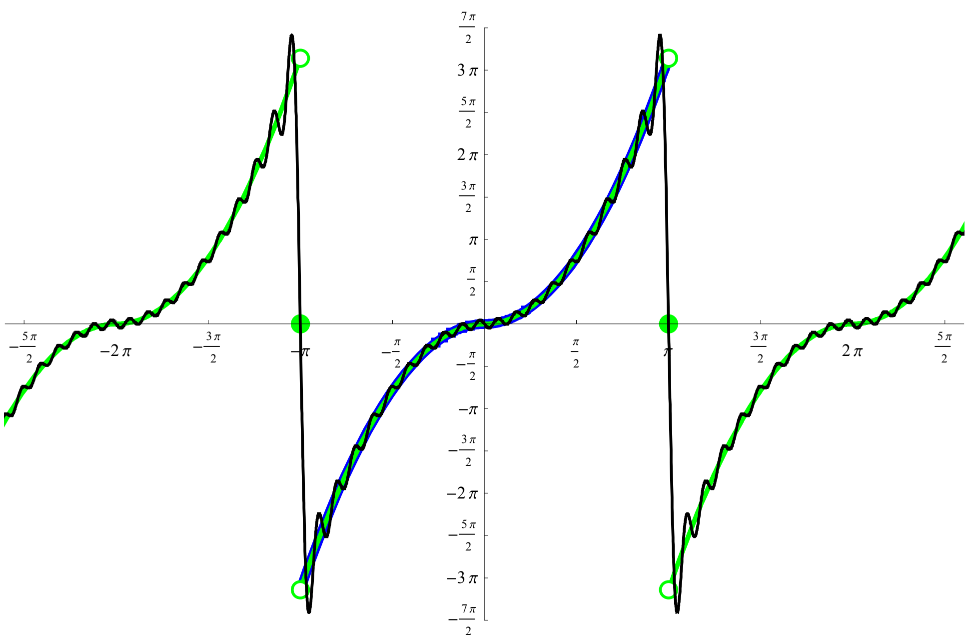
Thus the Fourier series of the given function is

$$\sum_{n=1}^{\infty} \left(-\frac{4}{n^3\,\pi} + \frac{4\,(-1)^n}{n^3\,\pi} - \frac{2\,(-1)^n\,\pi}{n}\right) \mathrm{Sin}[n\,x]$$

This series converges **pointwise** to the Fourier 2π-periodic extension of the function x Abs[x], restricted to $(-\pi,\pi]$, as illustrated in the following graph and manipulation

In[484]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, nn = 20;
 pic1 = Plot[{f7[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}];

 pic2 = Plot[{fft[f7[#] &, x, Pi]}, {x, -15, 10},
   PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];
 pic2a = Graphics[{
    {PointSize[0.02], Green, {Point[{# Pi, Pi^2}], Point[{# Pi, 0}], Point[{# Pi, -Pi^2}]} & /@
      Range[-11, 13, 2]}, {PointSize[0.014], White,
     {Point[{# Pi, Pi^2}], Point[{# Pi, -Pi^2}]} & /@ Range[-11, 13, 2]}
   }];
 pic3 = Plot[Evaluate[{Sum[(-4/(n^3 π) + 4 (-1)^n/(n^3 π) - 2 (-1)^n π/n) Sin[n x], {n, 1, nn}]}], {x, -12, 14},
   PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200]; Show[pic1, pic2, pic2a, pic3,
  PlotRange → {{-2.5 Pi, 2.5 Pi}, {-Pi^2 - .7, Pi^2 + .7}}, AspectRatio → 1 / GoldenRatio,
  Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-4 Pi, 4 Pi, Pi/2]}, ImageSize → is]]
```

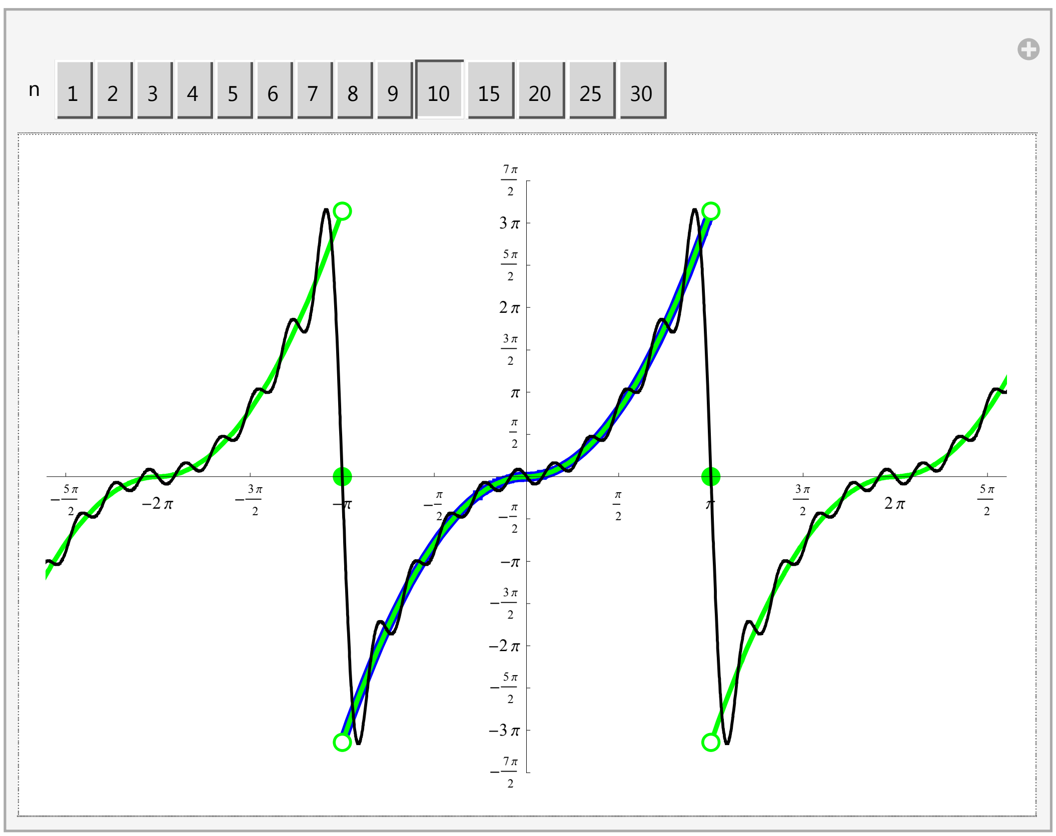
Out[484]=



Or, the same picture with Manipulate

In[485]:=
```
Module[{pic1, pic2, pic2a, pic3, nn},
 Manipulate[pic1 = Plot[{f7[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}];

  pic2 = Plot[{fft[f7[#] &, x, Pi]}, {x, -15, 10},
    PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];
  pic2a = Graphics[{
     {PointSize[0.02], Green, {Point[{# Pi, Pi^2}], Point[{# Pi, 0}], Point[{# Pi, -Pi^2}]} & /@
       Range[-11, 13, 2]}, {PointSize[0.014], White,
      {Point[{# Pi, Pi^2}], Point[{# Pi, -Pi^2}]} & /@ Range[-11, 13, 2]}
    }];
  pic3 = Plot[Evaluate[{Sum[(-4/(n^3 π) + 4 (-1)^n/(n^3 π) - 2 (-1)^n π/n) Sin[n x], {n, 1, nn}]}], {x, -12, 14},
    PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200]; Show[pic1, pic2, pic2a,
   pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {-Pi^2 - .7, Pi^2 + .7}}, AspectRatio → 1 / GoldenRatio,
   Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-4 Pi, 4 Pi, Pi/2]}, ImageSize → is],
  {{nn, 10, "n"}, Join[Range[10], Range[15, 30, 5]]}, ControlType → Setter]]
```

Out[485]=



■ Numerical series at special values of *x*

Example 8: x^2 UnitStep[x] on $-\pi < x \leq \pi$

In[486]:=
```
Clear[f8];

f8[x_] = x^2 UnitStep[x];
```

on the interval $(-\pi,\pi]$

The coefficient a_0

In[488]:=
```
FullSimplify[1/(2 Pi) Integrate[x^2, {x, 0, Pi}]]
```

Out[488]= $\frac{\pi^2}{6}$

The coefficients a_n

In[489]:=
```
FullSimplify[1/Pi Integrate[x^2 Cos[n x], {x, 0, Pi}], And[n ∈ Integers, n > 0]]
```

Out[489]= $\frac{2\,(-1)^n}{n^2}$

The coefficients b_n

In[490]:=
```
FullSimplify[1/Pi Integrate[x^2 Sin[n x], {x, 0, Pi}], And[n ∈ Integers, n > 0]]
```

Out[490]= $\frac{-2 + (-1)^n\left(2 - n^2 \pi^2\right)}{n^3 \pi}$

In[491]:=
```
Expand[(-2 + (-1)^n (2 - n^2 π^2))/(n^3 π)]
```

Out[491]= $-\frac{2}{n^3 \pi} + \frac{2\,(-1)^n}{n^3 \pi} - \frac{(-1)^n \pi}{n}$

Thus the Fourier series of the given function is

$$\frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left(\frac{2\,(-1)^n}{n^2}\,\mathrm{Cos}[n\,x] + \left(-\frac{2}{n^3\pi} + \frac{2\,(-1)^n}{n^3\pi} - \frac{(-1)^n\pi}{n}\right)\mathrm{Sin}[n\,x]\right)$$

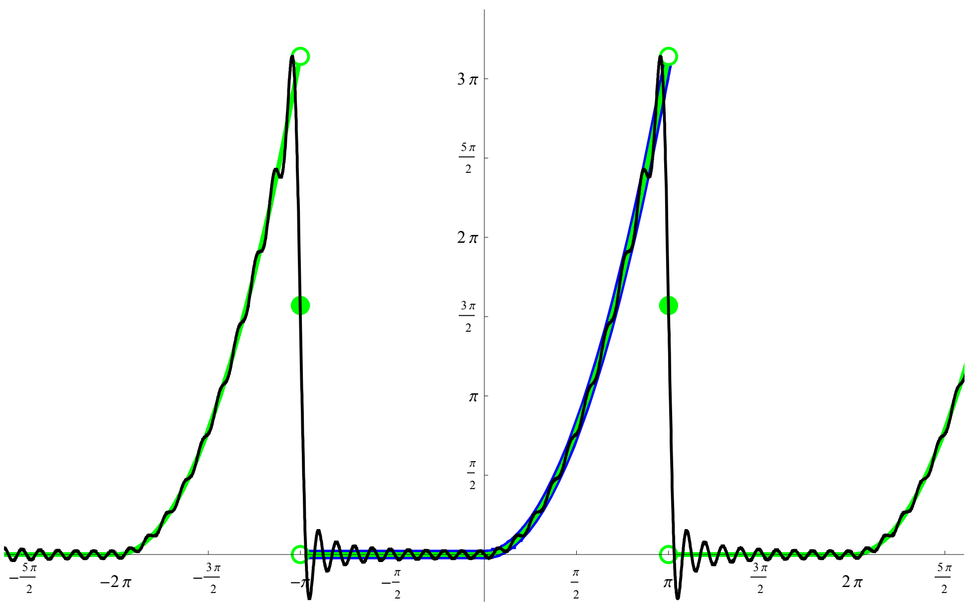
This series converges **pointwise** to the Fourier 2π-periodic extension of the function x^2 UnitStep[x], restricted to $(-\pi,\pi]$, as illustrated in the following graph and manipulation

In[492]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, nn = 20;
 pic1 = Plot[{f8[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}];

 pic2 = Plot[Evaluate[{fft[f8[#] &, x, Pi]}], {x, -20, 20},
   PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-11 Pi, 14 Pi, 2 Pi],
   PlotRange → {{-2.5 Pi, 2.5 Pi}, {0 - .7, Pi^2 + .7}}];

 pic2a = Graphics[{
    {PointSize[0.02], Green,
     {Point[{# Pi, Pi^2}], Point[{# Pi, 0}], Point[{# Pi, Pi^2 / 2}]} & /@ Range[-11, 13, 2]},
    {PointSize[0.014], White, {Point[{# Pi, Pi^2}], Point[{# Pi, 0}]} & /@ Range[-11, 13, 2]}
   }];
```

$$\text{pic3 = Plot}\left[\text{Evaluate}\left[\left\{\frac{\pi^2}{6}+\sum_{n=1}^{nn}\left(\frac{2(-1)^n}{n^2}\text{Cos}[n\,x]+\left(-\frac{2}{n^3\pi}+\frac{2(-1)^n}{n^3\pi}-\frac{(-1)^n\pi}{n}\right)\text{Sin}[n\,x]\right)\right\}\right],\right.$$

```
   {x, -12, 14}, PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200];
 Show[pic1, pic2, pic2a, pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {0 - .7, Pi^2 + .7}},
  AspectRatio → 1 / GoldenRatio,
  Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-4 Pi, 4 Pi, Pi/2]}, ImageSize → is]]
```

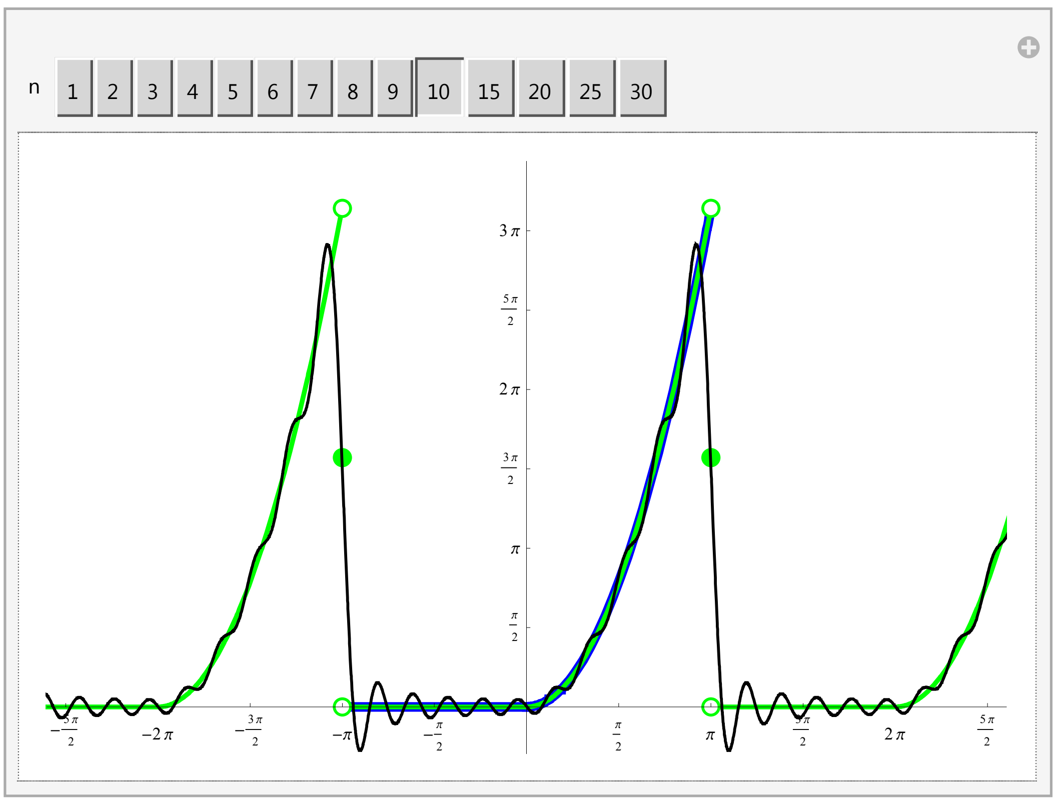
Out[492]=



Or, the same picture with Manipulate

In[493]:=
```
Module[{pic1, pic2, pic2a, pic3, nn},
 Manipulate[pic1 = Plot[{f8[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}];

  pic2 = Plot[Evaluate[{fft[f8[#] &, x, Pi]}], {x, -20, 20},
    PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-11 Pi, 14 Pi, 2 Pi],
    PlotRange → {{-2.5 Pi, 2.5 Pi}, {0 - .7, Pi^2 + .7}}];

  pic2a = Graphics[{
     {PointSize[0.02], Green,
      {Point[{# Pi, Pi^2}], Point[{# Pi, 0}], Point[{# Pi, Pi^2 / 2}]} & /@ Range[-11, 13, 2]},
     {PointSize[0.014], White, {Point[{# Pi, Pi^2}], Point[{# Pi, 0}]} & /@ Range[-11, 13, 2]}
    }];
  pic3 = Plot[Evaluate[{π^2/6 + Sum[(2 (-1)^n/n^2 Cos[n x] + (-2/(n^3 π) + 2 (-1)^n/(n^3 π) - (-1)^n π/n) Sin[n x]), {n, 1, nn}]}],
    {x, -12, 14}, PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200];
  Show[pic1, pic2, pic2a, pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {0 - .7, Pi^2 + .7}},
   AspectRatio → 1 / GoldenRatio,
   Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-4 Pi, 4 Pi, Pi/2]}, ImageSize → is],
  {{nn, 10, "n"}, Join[Range[10], Range[15, 30, 5]]}, ControlType → Setter]]
```

Out[493]=



- **Numerical series at special values of *x***

Example 9: $x(\pi - \text{Abs}[x])$ on $-\pi < x \leq \pi$

In[494]:=
```
Clear[f9];

f9[x_] = x (Pi - Abs[x]);
```

on the interval $(-\pi,\pi]$

The coefficient a_0

In[496]:=
```
FullSimplify[1/(2 Pi) Integrate[f9[x], {x, -Pi, Pi}]]
```

Out[496]= 0

The coefficients a_n

In[497]:=
```
FullSimplify[1/Pi Integrate[f9[x] Cos[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[497]= 0

The coefficients b_n

In[498]:=
```
FullSimplify[1/Pi Integrate[f9[x] Sin[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[498]= $-\frac{4\,(-1+(-1)^n)}{n^3\,\pi}$

This formula simplifies to $\frac{8}{n^3\,\pi}$ for n odd and 0 for n even.

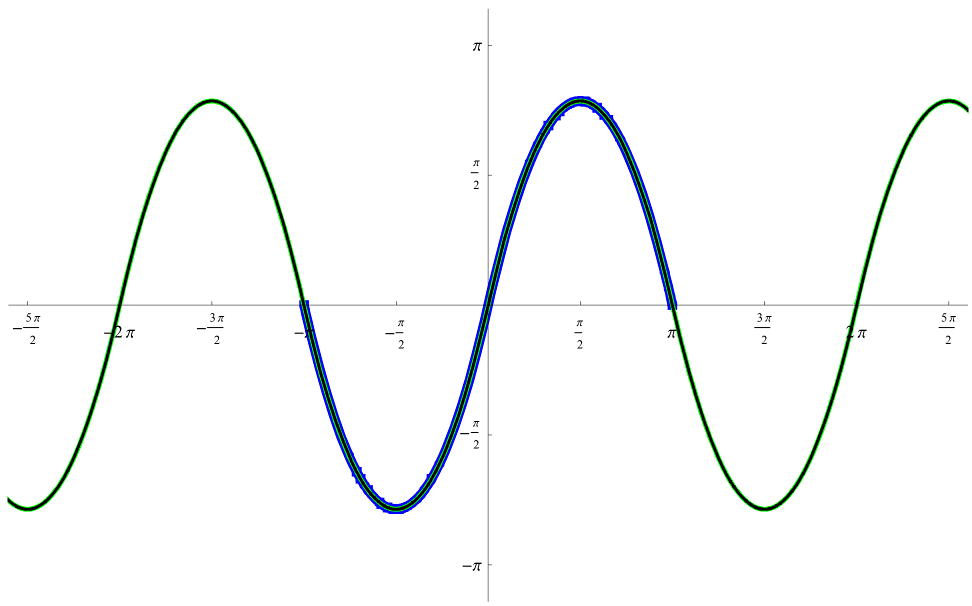
Thus the Fourier series of the given function is

$$\frac{8}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2\,k-1)^3}\,\text{Sin}[(2\,k-1)\,x]$$

This series converges **uniformly** to the Fourier 2π-periodic extension of the function $x(\pi - \text{Abs}[x])$, restricted to $(-\pi,\pi]$, as illustrated in the following graph and manipulation

In[499]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, nn = 10;
 pic1 = Plot[{f9[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}];

 pic2 = Plot[{fft[f9[#] &, x, Pi]}, {x, -15, 10},
   PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

 (* pic2a=Graphics[{
     {PointSize[0.02],Green,
      {Point[{# Pi,Pi^2}],Point[{# Pi,0}],Point[{#Pi,-Pi^2}]}&/@Range[-11,13,2]},
     {PointSize[0.014],White,{Point[{# Pi,Pi^2}],Point[{#Pi,-Pi^2}]}&/@Range[-11,13,2]}
    }]; *)
 pic3 = Plot[Evaluate[{8/π Sum[1/(2 k - 1)^3 Sin[(2 k - 1) x], {k, 1, Floor[nn/2]}]}], {x, -12, 14},
   PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200]; Show[pic1, pic2, pic3,
  PlotRange → {{-2.5 Pi, 2.5 Pi}, {-Pi - .3, Pi + .3}}, AspectRatio → 1 / GoldenRatio,
  Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-4 Pi, 4 Pi, Pi/2]}, ImageSize → is]]
```

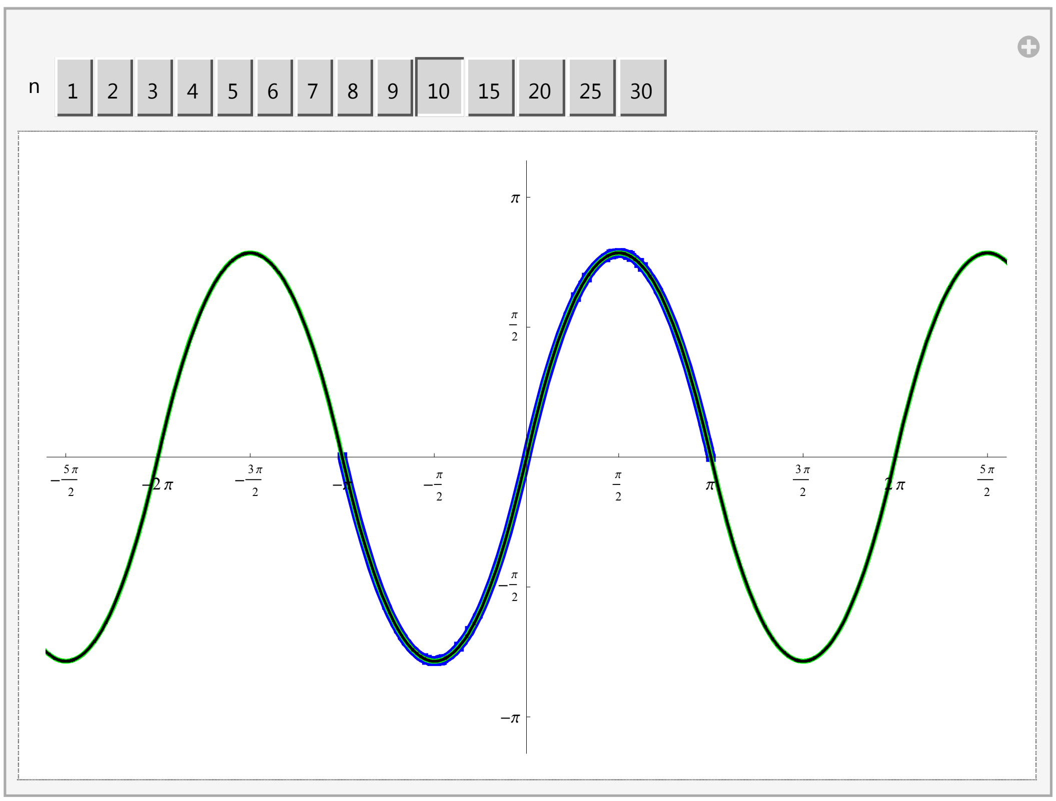
Out[499]=



Or, the same picture with Manipulate

In[500]:=
```
Module[{pic1, pic2, pic2a, pic3, nn},
 Manipulate[pic1 = Plot[{f9[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}];

  pic2 = Plot[{fft[f9[#] &, x, Pi]}, {x, -15, 10},
    PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

  (* pic2a=Graphics[{
      {PointSize[0.02],Green,
       {Point[{# Pi,Pi^2}],Point[{# Pi,0}],Point[{#Pi,-Pi^2}]}&/@Range[-11,13,2]},
      {PointSize[0.014],White,{Point[{# Pi,Pi^2}],Point[{#Pi,-Pi^2}]}&/@Range[-11,13,2]}
     }]; *)
  pic3 = Plot[Evaluate[{8/π Sum[1/(2 k - 1)^3 Sin[(2 k - 1) x], {k, 1, Ceiling[nn/2]}]}], {x, -12, 14},
    PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200]; Show[pic1, pic2, pic3,
   PlotRange → {{-2.5 Pi, 2.5 Pi}, {-Pi - .3, Pi + .3}}, AspectRatio → 1 / GoldenRatio,
   Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-4 Pi, 4 Pi, Pi/2]}, ImageSize → is],
  {{nn, 10, "n"}, Join[Range[10], Range[15, 30, 5]]}, ControlType → Setter]]
```

Out[500]=



- **Numerical series at special values of *x***
- $x = \pi/2$

Notice that the convergence theorem implies that for a specific $x = \pi/2$ the following numerical series

$$\frac{8}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2\,k\,-1)^3}\ \mathrm{Sin}\left[(2\,k-1)\ *\frac{\pi}{2}\right]$$

which equals

$$\frac{8}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2\,k\,-1)^3}$$

converges to $\frac{\pi^{\wedge}2}{4}$.

Thus

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2\,k\,-1)^3}=\frac{\pi^3}{32}$$

Mathematica knows this formula

In[501]:= $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2\,k\,-1)^3}$

Out[501]= $\frac{\pi^3}{32}$

Example 10: Cos[*x*] Sign[*x*] on $-\pi < x \leq \pi$

In[502]:=
```
Clear[f10];

f10[x_] = Cos[x] Sign[x];
```

on the interval (-π,π]

The coefficient a_0 is 0

In[504]:= `FullSimplify[` $\frac{1}{2\ \mathrm{Pi}}$ `Integrate[f10[x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]`

Out[504]= 0

The coefficients a_n are 0

In[505]:= `FullSimplify[` $\frac{1}{\mathrm{Pi}}$ `Integrate[f10[x] Cos[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]`

Out[505]= 0

The coefficients b_n

In[506]:= `FullSimplify[` $\frac{1}{\mathrm{Pi}}$ `Integrate[f10[x] Sin[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]`

Out[506]= $\frac{2\ (1+(-1)^n)\ n}{\left(-1+n^2\right)\ \pi}$

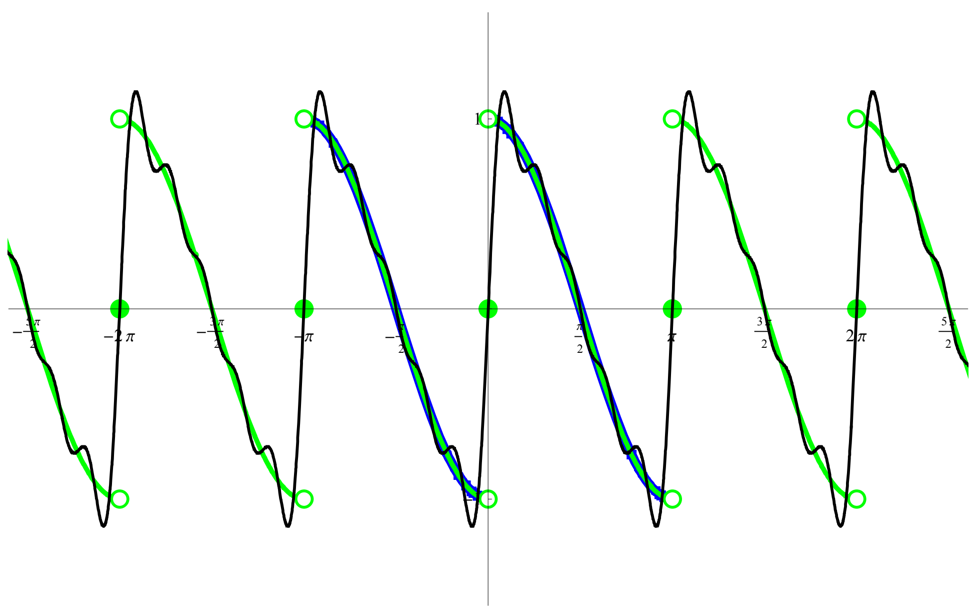
This formula simplifies; for even n to $\frac{4\,n}{\pi\left(n^2-1\right)}$ and for odd n to 0. Thus the Fourier series of the given function is

$$\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{2\,k}{4\,k^2-1}\ \mathrm{Sin}[(2\,k)\ x]$$

This series converges pointwise to the Fourier 2π-periodic extension of Cos[x], as illustrated in the following graph and manipulation

In[507]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, nn = 10; pic1 =
  Plot[{f10[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}, Exclusions → {0}];

 pic2 = Plot[{fft[f10[#] &, x, Pi]}, {x, -5, 10},
   PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

 pic2a = Graphics[{
    {PointSize[0.02], Green,
     {Point[{# Pi, -1}], Point[{# Pi, 1}], Point[{# Pi, 0}]} & /@ Range[-10, 13, 1]},
    {PointSize[0.014], White, {Point[{# Pi, -1}], Point[{# Pi, 1}]} & /@ Range[-10, 13, 1]}
   }];

 pic3 = Plot[Evaluate[{4/π Sum[(2 k)/(4 k^2 - 1) Sin[(2 k) x], {k, 1, Ceiling[nn/2]}]}],
   {x, -12, 14}, PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200];
 Show[pic1, pic2, pic2a, pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {-1.5, 1.5}},
  Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-2, 2, 1]}, ImageSize → is]]
```

Out[507]=



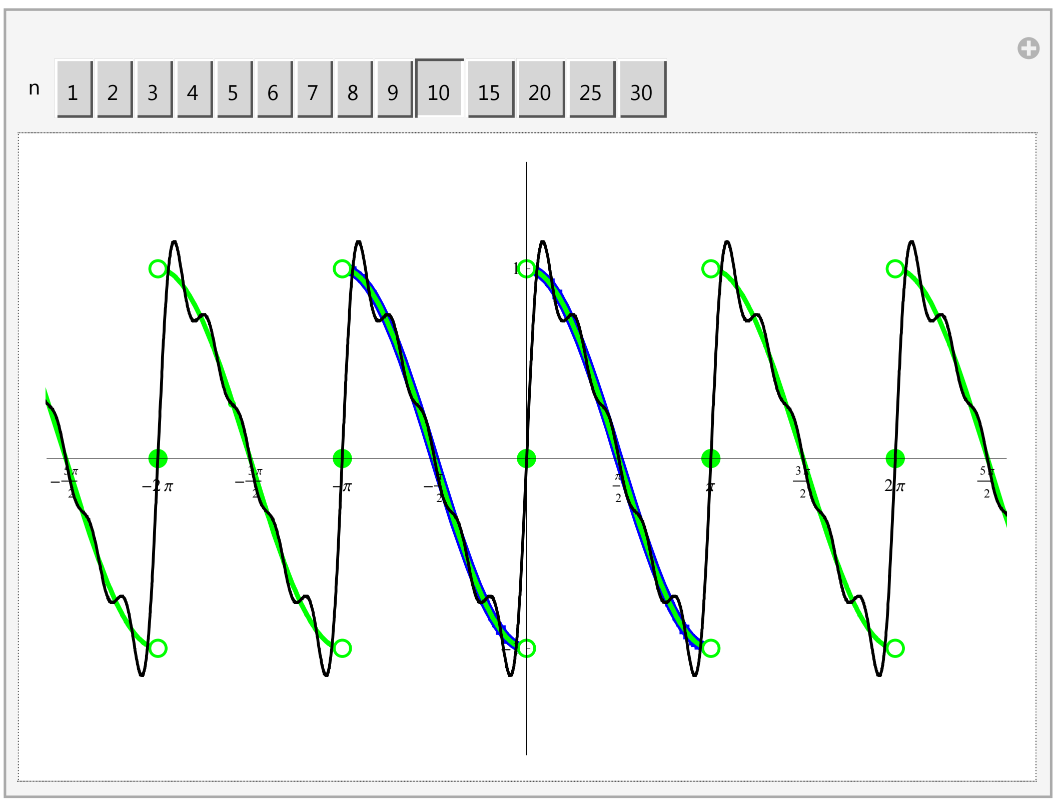
Or, the same picture with Manipulate

In[508]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, Manipulate[pic1 =
   Plot[{f10[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}, Exclusions → {0}];

  pic2 = Plot[{fft[f10[#] &, x, Pi]}, {x, -5, 10},
    PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

  pic2a = Graphics[{
     {PointSize[0.02], Green,
      {Point[{# Pi, -1}], Point[{# Pi, 1}], Point[{# Pi, 0}]} & /@ Range[-10, 13, 1]},
     {PointSize[0.014], White, {Point[{# Pi, -1}], Point[{# Pi, 1}]} & /@ Range[-10, 13, 1]}
    }];
```

$$\text{pic3 = Plot}\left[\text{Evaluate}\left[\left\{\frac{4}{\pi}\sum_{k=1}^{\text{Ceiling[nn/2]}}\frac{2\,k}{4\,k^2-1}\,\text{Sin[(2 k) x]}\right\}\right],\right.$$

```
    {x, -12, 14}, PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200];
  Show[pic1, pic2, pic2a, pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {-1.5, 1.5}},
   Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-2, 2, 1]}, ImageSize → is],
  {{nn, 10, "n"}, Join[Range[10], Range[15, 30, 5]]}, ControlType → Setter]]
```

Out[508]=



Example 11: Sin[x] Sign[x] on $-\pi < x \leq \pi$

In[509]:=
```
Clear[f11];

f11[x_] = Sign[x] Sin[x];
```

on the interval $(-\pi,\pi]$

The coefficient a_0

In[511]:=
```
FullSimplify[1/(2 Pi) Integrate[f11[x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[511]= $\frac{2}{\pi}$

The coefficients a_n

In[512]:=
```
FullSimplify[1/Pi Integrate[f11[x] Cos[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[512]= $-\frac{2\,(1+(-1)^n)}{(-1+n^2)\,\pi}$

This formula simplifies; for even n to $\frac{4}{\pi(n^2-1)}$ and for odd n to 0.

The coefficients b_n

In[513]:=
```
FullSimplify[1/Pi Integrate[f11[x] Sin[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[513]= 0

Thus the Fourier series of the given function is

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4\,k^2-1}\,\mathrm{Cos}[(2\,k)\,x]$$

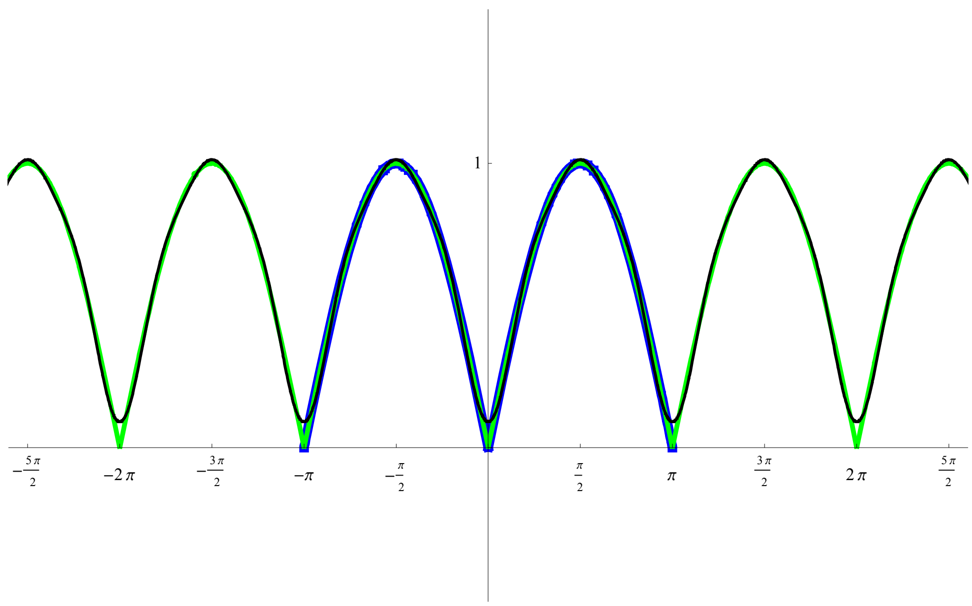
This series converges **uniformly** to the Fourier 2π-periodic even extension of Sin[x], as illustrated in the following graph and manipulation

In[514]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, nn = 5; pic1 =
  Plot[{f11[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}, Exclusions → {0}];

 pic2 = Plot[{fft[f11[#] &, x, Pi]}, {x, -5, 10},
   PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

 (* pic2a=Graphics[{
    {PointSize[0.02],Green,
     {Point[{# Pi,-1}],Point[{#Pi,1}],Point[{# Pi,0}]}&/@Range[-10,13,1]},
    {PointSize[0.014],White,{Point[{# Pi,-1}],Point[{#Pi,1}]}&/@Range[-10,13,1]}
   }];*)
```

$$\text{pic3 = Plot}\left[\text{Evaluate}\left[\left\{\frac{2}{\pi}-\frac{4}{\pi}\sum_{k=1}^{\text{Ceiling[nn/2]}}\frac{1}{4\,k^2-1}\,\text{Cos}[(2\,k)\,x]\right\}\right],\right.$$

```
  {x, -12, 14}, PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200];
 Show[pic1, pic2, pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {-0.5, 1.5}},
  Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-2, 2, 1]}, ImageSize → is]]
```

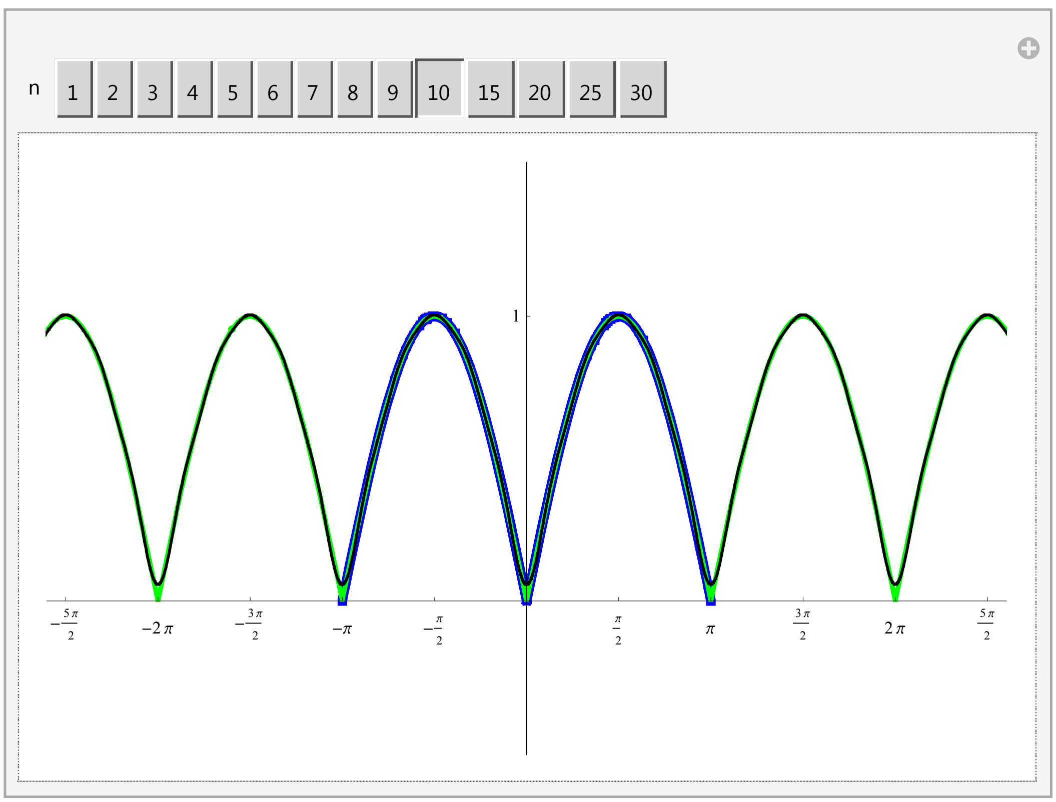
Out[514]=



Or, the same picture with Manipulate

In[515]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, Manipulate[pic1 =
   Plot[{f11[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}, Exclusions → {0}];

  pic2 = Plot[{fft[f11[#] &, x, Pi]}, {x, -5, 10},
    PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10 Pi, 14 Pi, Pi]];

  (* pic2a=Graphics[{
     {PointSize[0.02],Green,
      {Point[{# Pi,-1}],Point[{#Pi,1}],Point[{# Pi,0}]}&/@Range[-10,13,1]},
     {PointSize[0.014],White,{Point[{# Pi,-1}],Point[{#Pi,1}]}&/@Range[-10,13,1]}
    }];*)

  pic3 = Plot[Evaluate[{2/π - 4/π Sum[1/(4 k^2 - 1) Cos[(2 k) x], {k, 1, Ceiling[nn/2]}]}],
    {x, -12, 14}, PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200];
  Show[pic1, pic2, pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {-0.5, 1.5}},
   Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-2, 2, 1]}, ImageSize → is],
  {{nn, 10, "n"}, Join[Range[10], Range[15, 30, 5]]}, ControlType → Setter]]
```

Out[515]=



- **Numerical series at special values of *x***
- $x = \pi/2$

Notice that the convergence theorem implies that for a specific $x = \pi/2$ the following numerical series

$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{4\,k^2 - 1} \text{Cos}\left[(2\,k)\,\frac{\pi}{2}\right]$$

which equals

$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{4\,k^2 - 1}$$

converges to 1.

Thus

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{4\,k^2 - 1} = \frac{\pi}{4}\left(\frac{2}{\pi} - 1\right) = \frac{2 - \pi}{4}$$

Mathematica knows this formula

In[516]:=
$$\sum_{k=1}^{\infty} \frac{(-1)^k}{4\,k^2 - 1}$$

Out[516]=
$$\frac{2 - \pi}{4}$$

Example 12: Exp[x] on $-\pi < x \leq \pi$

In[517]:=
```
Clear[f12];

f12[x_] = Exp[x];
```

on the interval $(-\pi,\pi]$

The coefficient a_0

In[519]:=
```
FullSimplify[1/(2 Pi) Integrate[Exp[x], {x, -Pi, Pi}]]
```

Out[519]=
$$\frac{\text{Sinh}[\pi]}{\pi}$$

The coefficients a_n

In[520]:=
```
FullSimplify[1/Pi Integrate[Exp[x] Cos[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[520]=
$$\frac{2\,(-1)^n\,\text{Sinh}[\pi]}{\pi + n^2\,\pi}$$

The coefficients b_n

In[521]:=
```
FullSimplify[1/Pi Integrate[Exp[x] Sin[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[521]=
$$-\frac{2\,(-1)^n\,n\,\text{Sinh}[\pi]}{\pi + n^2\,\pi}$$

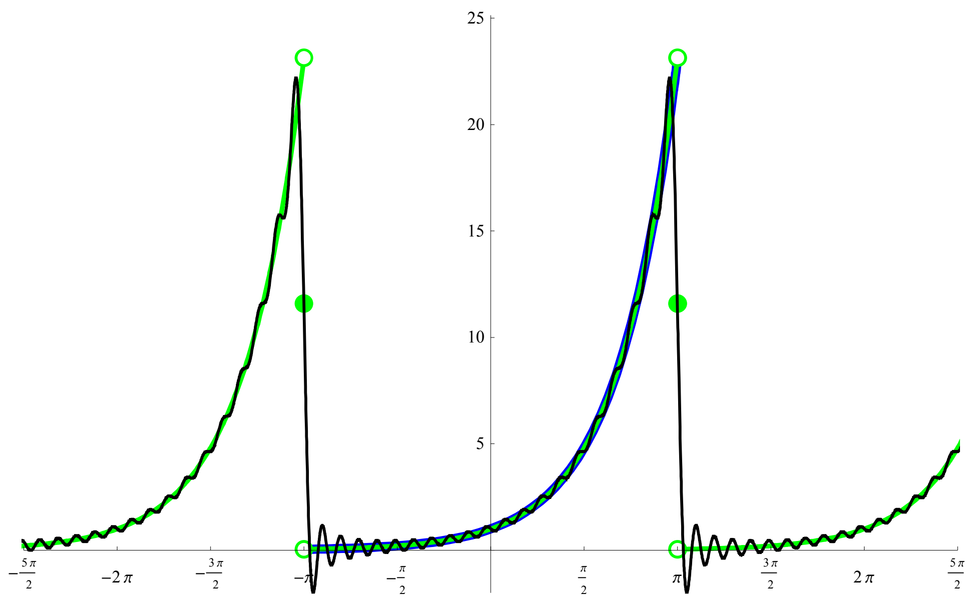
Thus the Fourier series of the given function is

$$\frac{\text{Sinh}[\pi]}{\pi} + \frac{2\,\text{Sinh}[\pi]}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{1 + n^2}\,(\text{Cos}[n\,x] - n\,\text{Sin}[n\,x])$$

This series converges **pointwise** to the Fourier 2π-periodic extension of the function Exp[x], restricted to (-π,π], as illustrated in the following graph and manipulation

In[522]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, nn = 20; pic1 =
  Plot[{f12[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}, PlotRange → All];

 pic2 = Plot[Evaluate[{fft[f12[#] &, x, Pi]}],
   {x, -20, 20}, PlotStyle → {{Thickness[0.005], Green}},
   Exclusions → Range[-11 Pi, 14 Pi, 2 Pi], PlotRange → All];

 pic2a = Graphics[{
    {PointSize[0.02], Green,
     {Point[{# Pi, Cosh[Pi]}], Point[{# Pi, Exp[-Pi]}], Point[{# Pi, Exp[Pi]}]} & /@
      Range[-11, 13, 2]}, {PointSize[0.014], White,
     {Point[{# Pi, Exp[-Pi]}], Point[{# Pi, Exp[Pi]}]} & /@ Range[-11, 13, 2]}
   }];
 pic3 = Plot[Evaluate[{Sinh[π]/π + (2 Sinh[π])/π Sum[(-1)^n/(1 + n^2) (Cos[n x] - n Sin[n x]), {n, 1, nn}]}], {x, -12, 14},
   PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200]; Show[pic1, pic2, pic2a, pic3,
  PlotRange → {{-2.5 Pi, 2.5 Pi}, {0 - 2, Exp[Pi] + 2}}, AspectRatio → 1 / GoldenRatio,
  Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[0, 40, 5]}, ImageSize → is]]
```

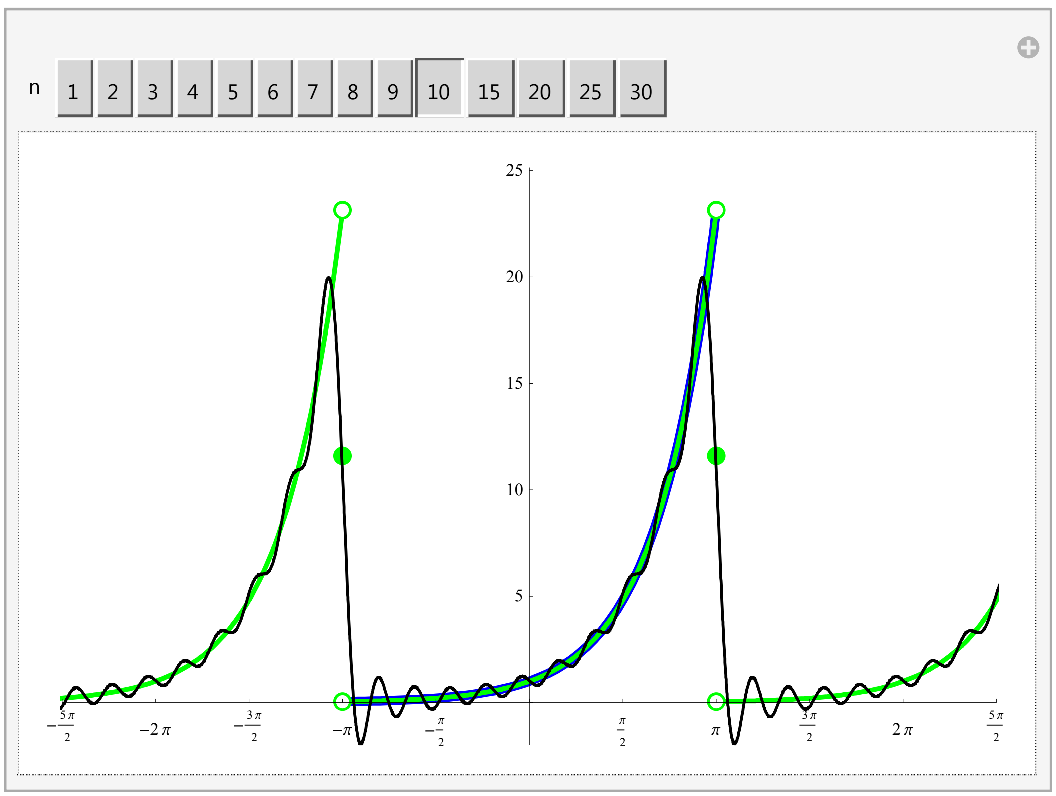
Out[522]=



Or, the same picture with Manipulate

In[523]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, Manipulate[pic1 =
   Plot[{f12[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}, PlotRange → All];

  pic2 = Plot[Evaluate[{fft[f12[#] &, x, Pi]}], {x, -20, 20}, PlotStyle →
     {{Thickness[0.005], Green}}, Exclusions → Range[-11 Pi, 14 Pi, 2 Pi], PlotRange → All];

  pic2a = Graphics[{
     {PointSize[0.02], Green,
      {Point[{# Pi, Cosh[Pi]}], Point[{# Pi, Exp[-Pi]}], Point[{# Pi, Exp[Pi]}]} & /@
       Range[-11, 13, 2]}, {PointSize[0.014], White,
      {Point[{# Pi, Exp[-Pi]}], Point[{# Pi, Exp[Pi]}]} & /@ Range[-11, 13, 2]}
    }];
  pic3 = Plot[Evaluate[{Sinh[π]/π + 2 Sinh[π]/π Sum[(-1)^n/(1 + n^2) (Cos[n x] - n Sin[n x]), {n, 1, nn}]}], {x, -12, 14},
    PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200]; Show[pic1, pic2, pic2a,
   pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {0 - 2, Exp[Pi] + 2}}, AspectRatio → 1 / GoldenRatio,
   Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[0, 40, 5]}, ImageSize → is],
  {{nn, 10, "n"}, Join[Range[10], Range[15, 30, 5]]}, ControlType → Setter]]
```

Out[523]=



■ Numerical series at special values of *x*

■ $x = \pi$

Notice that the convergence theorem implies that for a specific $x = \pi$ the following numerical series

$$\frac{\text{Sinh}[\pi]}{\pi} + \frac{2\,\text{Sinh}[\pi]}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} (\text{Cos}[n\,\pi] - n\,\text{Sin}[n\,\pi])$$

which equals

$$\frac{\text{Sinh}[\pi]}{\pi} + \frac{2\,\text{Sinh}[\pi]}{\pi} \sum_{n=1}^{\infty} \frac{1}{1+n^2}$$

converges to Cosh[π].

Thus

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{\pi}{2\,\text{Sinh}[\pi]} \left(\text{Cosh}[\pi] - \frac{\text{Sinh}[\pi]}{\pi}\right) = \frac{1}{2} (\pi\,\text{Coth}[\pi] - 1)$$

Mathematica knows this formula

In[524]:=
$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}$$

Out[524]=
$$\frac{1}{2} (-1 + \pi\,\text{Coth}[\pi])$$

- **$x = 0$**

Notice that the convergence theorem implies that for the specific $x = 0$ the following numerical series

$$\frac{\text{Sinh}[\pi]}{\pi} + \frac{2\,\text{Sinh}[\pi]}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} (\text{Cos}[n\,0] - n\,\text{Sin}[n\,0])$$

which equals

$$\frac{\text{Sinh}[\pi]}{\pi} + \frac{2\,\text{Sinh}[\pi]}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}$$

converges to 1.

Thus

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi}{2\,\text{Sinh}[\pi]} \left(1 - \frac{\text{Sinh}[\pi]}{\pi}\right) = \frac{1}{2} \left(\frac{\pi}{\text{Sinh}[\pi]} - 1\right)$$

Mathematica knows this formula

In[525]:=
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}$$

Out[525]=
$$\frac{1}{2} (-1 + \pi\,\text{Csch}[\pi])$$

Example 13: Cosh[x] on $-\pi < x \le \pi$

In[526]:=
```
Clear[f13];

f13[x_] = Cosh[x];
```

on the interval (-π,π]

Since Cosh[x] is the even part of Exp[x], we know the constant coefficient and the coefficients with Cos functions will be identical to the corresponding coefficients calculated for Exp[x].

The coefficient a_0

In[528]:=
```
FullSimplify[1/(2 Pi) Integrate[Cosh[x], {x, -Pi, Pi}]]
```

Out[528]= $\frac{\text{Sinh}[\pi]}{\pi}$

The coefficients a_n

In[529]:=
```
FullSimplify[1/Pi Integrate[Cosh[x] Cos[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[529]= $\frac{2\,(-1)^n\,\text{Sinh}[\pi]}{\pi + n^2\,\pi}$

The coefficients b_n

In[530]:=
```
FullSimplify[1/Pi Integrate[Cosh[x] Sin[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[530]= 0

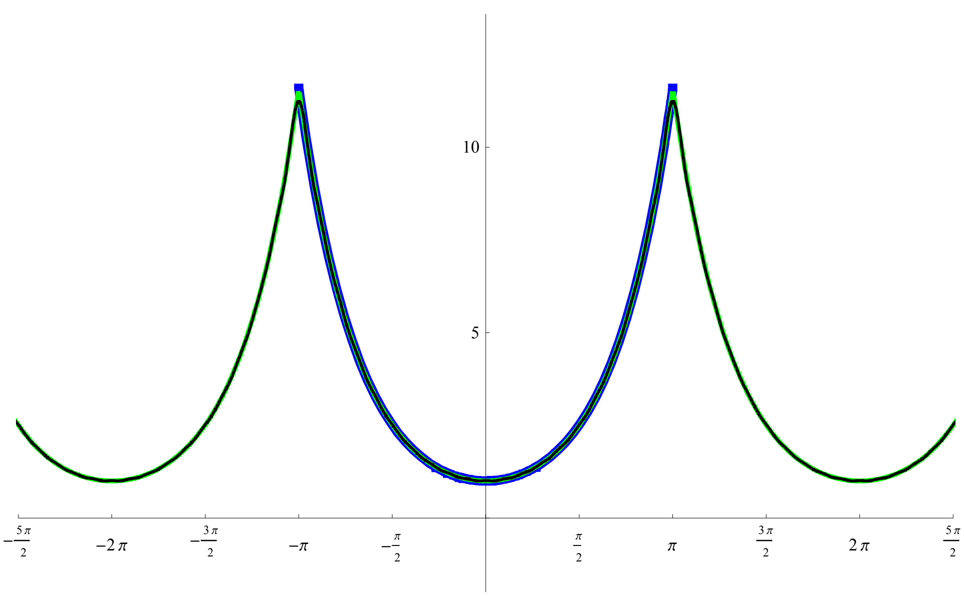
Thus the Fourier series of the given function is

$$\frac{\text{Sinh}[\pi]}{\pi} + \frac{2\,\text{Sinh}[\pi]}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}\,\text{Cos}[n\,x]$$

This series converges **uniformly** to the Fourier 2π-periodic extension of the function Cosh[x], restricted to $(-\pi,\pi]$, as illustrated in the following graph and manipulation

In[531]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, nn = 20; pic1 =
  Plot[{f13[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}, PlotRange → All];

 pic2 = Plot[Evaluate[{fft[f13[#] &, x, Pi]}],
   {x, -20, 20}, PlotStyle → {{Thickness[0.005], Green}},
   Exclusions → Range[-11 Pi, 14 Pi, 2 Pi], PlotRange → All];
 (*
 pic2a=Graphics[{
     {PointSize[0.02],Green,
      {Point[{# Pi,Cosh[Pi]}],Point[{# Pi,Exp[-Pi]}],Point[{#Pi,Exp[Pi]}]}&/@
       Range[-11,13,2]},{PointSize[0.014],White,
      {Point[{# Pi,Exp[-Pi]}],Point[{#Pi,Exp[Pi]}]}&/@Range[-11,13,2]}
    }];
 *)
 pic3 = Plot[Evaluate[{Sinh[π]/π + 2 Sinh[π]/π Sum[(-1)^n/(1 + n^2) Cos[n x], {n, 1, nn}]}],
   {x, -12, 14}, PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200];
 Show[pic1, pic2, pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {0 - 2, Cosh[Pi] + 2}},
  AspectRatio → 1 / GoldenRatio, AxesOrigin → 0,
  Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[0, 40, 5]}, ImageSize → is]]
```

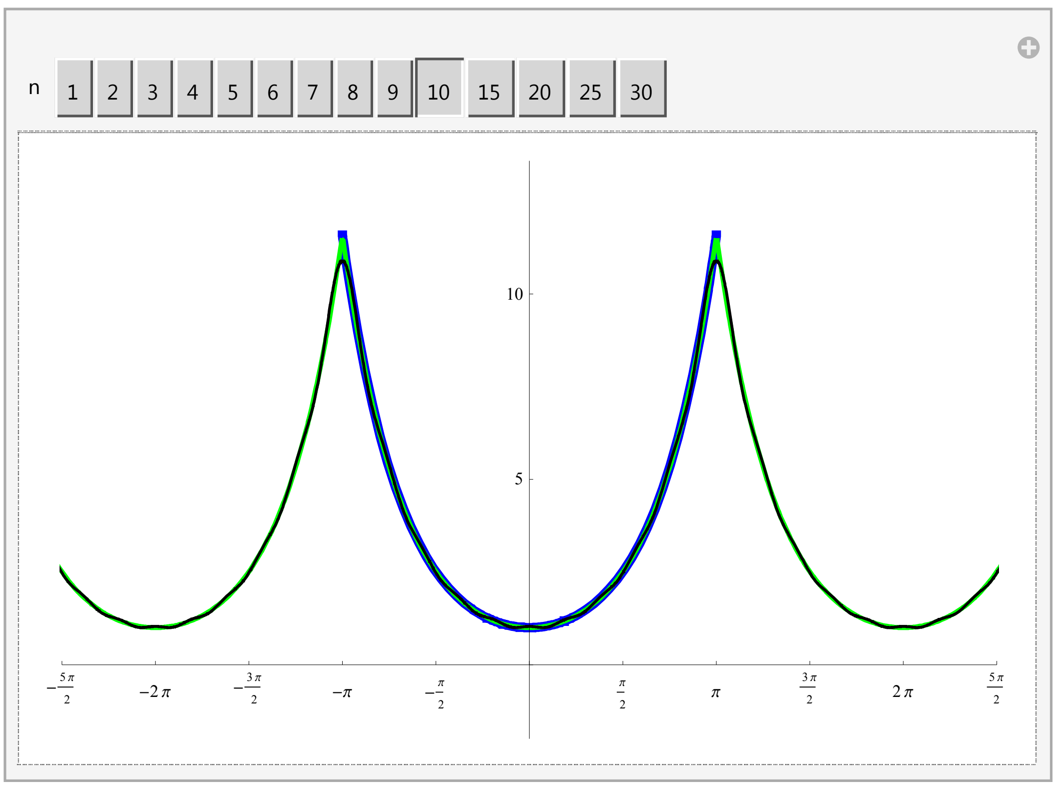
Out[531]=



Or, the same picture with Manipulate

In[532]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, Manipulate[pic1 =
   Plot[{f13[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}, PlotRange → All];

  pic2 = Plot[Evaluate[{fft[f13[#] &, x, Pi]}], {x, -20, 20}, PlotStyle →
     {{Thickness[0.005], Green}}, Exclusions → Range[-11 Pi, 14 Pi, 2 Pi], PlotRange → All];
  (*
  pic2a=Graphics[{
      {PointSize[0.02],Green,{Point[{# Pi,Cosh[Pi]}],Point[{# Pi,Exp[-Pi]}],
          Point[{#Pi,Exp[Pi]}]}&/@Range[-11,13,2]},{PointSize[0.014],White,
       {Point[{# Pi,Exp[-Pi]}],Point[{#Pi,Exp[Pi]}]}&/@Range[-11,13,2]}
     }];
  *)
  pic3 = Plot[Evaluate[{Sinh[π]/π + 2 Sinh[π]/π Sum[(-1)^n/(1 + n^2) Cos[n x], {n, 1, nn}]}],
    {x, -12, 14}, PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200];
  Show[pic1, pic2, pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {0 - 2, Cosh[Pi] + 2}},
   AspectRatio → 1 / GoldenRatio, AxesOrigin → 0,
   Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[0, 40, 5]}, ImageSize → is],
  {{nn, 10, "n"}, Join[Range[10], Range[15, 30, 5]]}, ControlType → Setter]]
```

Out[532]=



- **Numerical series at special values of *x***

Example 14: Sinh[*x*] on $-\pi < x \leq \pi$

In[533]:=
```
Clear[f14];

f14[x_] = Sinh[x];
```

on the interval $(-\pi,\pi]$

The coefficient a_0

In[535]:=
```
FullSimplify[1/(2 Pi) Integrate[Sinh[x], {x, -Pi, Pi}]]
```

Out[535]= 0

The coefficients a_n

In[536]:=
```
FullSimplify[1/Pi Integrate[Sinh[x] Cos[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[536]= 0

The coefficients b_n (identical to the corresponding coefficient for Exp[x])

In[537]:=
```
FullSimplify[1/Pi Integrate[Sinh[x] Sin[n x], {x, -Pi, Pi}], And[n ∈ Integers, n > 0]]
```

Out[537]= $-\frac{2\,(-1)^n\, n\, \mathrm{Sinh}[\pi]}{\pi + n^2\,\pi}$

Thus the Fourier series of the given function is

$$\frac{2\,\mathrm{Sinh}[\pi]}{\pi}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,n}{1+n^2}\,\mathrm{Sin}[n\,x]\right)$$

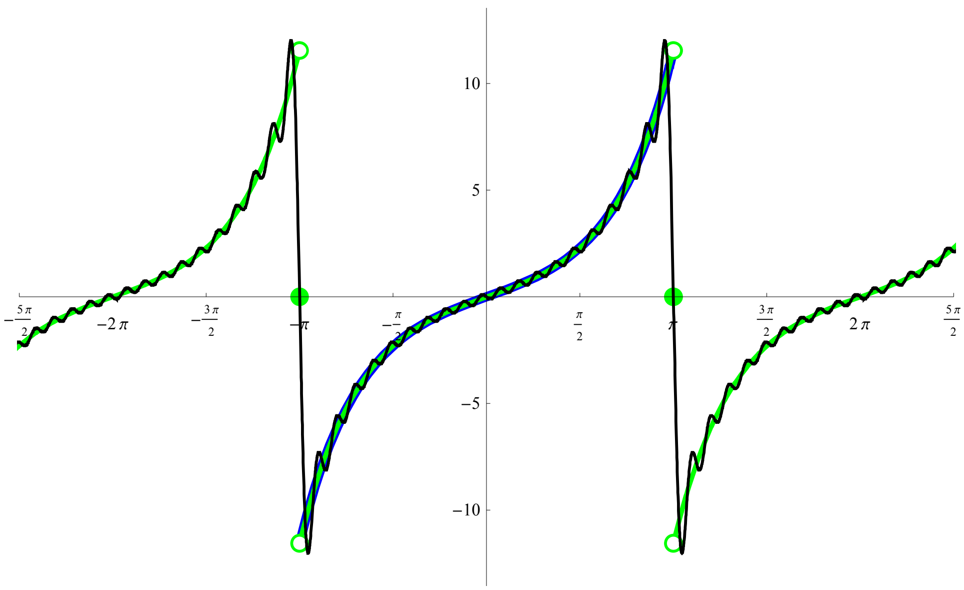
This series converges **pointwise** to the Fourier 2π-periodic extension of the function Sinh[*x*], restricted to $(-\pi,\pi]$, as illustrated in the following graph and manipulation

In[538]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, nn = 20; pic1 =
  Plot[{f14[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}, PlotRange → All];

 pic2 = Plot[Evaluate[{fft[f14[#] &, x, Pi]}],
   {x, -20, 20}, PlotStyle → {{Thickness[0.005], Green}},
   Exclusions → Range[-11 Pi, 14 Pi, 2 Pi], PlotRange → All];

 pic2a = Graphics[{
    {PointSize[0.02], Green,
     {Point[{# Pi, 0}], Point[{# Pi, Sinh[-Pi]}], Point[{# Pi, Sinh[Pi]}]} & /@
      Range[-11, 13, 2]}, {PointSize[0.014], White,
     {Point[{# Pi, Sinh[-Pi]}], Point[{# Pi, Sinh[Pi]}]} & /@ Range[-11, 13, 2]}
   }];
```

$$\text{pic3 = Plot}\left[\text{Evaluate}\left[\left\{\frac{2\,\text{Sinh}[\pi]}{\pi}\sum_{n=1}^{nn}\frac{(-1)^{n+1}\,n}{1+n^2}\,\text{Sin}[n\,x]\right\}\right],\ \{x, -12, 14\},\right.$$

```
   PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200]; Show[pic1, pic2, pic2a, pic3,
  PlotRange → {{-2.5 Pi, 2.5 Pi}, {-Sinh[Pi] - 2, Sinh[Pi] + 2}}, AspectRatio → 1 / GoldenRatio,
  Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-40, 40, 5]}, ImageSize → is]]
```

Out[538]=



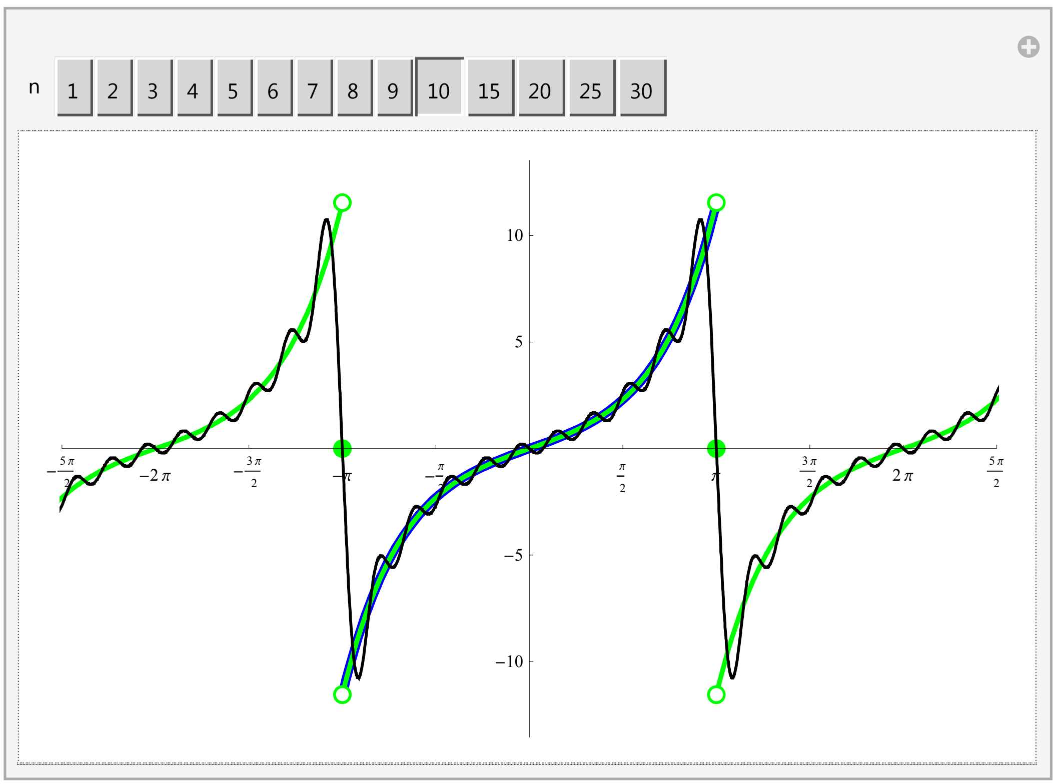
Or, the same picture with Manipulate

In[539]:=
```
Module[{pic1, pic2, pic2a, pic3, nn}, Manipulate[pic1 =
   Plot[{f14[x]}, {x, -Pi, Pi}, PlotStyle → {{Thickness[0.01], Blue}}, PlotRange → All];

  pic2 = Plot[Evaluate[{fft[f14[#] &, x, Pi]}], {x, -20, 20}, PlotStyle →
     {{Thickness[0.005], Green}}, Exclusions → Range[-11 Pi, 14 Pi, 2 Pi], PlotRange → All];

  pic2a = Graphics[{
     {PointSize[0.02], Green,
      {Point[{# Pi, 0}], Point[{# Pi, Sinh[-Pi]}], Point[{# Pi, Sinh[Pi]}]} & /@
       Range[-11, 13, 2]}, {PointSize[0.014], White,
      {Point[{# Pi, Sinh[-Pi]}], Point[{# Pi, Sinh[Pi]}]} & /@ Range[-11, 13, 2]}
    }];
```

$$\texttt{pic3 = Plot}\left[\texttt{Evaluate}\left[\left\{\frac{2\,\texttt{Sinh}[\pi]}{\pi}\sum_{n=1}^{nn}\frac{(-1)^{n+1}\,n}{1+n^2}\,\texttt{Sin}[n\,x]\right\}\right],\right.$$

```
    {x, -12, 14}, PlotStyle → {{Thickness[0.003], Black}}, PlotPoints → 200];
  Show[pic1, pic2, pic2a, pic3, PlotRange → {{-2.5 Pi, 2.5 Pi}, {-Sinh[Pi] - 2, Sinh[Pi] + 2}},
   AspectRatio → 1 / GoldenRatio,
   Ticks → {Range[-10 Pi, 10 Pi, Pi/2], Range[-40, 40, 5]}, ImageSize → is],
  {{nn, 10, "n"}, Join[Range[10], Range[15, 30, 5]]}, ControlType → Setter]]
```

Out[539]=



■ Numerical series at special values of *x*

■ $x = \pi / 2$

Notice that the convergence theorem implies that for a specific $x = \pi$ the following numerical series

$$\frac{2\,\text{Sinh}[\pi]}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,n}{1+n^2}\,\text{Sin}\left[n\,\frac{\pi}{2}\right]$$

which equals

$$\frac{\text{Sinh}[\pi]}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\,(2\,k-1)}{2\,k^2-2\,k+1}$$

converges to Sinh[π/2].

Thus

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\,(2\,k-1)}{2\,k^2-2\,k+1}=\frac{\pi\,\text{Sinh}[\pi/2]}{\text{Sinh}[\pi]}=\frac{\pi\,\text{Sinh}[\pi/2]}{2\,\text{Sinh}[\pi/2]\,\text{Cosh}[\pi/2]}=\frac{\pi}{2\,\text{Cosh}[\pi/2]}$$

Mathematica knows this formula

In[540]:= $$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\,(2\,k-1)}{2\,k^2-2\,k+1}$$

Out[540]= $$\frac{1}{2}\,\pi\,\text{Sech}\left[\frac{\pi}{2}\right]$$