Examples of Fourier series

Preliminaries

Below is the definition of a periodic extension of a function defined on $(-L, L]$. This definition takes a function as a variable. The function has to be inputted as a so called pure function (that is instead of the variable we put # and the formula ends with &).
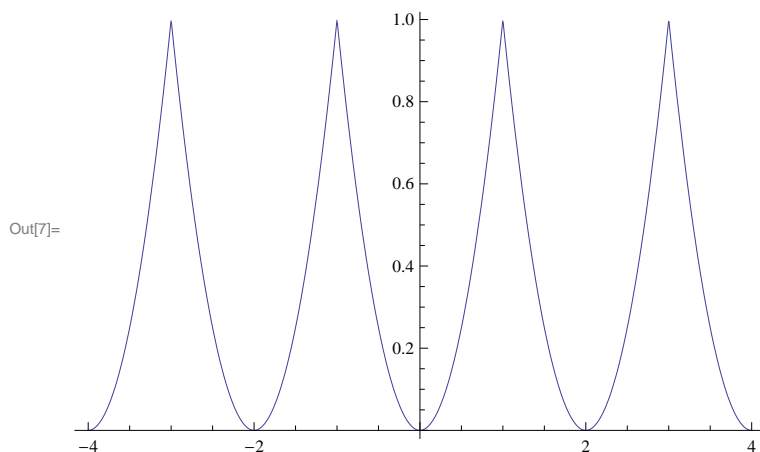
In[1]:= **Clear[ff, x, lL];**

$$\mathtt{fft[ff\_, x\_, lL\_] := ff\left[x - \left(Ceiling\left[\frac{x - (-lL)}{2\ lL}\right] - 1\right)(2\ lL)\right]}$$

In[5]:= $\mathtt{fft\left[\#^2\ \&,\ x,\ 1\right]}$

Out[5]= $\left(x - 2\left(-1 + Ceiling\left[\dfrac{1 + x}{2}\right]\right)\right)^2$

In[7]:= $\mathtt{Plot\left[fft\left[\#^2\ \&,\ x,\ 1\right],\ \{x,\ -4,\ 4\}\right]}$

Out[7]=

Example -1

In[113]:= `Clear[ca0, ca, cb, ff, n, lL, nn];`

`ff[x_] = Sign[x];`

`nn = 10;`

`cb[n_, lL_] = FullSimplify[`
`    1/lL Integrate[ff[x] Sin[(n Pi)/lL x], {x, -lL, lL}], And[lL > 0, n ∈ Integers, n > 0]]`

`ca0 = FullSimplify[1/(2 lL) Integrate[ff[x], {x, -lL, lL}], And[lL > 0]]`

`ca[n_, lL_] = FullSimplify[`
`    1/lL Integrate[ff[x] Cos[(n Pi)/lL x], {x, -lL, lL}], And[lL > 0, n ∈ Integers, n > 0]]`

Out[116]= $-\dfrac{2\left(-1+(-1)^{n}\right)}{n\,\pi}$

Out[117]= 0

Out[118]= 0

In[119]:= `lL = 1; Module[{pic1, pic2, pic2a, pic3},`

```
pic1 = Plot[{ff[x]}, {x, -lL, lL}, PlotStyle → {{Thickness[0.01], Blue}},
   PlotRange → {{-4, 7}, {-1.5, 1.5}}]; pic2 = Plot[{fft[ff[#] &, x, lL]}, {x, -5, 10},
   PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10, 14, 1],
   PlotRange → {{-4, 7}, {-1.5, 1.5}}, AspectRatio → Automatic];

pic2a = Graphics[{
    {PointSize[0.02], Green,
     {Point[{#, -1}], Point[{#, 1}], Point[{#, 0}]} & /@ Range[-10, 13, 1]},
    {PointSize[0.014], White, {Point[{#, -1}], Point[{#, 1}]} & /@ Range[-10, 13, 1]}
```

$$\}]; \text{pic3} = \text{Plot}\left[\text{Evaluate}\left[\left\{\text{ca0} + \text{Sum}\left[\text{ca}[n, lL] \text{ Cos}\left[\frac{n \text{ Pi}}{lL} x\right], \{n, 1, nn\}\right] + \right.\right.$$

$$\left.\left.\text{Sum}\left[\text{cb}[n, lL] \text{ Sin}\left[\frac{n \text{ Pi}}{lL} x\right], \{n, 1, nn\}\right]\right\}\right], \{x, -12, 14\},$$

```
   PlotStyle → {{Thickness[0.003], Black}}, PlotRange → {{-4, 7}, {-1.5, 1.5}}];

Show[pic1, pic2, pic2a, pic3]
```

Out[119]=
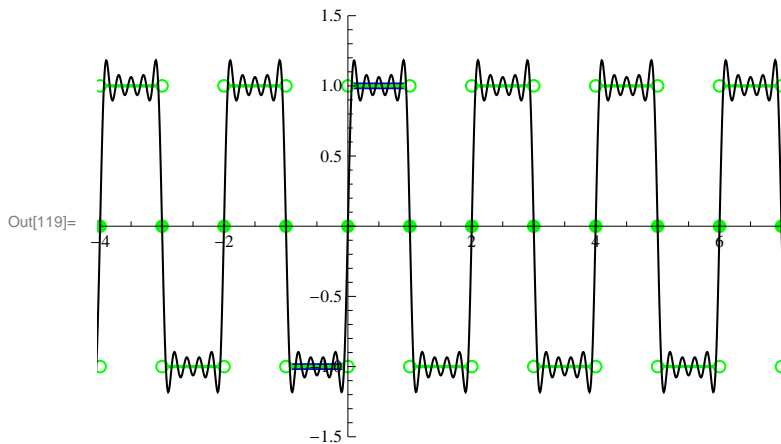
Example 0

In[120]:= `Clear[ca0, ca, cb, ff, n, lL, nn];`

`ff[x_] = UnitStep[x];`

`nn = 10;`

`cb[n_, lL_] = FullSimplify[`
$\frac{1}{lL}$ `Integrate[ff[x] Sin[`$\frac{n\,Pi}{lL}$ `x], {x, -lL, lL}], And[lL > 0, n ∈ Integers, n > 0]]`

`ca0 = FullSimplify[`$\frac{1}{2\,lL}$ `Integrate[ff[x], {x, -lL, lL}], And[lL > 0]]`

`ca[n_, lL_] = FullSimplify[`
$\frac{1}{lL}$ `Integrate[ff[x] Cos[`$\frac{n\,Pi}{lL}$ `x], {x, -lL, lL}], And[lL > 0, n ∈ Integers, n > 0]]`

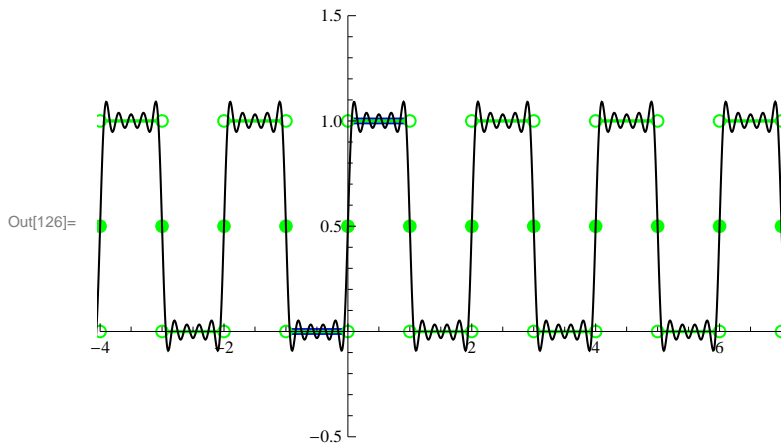Out[123]= $-\dfrac{-1 + (-1)^n}{n\,\pi}$

Out[124]= $\dfrac{1}{2}$

Out[125]= 0

In[126]:= `lL = 1; Module[{pic1, pic2, pic2a, pic3},`

```
    pic1 = Plot[{ff[x]}, {x, -lL, lL}, PlotStyle → {{Thickness[0.01], Blue}},
       PlotRange → {{-4, 7}, {-.5, 1.5}}]; pic2 = Plot[{fft[ff[#] &, x, lL]}, {x, -5, 10},
       PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10, 14, 1],
       PlotRange → {{-4, 7}, {-.5, 1.5}}, AspectRatio → Automatic];

    pic2a = Graphics[{
        {PointSize[0.02], Green,
         {Point[{#, 0}], Point[{#, 1}], Point[{#, 1 / 2}]} & /@ Range[-10, 13, 1]},
        {PointSize[0.014], White, {Point[{#, 0}], Point[{#, 1}]} & /@ Range[-10, 13, 1]}
```

$$\}]; \ pic3 = Plot\left[Evaluate\left[\left\{ca0 + Sum\left[ca[n, lL] \ Cos\left[\frac{n\,Pi}{lL}\,x\right], \{n, 1, nn\}\right] + \right.\right.$$

$$\left.\left.Sum\left[cb[n, lL] \ Sin\left[\frac{n\,Pi}{lL}\,x\right], \{n, 1, nn\}\right]\right\}\right], \{x, -12, 14\},$$

$$PlotStyle \to \{\{Thickness[0.003], Black\}\}, PlotRange \to \{\{-4, 7\}, \{-.5, 1.5\}\}\right];$$

`Show[pic1, pic2, pic2a, pic3]`

Out[126]=

Example 1

In[127]:= `Clear[ca0, ca, cb, ff, n, lL, nn];`

`ff[x_] = x;`

`nn = 20;`

$$cb[n\_, lL\_] = FullSimplify\Big[$$
$$\frac{1}{lL} \ Integrate\left[ff[x] \ Sin\left[\frac{n\,Pi}{lL}\,x\right], \{x, -lL, lL\}\right], And[lL > 0, n \in Integers, n > 0]\Big]$$

Out[130]= $-\dfrac{2\,(-1)^n\,lL}{n\,\pi}$

In[131]:= `lL = 2; Module[{pic1, pic2, pic2a, pic3},`

```
pic1 = Plot[{ff[x]}, {x, -lL, lL}, PlotStyle → {{Thickness[0.01], Blue}},
  PlotRange → {{-4, 11}, {-3, 3}}]; pic2 = Plot[{fft[ff[#] &, x, lL]}, {x, -5, 10},
  PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10, 14, 4],
  PlotRange → {{-4, 11}, {-3, 3}}, AspectRatio → Automatic];

pic2a = Graphics[{
   {PointSize[0.02], Green,
    {Point[{#, -2}], Point[{#, 2}], Point[{#, 0}]} & /@ Range[-10, 13, 4]},
   {PointSize[0.014], White, {Point[{#, -2}], Point[{#, 2}]} & /@ Range[-10, 13, 4]}
```

$$\}];\ pic3 = Plot\left[Evaluate\left[\left\{Sum\left[cb[n,\ lL]\ Sin\left[\frac{n\ Pi}{lL}\ x\right],\ \{n,\ 1,\ nn\}\right]\right\}\right],\right.$$

```
{x, -12, 14}, PlotStyle → {{Thickness[0.003], Black}},
PlotRange → {{-4, 11}, {-3, 3}}]; Show[pic1, pic2, pic2a, pic3]
```

Out[131]=
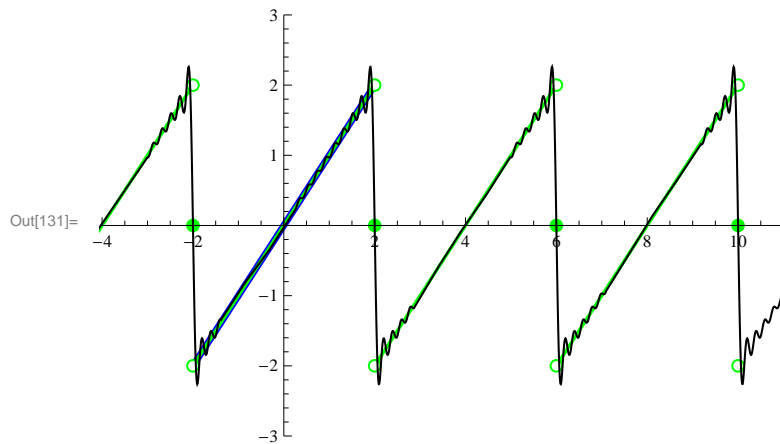
Example 2

In[132]:= `Clear[ca0, ca, cb, ff, n, lL, nn];`

`ff[x_] = Abs[x];`

`nn = 10;`

```
cb[n_, lL_] = FullSimplify[
    1/lL Integrate[ff[x] Sin[n Pi/lL x], {x, -lL, lL}], And[lL > 0, n ∈ Integers, n > 0]]
```

```
ca0 = FullSimplify[1/(2 lL) Integrate[ff[x], {x, -lL, lL}], And[lL > 0]]
```

```
ca[n_, lL_] = FullSimplify[
    1/lL Integrate[ff[x] Cos[n Pi/lL x], {x, -lL, lL}], And[lL > 0, n ∈ Integers, n > 0]]
```

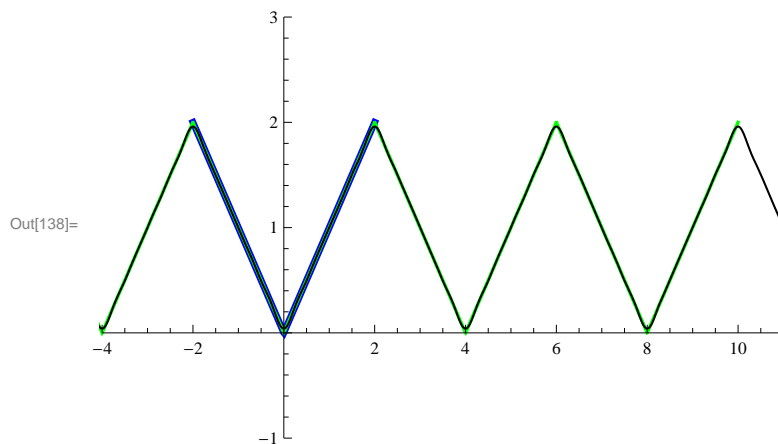Out[135]= 0

Out[136]= $\dfrac{lL}{2}$

Out[137]= $\dfrac{2 \left(-1 + (-1)^n\right) lL}{n^2 \pi^2}$

In[138]:= `lL = 2; Module[{pic1, pic2, pic2a, pic3},`

```
pic1 = Plot[{ff[x]}, {x, -lL, lL}, PlotStyle → {{Thickness[0.01], Blue}},
   PlotRange → {{-4, 11}, {-1, 3}}]; pic2 = Plot[{fft[ff[#] &, x, lL]}, {x, -5, 10},
   PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10, 14, 4],
   PlotRange → {{-4, 11}, {-1, 3}}, AspectRatio → Automatic]; pic2a = Graphics[{
     {PointSize[0.02], Green, {Point[{#, -2}], Point[{#, 2}], Point[{#, 0}]} & /@
       Range[-10, 13, 4]}, {PointSize[0.014], White,
      {Point[{#, -2}], Point[{#, 2}]} & /@ Range[-10, 13, 4]}
   }]; pic3 = Plot[Evaluate[{ca0 + Sum[ca[n, lL] Cos[(n Pi)/lL x], {n, 1, nn}] +
```

$$\mathrm{Sum}\left[\mathrm{cb}[n, lL]\ \mathrm{Sin}\left[\frac{n\ \mathrm{Pi}}{lL}\ x\right], \{n, 1, nn\}\right]\right],$$

```
   {x, -12, 14}, PlotStyle → {{Thickness[0.003], Black}},
   PlotRange → {{-4, 11}, {-1, 3}}]; Show[pic1, pic2, pic3]]
```

Out[138]=

Example 3

In[139]:=

```
Clear[ca0, ca, cb, ff, n, lL, nn];

ff[x_] = x UnitStep[x];

nn = 10;

cb[n_, lL_] = FullSimplify[
    1/lL Integrate[ff[x] Sin[n Pi/lL x], {x, -lL, lL}], And[lL > 0, n ∈ Integers, n > 0]]

ca0 = FullSimplify[1/(2 lL) Integrate[ff[x], {x, -lL, lL}], And[lL > 0]]

ca[n_, lL_] = FullSimplify[
    1/lL Integrate[ff[x] Cos[n Pi/lL x], {x, -lL, lL}], And[lL > 0, n ∈ Integers, n > 0]]
```

Out[142]= $-\dfrac{(-1)^n \, lL}{n \, \pi}$

Out[143]= $\dfrac{lL}{4}$

Out[144]= $\dfrac{(-1 + (-1)^n) \, lL}{n^2 \, \pi^2}$

In[145]:= `lL = 2; Module[{pic1, pic2, pic2a, pic3},`

```
    pic1 = Plot[{ff[x]}, {x, -lL, lL}, PlotStyle → {{Thickness[0.01], Blue}},
       PlotRange → {{-4, 11}, {-1, 3}}]; pic2 = Plot[{fft[ff[#] &, x, lL]}, {x, -5, 10},
       PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-10, 14, 4],
       PlotRange → {{-4, 11}, {-1, 3}}, AspectRatio → Automatic];

    pic2a = Graphics[{
       {PointSize[0.02], Green,
        {Point[{#, 0}], Point[{#, 2}], Point[{#, 1}]} & /@ Range[-10, 13, 4]},
       {PointSize[0.014], White, {Point[{#, 0}], Point[{#, 2}]} & /@ Range[-10, 13, 4]}
```

$$}]; pic3 = Plot\left[Evaluate\left[\left\{ca0 + Sum\left[ca[n, lL] Cos\left[\frac{n\,Pi}{lL}\,x\right], \{n, 1, nn\}\right] +\right.\right.$$

$$\left.\left.Sum\left[cb[n, lL] Sin\left[\frac{n\,Pi}{lL}\,x\right], \{n, 1, nn\}\right]\right\}\right], \{x, -12, 14\},$$

$$\left.PlotStyle → \{\{Thickness[0.003], Black\}\}, PlotRange → \{\{-4, 11\}, \{-1, 3\}\}\right];$$

`    Show[pic1, pic2, pic2a, pic3]`

Out[145]=
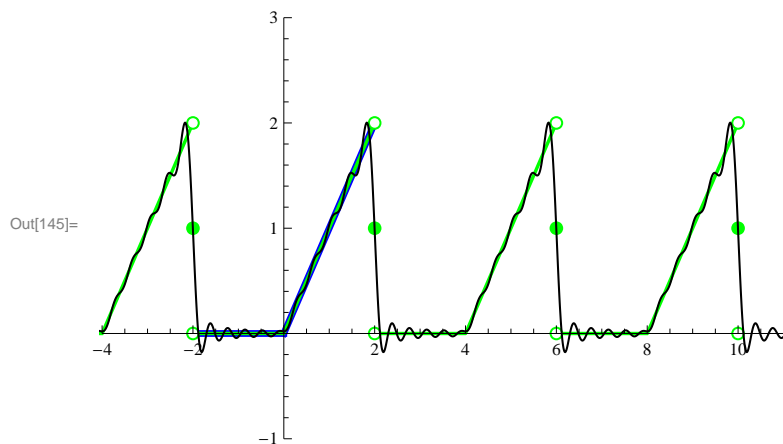
Example 4

In[146]:=

```
Clear[ca0, ca, cb, ff, n, lL, nn];

ff[x_] = x^2 UnitStep[x];

nn = 20;

cb[n_, lL_] = FullSimplify[
   1/lL Integrate[ff[x] Sin[(n Pi)/lL x], {x, -lL, lL}], And[lL > 0, n ∈ Integers, n > 0]]

ca0 = FullSimplify[1/(2 lL) Integrate[ff[x], {x, -lL, lL}], And[lL > 0]]

ca[n_, lL_] = FullSimplify[
   1/lL Integrate[ff[x] Cos[(n Pi)/lL x], {x, -lL, lL}], And[lL > 0, n ∈ Integers, n > 0]]
```

Out[149]= $-\frac{1}{n^3 \pi^3} lL^2 \left(2 + (-1)^n \left(-2 + n^2 \pi^2\right)\right)$

Out[150]= $\frac{lL^2}{6}$

Out[151]= $\frac{2 (-1)^n lL^2}{n^2 \pi^2}$

In[152]:= `lL = 1; Module[{pic1, pic2, pic2a, pic3}, pic1 = Plot[{ff[x]}, {x, -lL, lL},`

`    PlotStyle → {{Thickness[0.01], Blue}}, PlotRange → {{-4, 7}, {-.1, 1.1}}];`

`  pic2 = Plot[{fft[ff[#] &, x, lL]}, {x, -5, 10},`
`    PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-11, 14, 2],`
`    PlotRange → {{-4, 7}, {-.1, 1.1}}, AspectRatio → Automatic];`

`  pic2a = Graphics[{`
`     {PointSize[0.02], Green,`
`      {Point[{#, 0}], Point[{#, 1}], Point[{#, 1 / 2}]} & /@ Range[-11, 13, 2]},`
`     {PointSize[0.014], White, {Point[{#, 0}], Point[{#, 1}]} & /@ Range[-11, 13, 2]}`

$$}]; \text{pic3} = \text{Plot}\Big[\text{Evaluate}\Big[\Big\{ca0 + \text{Sum}\Big[ca[n, lL] \, \text{Cos}\Big[\frac{n\,Pi}{lL}\,x\Big], \{n, 1, nn\}\Big] +$$

$$\text{Sum}\Big[cb[n, lL] \, \text{Sin}\Big[\frac{n\,Pi}{lL}\,x\Big], \{n, 1, nn\}\Big]\Big\}\Big], \{x, -12, 14\},$$

`    PlotStyle → {{Thickness[0.003], Black}}, PlotRange → {{-4, 7}, {-.1, 1.1}}];`

`  Show[pic1, pic2, pic2a, pic3]`

Out[152]=
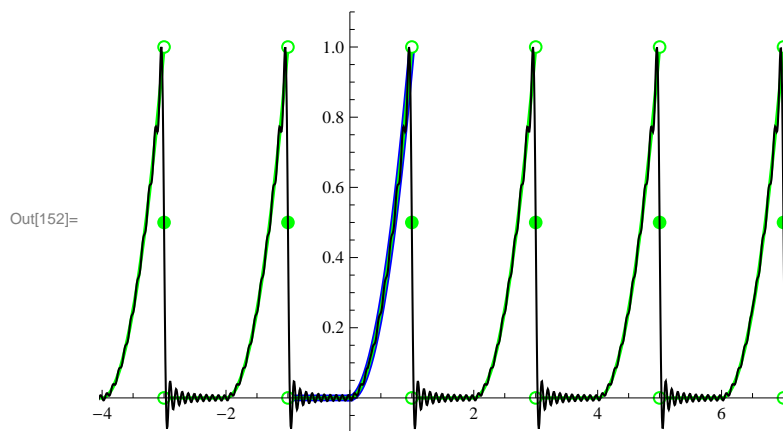
Example 5

In[153]:=

```
Clear[ca0, ca, cb, ff, n, lL, nn];

ff[x_] = x^2;

nn = 10;

cb[n_, lL_] = FullSimplify[
    1/lL Integrate[ff[x] Sin[(n Pi)/lL x], {x, -lL, lL}], And[lL > 0, n ∈ Integers, n > 0]]

ca0 = FullSimplify[1/(2 lL) Integrate[ff[x], {x, -lL, lL}], And[lL > 0]]

ca[n_, lL_] = FullSimplify[
    1/lL Integrate[ff[x] Cos[(n Pi)/lL x], {x, -lL, lL}], And[lL > 0, n ∈ Integers, n > 0]]
```

Out[156]= 0

Out[157]= $\dfrac{lL^2}{3}$

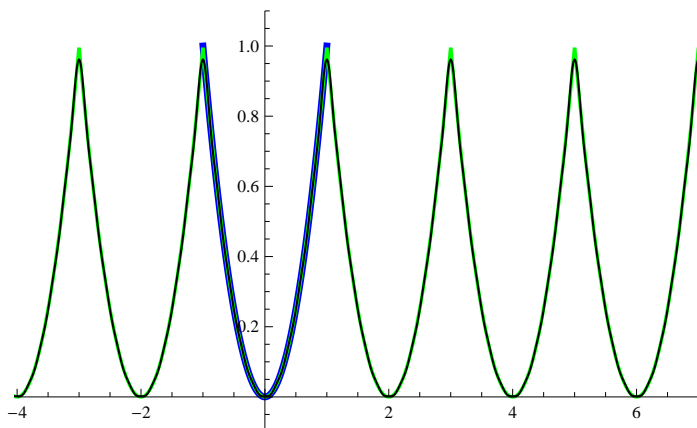Out[158]= $\dfrac{4 \, (-1)^n \, lL^2}{n^2 \, \pi^2}$

In[159]:= `lL = 1; Module[{pic1, pic2, pic2a, pic3}, pic1 = Plot[{ff[x]}, {x, -lL, lL},`

`PlotStyle → {{Thickness[0.01], Blue}}, PlotRange → {{-4, 7}, {-.1, 1.1}}];`

`pic2 = Plot[{fft[ff[#] &, x, lL]}, {x, -5, 10},`
`PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-11, 14, 2],`
`PlotRange → {{-4, 7}, {-.1, 1.1}}, AspectRatio → Automatic];`

`pic2a = Graphics[{`
`{PointSize[0.02], Green,`
`{Point[{#, -2}], Point[{#, 2}], Point[{#, 0}]} & /@ Range[-10, 13, 4]},`
`{PointSize[0.014], White, {Point[{#, -2}], Point[{#, 2}]} & /@ Range[-10, 13, 4]}`

$\}]; \text{pic3} = \text{Plot}\left[\text{Evaluate}\left[\left\{\text{ca0} + \text{Sum}\left[\text{ca}[n, lL]\, \text{Cos}\left[\frac{n\,Pi}{lL}\,x\right], \{n, 1, nn\}\right]\right.\right.\right. +$

$\left.\left.\left.\text{Sum}\left[\text{cb}[n, lL]\, \text{Sin}\left[\frac{n\,Pi}{lL}\,x\right], \{n, 1, nn\}\right]\right\}\right],\right.$

`{x, -12, 14}, PlotStyle → {{Thickness[0.003], Black}},`

$\left.\text{PlotRange} → \{\{-4, 7\}, \{-.1, 1.1\}\}\right]; \text{Show}[\text{pic1, pic2, pic3}]$

Out[159]=

Example 6

In[160]:=

```
Clear[ca0, ca, cb, ff, n, lL, nn];

ff[x_] = x^2 Sign[x];

nn = 10;

cb[n_, lL_] = FullSimplify[
    1/lL Integrate[ff[x] Sin[(n Pi)/lL x], {x, -lL, lL}], And[lL > 0, n ∈ Integers, n > 0]]

ca0 = FullSimplify[1/(2 lL) Integrate[ff[x], {x, -lL, lL}], And[lL > 0]]

ca[n_, lL_] = FullSimplify[
    1/lL Integrate[ff[x] Cos[(n Pi)/lL x], {x, -lL, lL}], And[lL > 0, n ∈ Integers, n > 0]]
```

Out[163]= $-\frac{1}{n^3 \pi^3} 2 lL^2 \left(2 + (-1)^n \left(-2 + n^2 \pi^2\right)\right)$

Out[164]= 0

Out[165]= 0

```
In[166]:= lL = 1; Module[{pic1, pic2, pic2a, pic3}, pic1 = Plot[{ff[x]}, {x, -lL, lL},
        PlotStyle → {{Thickness[0.01], Blue}}, PlotRange → {{-4, 7}, {-1.1, 1.1}}];

    pic2 = Plot[{fft[ff[#] &, x, lL]}, {x, -5, 10},
        PlotStyle → {{Thickness[0.005], Green}}, Exclusions → Range[-11, 14, 2],
        PlotRange → {{-4, 7}, {-1.1, 1.1}}, AspectRatio → Automatic];

    pic2a = Graphics[{
        {PointSize[0.02], Green,
         {Point[{#, -1}], Point[{#, 1}], Point[{#, 0}]} & /@ Range[-11, 13, 2]},
        {PointSize[0.014], White, {Point[{#, -1}], Point[{#, 1}]} & /@ Range[-11, 13, 2]}
      }]; pic3 = Plot[Evaluate[{ca0 + Sum[ca[n, lL] Cos[(n Pi)/lL x], {n, 1, nn}] +

          Sum[cb[n, lL] Sin[(n Pi)/lL x], {n, 1, nn}]}], {x, -12, 14},

        PlotStyle → {{Thickness[0.003], Black}}, PlotRange → {{-4, 7}, {-1.1, 1.1}}];

    Show[pic1, pic2, pic2a, pic3]]
```