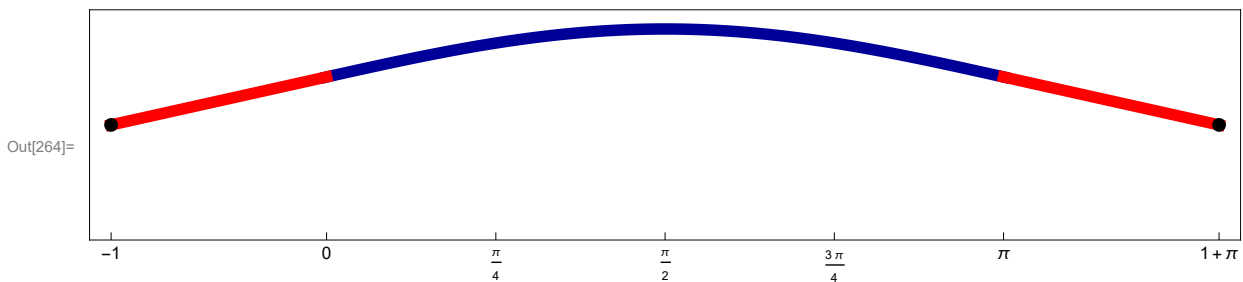


# Super glued strings

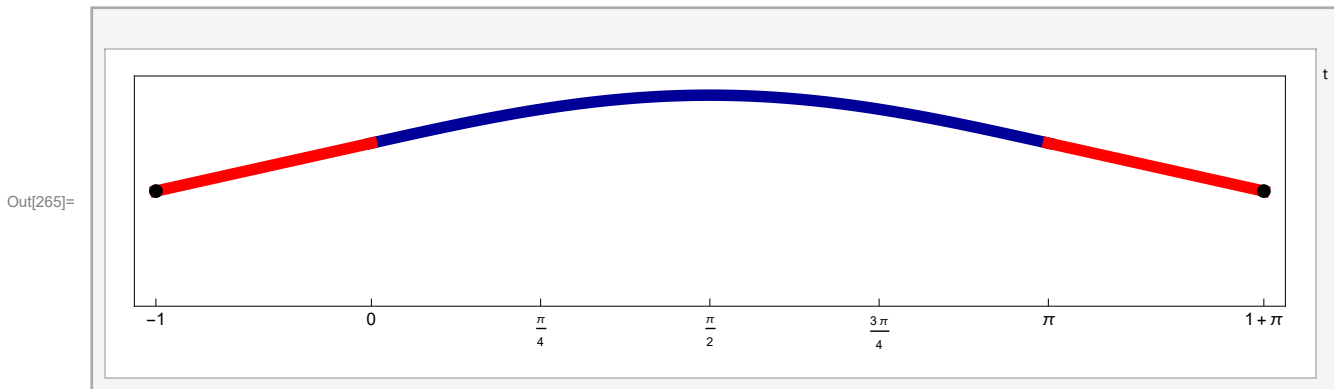
## Not a solution

The function illustrated here is not a physical solution of the vibrating string equation. The purpose of this section is to show the Mathematica principle of the animation. The actual solutions are more complicated functions.

```
In[264]:= Plot[1 + Sin[x], {x, 0, Pi}, PlotStyle -> {{Thickness[0.01], RGBColor[0, 0, 0.6]}},  
Epilog -> {{Red, Thickness[0.01], Line[{{-1, 0}, {0, 1}}], Line[{{Pi, 1}, {Pi + 1, 0}}]},  
{PointSize[0.012], Point[{-1, 0}], Point[{Pi + 1, 0}]}},  
PlotRange -> {{-1.1, Pi + 1.1}, {-2.4, 2.4}}, AspectRatio -> 1 / 5, Frame -> True,  
FrameTicks -> {{}, {}}, {Join[{-1, Pi + 1}, Range[0, Pi, Pi / 4]], {}},  
Axes -> False, ImageSize -> 600]
```



```
In[265]:= Manipulate[Plot[(1 + Sin[x]) Cos[t], {x, 0, Pi},  
PlotStyle -> {{Thickness[0.01], RGBColor[0, 0, 0.6]}}, Epilog -> {{Red,  
Thickness[0.01], Line[{{-1, 0}, {0, Cos[t]}]}, Line[{{Pi, Cos[t]}, {Pi + 1, 0}}]},  
{PointSize[0.012], Point[{-1, 0}], Point[{Pi + 1, 0}]}},  
PlotRange -> {{-1.1, Pi + 1.1}, {-2.4, 2.4}}, AspectRatio -> 1 / 5, Frame -> True,  
FrameTicks -> {{}, {}}, {Join[{-1, Pi + 1}, Range[0, Pi, Pi / 4]], {}},  
Axes -> False, ImageSize -> 600], {t, 0, 2 Pi, N[Pi / 64]}
```



## Solution 1

We want to solve

$$\partial_{tt}u[x, t] = \partial_{xx}u[x, t], \quad 0 \leq x \leq \pi, \quad t \geq 0, \quad u[0, t] - \partial_x u[0, t] = 0, \\ u[\pi, t] = 0, \quad u[x, 0] = \text{ff1}[x], \quad \partial_t u[x, 0] = 0.$$

So we need to solve the Sturm-Liouville eigenvalue problem

$$-y(x)'' = \lambda y(x), \quad 0 \leq x \leq \pi, \\ y'(0) - y(0) = 0, \\ y(\pi) = 0.$$

Notice that here we have

### No negative eigenvalues

We look for negative eigenvalues  $\lambda = -\mu^2$

$$-y(x)'' = -\mu^2 y(x), \quad 0 \leq x \leq \pi, \\ y'(0) - y(0) = 0, \\ y(\pi) = 0.$$

```
In[266]:= DSolve[y''[x] == μ² y[x], y[x], x]
```

```
Out[266]:= {{y[x] → e^{x μ} c₁ + e^{-x μ} c₂}}
```

I do not know how to force Mathematica into working with hyperbolic functions. So I force it by hand:

```
In[267]:= Simplify[(ExpToTrig[DSolve[y''[x] == μ² y[x], y[x], x]]) /. {C[1] + C[2] → C[1], C[1] - C[2] → C[2]}]
```

```
Out[267]:= {{y[x] → c₁ Cosh[x μ] + c₂ Sinh[x μ]}}
```

Substitute the fundamental solution ( $\text{Cosh}[\mu x] C[1] + \text{Sinh}[\mu x] C[2]$ ) into the boundary conditions and collect the expression with  $C[1]$  and  $C[2]$ . This will give us two homogeneous equations.

```
In[268]:= Collect[(Cosh[μ x] C[1] + Sinh[μ x] C[2]) /. {x → 0} - (D[(Cosh[μ x] C[1] + Sinh[μ x] C[2]), x] /. {x → 0}), {C[1], C[2]}]
```

```
Out[268]:= c₁ - μ c₂
```

```
In[269]:= Collect[(Cosh[μ x] C[1] + Sinh[μ x] C[2]) /. {x → Pi}, {C[1], C[2]}]
```

```
Out[269]:= c₁ Cosh[π μ] + c₂ Sinh[π μ]
```

We want nontrivial solutions for  $C[1]$  and  $C[2]$  of the system

$$c_1 - \mu c_2 = 0 \\ c_1 \text{Cosh}[\pi \mu] + c_2 \text{Sinh}[\pi \mu] = 0$$

For a nontrivial solution to exist we need the determinant to be equal to 0. We can collect the determinant by hand or automate that from the above formulas. By hand:

```
In[270]:= Det[{{1, -μ}, {Cosh[π μ], Sinh[π μ]}}]
```

```
Out[270]= μ Cosh[π μ] + Sinh[π μ]
```

Automate using the command Coefficient[]:

```
In[271]:= {{Coefficient[#, C[1]], Coefficient[#, C[2]]} &[
  ((Cosh[μ x] C[1] + Sinh[μ x] C[2]) /. {x → 0}) -
  ((D[(Cosh[μ x] C[1] + Sinh[μ x] C[2]), x]) /. {x → 0})],
  {Coefficient[#, C[1]], Coefficient[#, C[2]]} &[
  ((Cosh[μ x] C[1] + Sinh[μ x] C[2]) /. {x → Pi})]}
```

```
Out[271]= {{1, -μ}, {Cosh[π μ], Sinh[π μ]}}
```

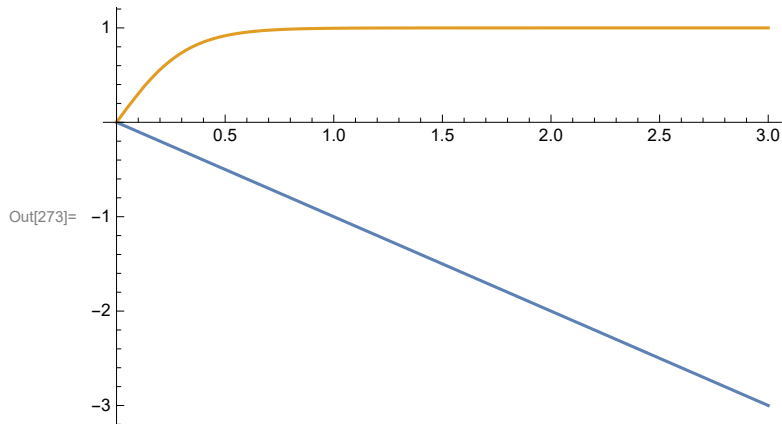
```
In[272]:= Det[{{Coefficient[#, C[1]], Coefficient[#, C[2]]} &[
  ((Cosh[μ x] C[1] + Sinh[μ x] C[2]) /. {x → 0}) -
  ((D[(Cosh[μ x] C[1] + Sinh[μ x] C[2]), x]) /. {x → 0})],
  {Coefficient[#, C[1]], Coefficient[#, C[2]]} &[
  ((Cosh[μ x] C[1] + Sinh[μ x] C[2]) /. {x → Pi})}]}
```

```
Out[272]= μ Cosh[π μ] + Sinh[π μ]
```

The above function is the function whose zeros give as  $-\mu^2$  the negative eigenvalues

It is easier to think of this equation as  $-\mu = \text{Tanh}[\pi \mu]$

```
In[273]:= Plot[{-μ, Tanh[Pi μ]}, {μ, 0, 3}]
```



Since the  $-\mu = \text{Tanh}[\pi \mu]$  has no positive solutions, there are no negative eigenvalues. Above is a powerful graphical verification of this claim. This claim follows formally from our knowledge about Tanh, it is in the first quadrant for positive  $\mu$ .

## Zero is not an eigenvalue

Is zero an eigenvalue?

$$-y(x)'' = 0, \quad 0 \leq x \leq \pi,$$

$$y'(0) - y(0) = 0,$$

$$y(\pi) = 0.$$

In[274]:= **DSolve**[y''[x] == 0, y[x], x]

Out[274]= {{y[x] → C<sub>1</sub> + x C<sub>2</sub>}}

Substitute into the boundary conditions

In[275]:= **Collect**[((C[1] + x C[2]) /. {x → 0}) - ((D[C[1] + x C[2], x]) /. {x → 0}), {C[1], C[2]}]

Out[275]= C<sub>1</sub> - C<sub>2</sub>

In[276]:= **Collect**[((C[1] + x C[2]) /. {x → Pi}), {C[1], C[2]}]

Out[276]= C<sub>1</sub> + π C<sub>2</sub>

We need a nontrivial solution for C[1] and C[2] of the system

$$C[1] - C[2] = 0$$

$$C[1] + \pi C[2] = 0$$

Since the determinant

In[277]:= **Det**[{{1, -1}, {1, Pi}}]

Out[277]= 1 + π

is nonzero, the system has no nontrivial solutions, thus zero is not an eigenvalue.

## Find positive eigenvalues

We look for positive eigenvalues  $\lambda = \mu^2$

$$-y(x)'' = \mu^2 y(x), \quad 0 \leq x \leq \pi,$$

$$y'(0) - y(0) = 0,$$

$$y(\pi) = 0.$$

In[278]:= **DSolve**[-y''[x] == μ<sup>2</sup> y[x], y[x], x]

Out[278]= {{y[x] → C<sub>1</sub> Cos[x μ] + C<sub>2</sub> Sin[x μ]}}

Substitute into the boundary conditions

In[279]:= **Collect**[((C[1] Cos[x μ] + C[2] Sin[x μ]) /. {x → 0}) - ((D[C[1] Cos[x μ] + C[2] Sin[x μ], x]) /. {x → 0}), {C[1], C[2]}]

Out[279]= C<sub>1</sub> - μ C<sub>2</sub>

In[280]:= **Collect**[((C[1] Cos[x μ] + C[2] Sin[x μ]) /. {x → Pi}), {C[1], C[2]}]

Out[280]= C<sub>1</sub> Cos[π μ] + C<sub>2</sub> Sin[π μ]

We want nontrivial solutions for C[1] and C[2] of the system

$$C[1] - \mu C[2] = 0$$

$$C[1] \cos[\pi \mu] + C[2] \sin[\pi \mu] = 0$$

For a nontrivial solution to exist we need the determinant to be equal to 0

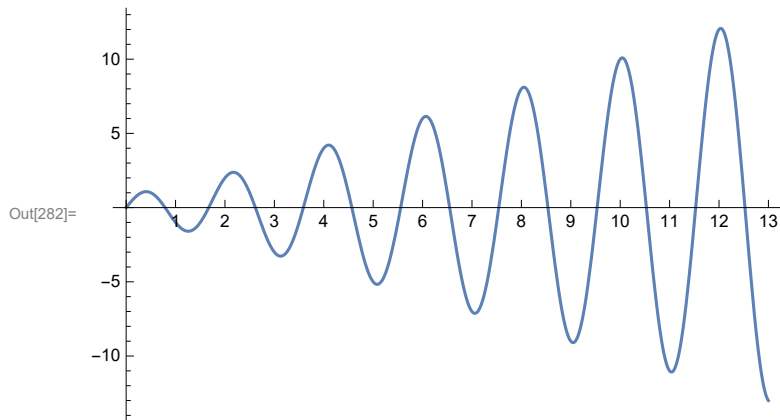
```
In[281]:= Det[{{1, -μ}, {Cos[π μ], Sin[π μ]}}]
```

```
Out[281]= μ Cos[π μ] + Sin[π μ]
```

The above function is the function whose zeros give as  $\mu^2$  the negative eigenvalues

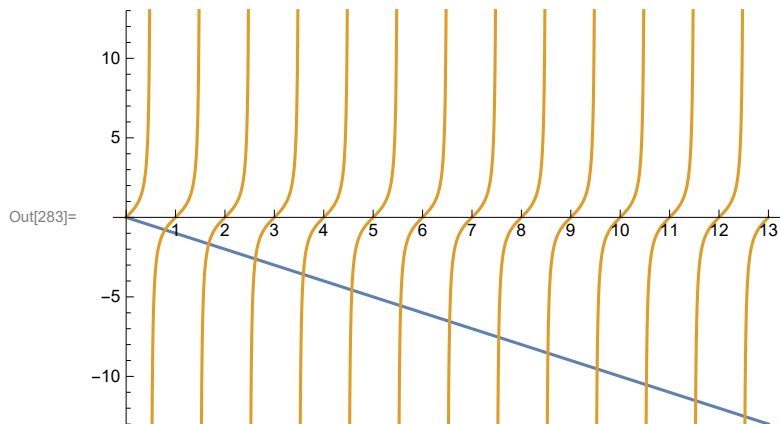
Getting solutions for  $\mu$  “visually”

```
In[282]:= Plot[μ Cos[π μ] + Sin[π μ], {μ, 0, 13}, Ticks → {Range[0, 20, 1], Automatic}]
```



Or, this plot might be more informative

```
In[283]:= Plot[{-μ, Tan[π μ]}, {μ, 0, 13},
  Ticks → {Range[0, 20, 1], Automatic}, PlotRange → {-13, 13}]
```



*Mathematica* has a command to find approximate solutions:

```
In[284]:= FindRoot[μ Cos[π μ] + Sin[π μ] == 0, {μ, .7}]
```

```
Out[284]= {μ → 0.787637}
```

In the command above we need to give an initial guess. Then *Mathematica* finds a much better approximation which is nearby. *Mathematica*'s approximate value above is given as a substitution rule. To get just the numeric value do

```
In[285]:= μ /. FindRoot[μ Cos[π μ] + Sin[π μ] == 0, {μ, .7}]
```

```
Out[285]= 0.787637
```

However, we need many solutions. To do that I make the above command as a pure function

```
In[286]:= ( $\mu$  /. FindRoot[ $\mu$  Cos[ $\pi$   $\mu$ ] + Sin[ $\pi$   $\mu$ ] == 0, { $\mu$ , #}]) &[0.7]
```

```
Out[286]= 0.787637
```

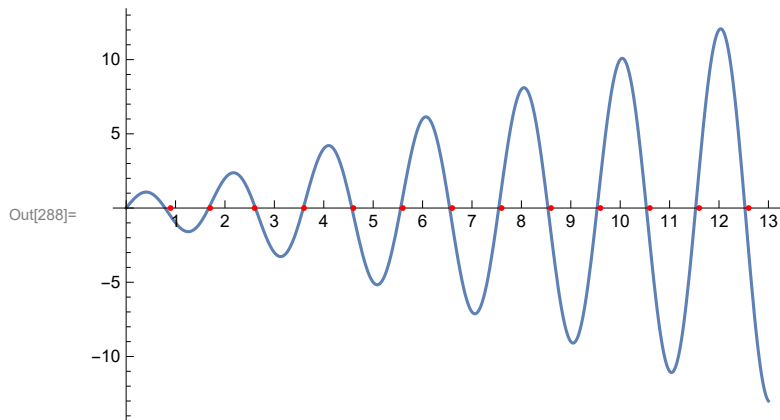
Next I make a list of initial guesses:

```
In[287]:= Join[{.9, 1.7}, Range[2.6, 20, 1]]
```

```
Out[287]= {0.9, 1.7, 2.6, 3.6, 4.6, 5.6, 6.6, 7.6, 8.6, 9.6,
  10.6, 11.6, 12.6, 13.6, 14.6, 15.6, 16.6, 17.6, 18.6, 19.6}
```

Next I verify that my initial guesses are close to the actual zeros of the function

```
In[288]:= Plot[ $\mu$  Cos[ $\pi$   $\mu$ ] + Sin[ $\pi$   $\mu$ ], { $\mu$ , 0, 13},
  Epilog -> {{Red, Point[{#, 0]}] & /@ Join[{.9, 1.7}, Range[2.6, 20, 1]]}},
  Ticks -> {Range[0, 20, 1], Automatic}]
```



Quite good guesses.

Then I “map” my “zero finding function” to this list

```
In[289]:= Map[( $\mu$  /. FindRoot[ $\mu$  Cos[ $\pi$   $\mu$ ] + Sin[ $\pi$   $\mu$ ] == 0, { $\mu$ , #}]) &, Join[{.9, 1.7}, Range[2.6, 20, 1]]]
```

```
Out[289]= {0.787637, 1.67161, 2.61621, 3.58655, 4.56859, 5.55668, 6.54824, 7.54196, 8.53712, 9.53327,
  10.5301, 11.5275, 12.5254, 13.5235, 14.5219, 15.5205, 16.5192, 17.5182, 18.5172, 19.5163}
```

Next I name these approximate eigenvalues. The named list of many  $\mu$ -s will be essential in what follows.

```
In[290]:= Clear[eves1];
eves1 =
```

```
Map[( $\mu$  /. FindRoot[ $\mu$  Cos[ $\pi$   $\mu$ ] + Sin[ $\pi$   $\mu$ ] == 0, { $\mu$ , #}]) &, Join[{.9, 1.7}, Range[2.6, 20, 1]]]
```

```
Out[291]= {0.787637, 1.67161, 2.61621, 3.58655, 4.56859, 5.55668, 6.54824, 7.54196, 8.53712, 9.53327,
  10.5301, 11.5275, 12.5254, 13.5235, 14.5219, 15.5205, 16.5192, 17.5182, 18.5172, 19.5163}
```

```
In[292]:= Length[eves1]
```

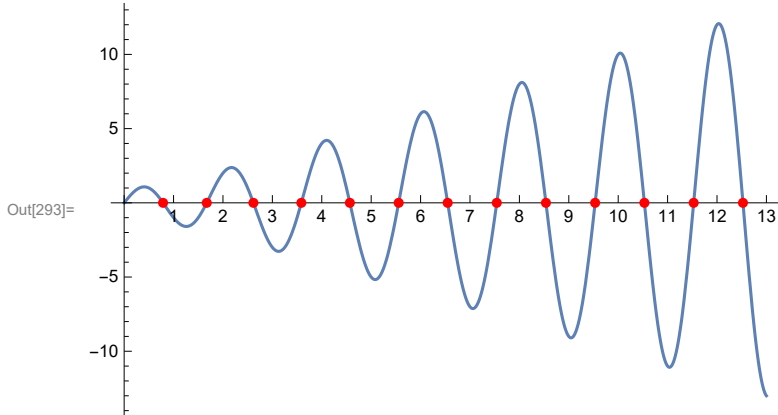
```
Out[292]= 20
```

Thus we found twenty  $\mu$ -s for the eigenvalues.

I visually inspect that there are no repetitions.

Now I can check graphically that I got the right solutions

```
In[293]:= Plot[ $\mu \text{Cos}[\pi \mu] + \text{Sin}[\pi \mu]$ , { $\mu$ , 0, 13},
  Epilog -> {{Red, PointSize[0.015], Point[{#, 0]} & /@ eves1}},
  Ticks -> {Range[0, 20, 1], Automatic}]
```



Looks good.

Let us apply the original function to these approximations:

```
In[294]:= (# Cos[ $\pi$  #] + Sin[ $\pi$  #]) & /@ eves1
```

Out[294]=  $\{2.22045 \times 10^{-16}, 1.55431 \times 10^{-15}, 3.88578 \times 10^{-15}, 6.66134 \times 10^{-16}, -1.11022 \times 10^{-15},$   
 $1.13243 \times 10^{-14}, -1.21014 \times 10^{-14}, -1.39888 \times 10^{-14}, 1.05471 \times 10^{-14}, 1.72085 \times 10^{-14},$   
 $4.94049 \times 10^{-14}, 5.9952 \times 10^{-15}, -1.88738 \times 10^{-14}, -2.64233 \times 10^{-14}, -5.75096 \times 10^{-14},$   
 $4.71845 \times 10^{-14}, 4.996 \times 10^{-14}, 2.20712 \times 10^{-13}, 1.9984 \times 10^{-15}, 1.07914 \times 10^{-13}\}$

```
In[295]:= Chop[(# Cos[ $\pi$  #] + Sin[ $\pi$  #]) & /@ eves1]
```

Out[295]=  $\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$

Quite good!

Based on the matrix that determined the eigenvalues we can now define the eigenfunctions: With  $\mu$  being the solution of  $\mu \text{Cos}[\pi \mu] + \text{Sin}[\pi \mu] = 0$ , a corresponding eigenfunction is found by solving for  $C_1$  and  $C_2$  the following system

```
In[296]:= MatrixForm[{{1, - $\mu$ }, {Cos[ $\pi \mu$ ], Sin[ $\pi \mu$ ]}}]
```

Out[296]/MatrixForm=

$$\begin{pmatrix} 1 & -\mu \\ \text{Cos}[\pi \mu] & \text{Sin}[\pi \mu] \end{pmatrix}$$

Since the determinant of this system is zero, the system has many solutions. A simple solution is  $C_1 = \mu$ ,  $C_2 = 1$ . Below are twenty eigenfunctions corresponding to the eigenvalues that we found.

```
In[297]= Clear[efs1];
```

```
efs1[x_] = (# Cos[# x] + Sin[# x]) & /@ eves1
```

```
Out[298]= {0.787637 Cos[0.787637 x] + Sin[0.787637 x], 1.67161 Cos[1.67161 x] + Sin[1.67161 x],
2.61621 Cos[2.61621 x] + Sin[2.61621 x], 3.58655 Cos[3.58655 x] + Sin[3.58655 x],
4.56859 Cos[4.56859 x] + Sin[4.56859 x], 5.55668 Cos[5.55668 x] + Sin[5.55668 x],
6.54824 Cos[6.54824 x] + Sin[6.54824 x], 7.54196 Cos[7.54196 x] + Sin[7.54196 x],
8.53712 Cos[8.53712 x] + Sin[8.53712 x], 9.53327 Cos[9.53327 x] + Sin[9.53327 x],
10.5301 Cos[10.5301 x] + Sin[10.5301 x], 11.5275 Cos[11.5275 x] + Sin[11.5275 x],
12.5254 Cos[12.5254 x] + Sin[12.5254 x], 13.5235 Cos[13.5235 x] + Sin[13.5235 x],
14.5219 Cos[14.5219 x] + Sin[14.5219 x], 15.5205 Cos[15.5205 x] + Sin[15.5205 x],
16.5192 Cos[16.5192 x] + Sin[16.5192 x], 17.5182 Cos[17.5182 x] + Sin[17.5182 x],
18.5172 Cos[18.5172 x] + Sin[18.5172 x], 19.5163 Cos[19.5163 x] + Sin[19.5163 x]}
```

Here is, for example, the third eigenfunction

```
In[299]= efs1[x][[3]]
```

```
Out[299]= 2.61621 Cos[2.61621 x] + Sin[2.61621 x]
```

Verify the eigenvalue equation:

```
In[300]= FullSimplify[Table[D[efs1[x][[k]], {x, 2}] + (eves1[[k]])^2 efs1[x][[k]], {k, 1, Length[eves1]}]]
```

```
Out[300]= {0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.}
```

Verify the boundary conditions

```
In[301]= Table[Chop[(((efs1[x][[k]]) /. {x -> 0}) - (D[efs1[x][[k]], {x, 1}] /. {x -> 0}),
((efs1[x][[k]]) /. {x -> Pi})}], {k, 1, Length[eves1]}]
```

```
Out[301]= {{0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0},
{0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}}
```

Let us verify if the eigenfunctions are mutually orthogonal:



```
In[302]= MatrixForm[Chop[Table[NIntegrate[efs1[x][[j]] × efs1[x][[k]], {x, 0, Pi}, MaxRecursion → 200, AccuracyGoal → 10, PrecisionGoal → 10], {j, 1, Length[eves1]}, {k, 1, Length[eves1]}]]]
```

```
Out[302]/MatrixForm=
{
  3.04528    0          0          0          0          0          0          0          0          0          0
    0    6.46002    0          0          0          0          0          0          0          0          0
    0    0    12.8222    0          0          0          0          0          0          0          0
    0    0    0    22.2765    0          0          0          0          0          0          0
    0    0    0    0    34.8565    0          0          0          0          0          0
    0    0    0    0    0    50.5717    0          0          0          0          0
    0    0    0    0    0    0    69.4256    0          0          0          0
    0    0    0    0    0    0    0    91.4195    0          0          0
    0    0    0    0    0    0    0    0    116.554    0          0
    0    0    0    0    0    0    0    0    0    144.83    0
    0    0    0    0    0    0    0    0    0    0    176.24
    0    0    0    0    0    0    0    0    0    0          0
    0    0    0    0    0    0    0    0    0    0          0
    0    0    0    0    0    0    0    0    0    0          0
    0    0    0    0    0    0    0    0    0    0          0
    0    0    0    0    0    0    0    0    0    0          0
    0    0    0    0    0    0    0    0    0    0          0
    0    0    0    0    0    0    0    0    0    0          0
    0    0    0    0    0    0    0    0    0    0          0
    0    0    0    0    0    0    0    0    0    0          0
    0    0    0    0    0    0    0    0    0    0          0
}
```

For later use we need the squares of the norms of the eigenfunctions

```
In[303]= Clear [efns1];
```

```
efns1 = Chop[Table[NIntegrate[efs1[x][[j]] × efs1[x][[j]], {x, 0, Pi}, MaxRecursion → 200, AccuracyGoal → 10, PrecisionGoal → 10], {j, 1, Length[eves1]}]]
```

```
Out[304]= {3.04528, 6.46002, 12.8222, 22.2765, 34.8565, 50.5717, 69.4256, 91.4195, 116.554, 144.83, 176.247, 210.805, 248.505, 289.346, 333.328, 380.453, 430.718, 484.126, 540.674, 600.365}
```

## Give initial conditions

Next we give our initial condition, the initial shape of the string. Notice that the initial velocity is 0, so we do not need to work with sine part of the time components.

```
In[305]= Clear [ff1];
```

$$ff1[x_] = \frac{1}{3} \left( x - \frac{\pi}{-1 - 2\pi} \right) (x - 1) (x - \text{Pi})$$

```
Out[306]=
```

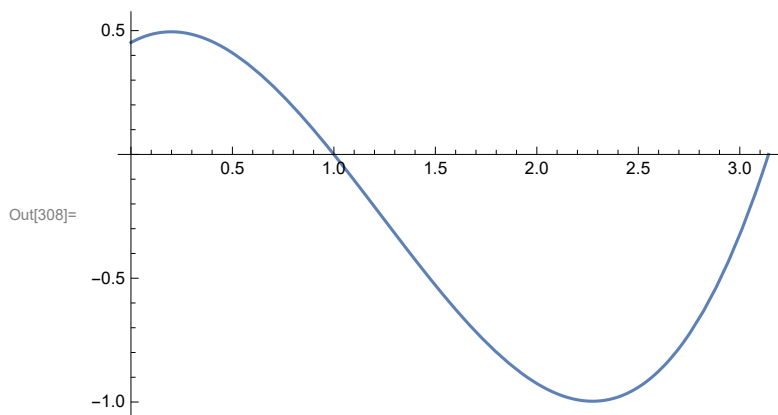
$$\frac{1}{3} (-1 + x) (-\pi + x) \left( -\frac{\pi}{-1 - 2\pi} + x \right)$$

I chose this function so that it satisfies the BCs.

```
In[307]= Simplify[ff1'[0] - ff1[0] == 0]
```

```
Out[307]= True
```

In[308]:= **Plot**[ff1[x], {x, 0, Pi}]



Here are the coefficients in the Fourier expansion for ff1

In[309]:= **Clear**[coefs1];

**coefs1 = Chop**[Table[NIntegrate[(ff1[x]) efs1[x][[j]], {x, 0, Pi},  
MaxRecursion → 200, AccuracyGoal → 10, PrecisionGoal → 10], {j, 1, Length[eves1]}]]

Out[309]= {-0.76951, 2.54647, -0.346054, 0.404428, -0.139934, 0.15144, -0.0739445,  
0.0779903, -0.0454394, 0.0473117, -0.0306808, 0.0316973, -0.0220841, 0.0226967,  
-0.0166473, 0.0170447, -0.0129939, 0.0132664, -0.0104223, 0.0106171}

I like to keep the above coefficients and the squares of the norms separate. In fact in the expansion we need to divide the above coefficients with the squares of the norms. Here is how we do it (somewhat weird, a list divided by a list, but it works)

**coefs1**

In[310]:=

**efns1**

Out[310]= {-0.25269, 0.39419, -0.0269886, 0.0181549, -0.00401456,  
0.00299455, -0.00106509, 0.000853103, -0.000389857, 0.000326671,  
-0.000174079, 0.000150363, -0.000088868, 0.0000784415, -0.0000499425,  
0.0000448011, -0.0000301681, 0.0000274027, -0.0000192765, 0.0000176844}

Next we define our approximation of the solution. Remember, our few solutions in x are in the table efs1[x][[k], k={1,...,20}, the corresponding time components are Cos[eves1[[k]] t], where eves1 is the list of twenty  $\mu$ -s. To get the solution these few functions need to be scaled by the coefficients in

$\frac{\text{coefs1}[[k]]}{\text{efns1}[[k]]}$ . So, our approximation of the solution is

```
In[311]:= Sum[ $\frac{\text{coefs1}[[k]]}{\text{efns1}[[k]]}$  efs1[x][[k]] * Cos[eves1[[k]] t], {k, 1, Length[eves1]}]
Out[311]:= -0.25269 Cos[0.787637 t] (0.787637 Cos[0.787637 x] + Sin[0.787637 x]) +
0.39419 Cos[1.67161 t] (1.67161 Cos[1.67161 x] + Sin[1.67161 x]) -
0.0269886 Cos[2.61621 t] (2.61621 Cos[2.61621 x] + Sin[2.61621 x]) +
0.0181549 Cos[3.58655 t] (3.58655 Cos[3.58655 x] + Sin[3.58655 x]) -
0.00401456 Cos[4.56859 t] (4.56859 Cos[4.56859 x] + Sin[4.56859 x]) +
0.00299455 Cos[5.55668 t] (5.55668 Cos[5.55668 x] + Sin[5.55668 x]) -
0.00106509 Cos[6.54824 t] (6.54824 Cos[6.54824 x] + Sin[6.54824 x]) +
0.000853103 Cos[7.54196 t] (7.54196 Cos[7.54196 x] + Sin[7.54196 x]) -
0.000389857 Cos[8.53712 t] (8.53712 Cos[8.53712 x] + Sin[8.53712 x]) +
0.000326671 Cos[9.53327 t] (9.53327 Cos[9.53327 x] + Sin[9.53327 x]) -
0.000174079 Cos[10.5301 t] (10.5301 Cos[10.5301 x] + Sin[10.5301 x]) +
0.000150363 Cos[11.5275 t] (11.5275 Cos[11.5275 x] + Sin[11.5275 x]) -
0.000088868 Cos[12.5254 t] (12.5254 Cos[12.5254 x] + Sin[12.5254 x]) +
0.0000784415 Cos[13.5235 t] (13.5235 Cos[13.5235 x] + Sin[13.5235 x]) -
0.0000499425 Cos[14.5219 t] (14.5219 Cos[14.5219 x] + Sin[14.5219 x]) +
0.0000448011 Cos[15.5205 t] (15.5205 Cos[15.5205 x] + Sin[15.5205 x]) -
0.0000301681 Cos[16.5192 t] (16.5192 Cos[16.5192 x] + Sin[16.5192 x]) +
0.0000274027 Cos[17.5182 t] (17.5182 Cos[17.5182 x] + Sin[17.5182 x]) -
0.0000192765 Cos[18.5172 t] (18.5172 Cos[18.5172 x] + Sin[18.5172 x]) +
0.0000176844 Cos[19.5163 t] (19.5163 Cos[19.5163 x] + Sin[19.5163 x])
```

There is much more compact way of delivering the same result without using summation. In fact that sum is written as a dot product.

```
In[312]:= (efs1[x] * ((Cos[# t]) & /@ eves1)) .  $\begin{pmatrix} \text{coefs1} \\ \text{efns1} \end{pmatrix}$ 
Out[312]:= -0.25269 Cos[0.787637 t] (0.787637 Cos[0.787637 x] + Sin[0.787637 x]) +
0.39419 Cos[1.67161 t] (1.67161 Cos[1.67161 x] + Sin[1.67161 x]) -
0.0269886 Cos[2.61621 t] (2.61621 Cos[2.61621 x] + Sin[2.61621 x]) +
0.0181549 Cos[3.58655 t] (3.58655 Cos[3.58655 x] + Sin[3.58655 x]) -
0.00401456 Cos[4.56859 t] (4.56859 Cos[4.56859 x] + Sin[4.56859 x]) +
0.00299455 Cos[5.55668 t] (5.55668 Cos[5.55668 x] + Sin[5.55668 x]) -
0.00106509 Cos[6.54824 t] (6.54824 Cos[6.54824 x] + Sin[6.54824 x]) +
0.000853103 Cos[7.54196 t] (7.54196 Cos[7.54196 x] + Sin[7.54196 x]) -
0.000389857 Cos[8.53712 t] (8.53712 Cos[8.53712 x] + Sin[8.53712 x]) +
0.000326671 Cos[9.53327 t] (9.53327 Cos[9.53327 x] + Sin[9.53327 x]) -
0.000174079 Cos[10.5301 t] (10.5301 Cos[10.5301 x] + Sin[10.5301 x]) +
0.000150363 Cos[11.5275 t] (11.5275 Cos[11.5275 x] + Sin[11.5275 x]) -
0.000088868 Cos[12.5254 t] (12.5254 Cos[12.5254 x] + Sin[12.5254 x]) +
0.0000784415 Cos[13.5235 t] (13.5235 Cos[13.5235 x] + Sin[13.5235 x]) -
0.0000499425 Cos[14.5219 t] (14.5219 Cos[14.5219 x] + Sin[14.5219 x]) +
0.0000448011 Cos[15.5205 t] (15.5205 Cos[15.5205 x] + Sin[15.5205 x]) -
0.0000301681 Cos[16.5192 t] (16.5192 Cos[16.5192 x] + Sin[16.5192 x]) +
0.0000274027 Cos[17.5182 t] (17.5182 Cos[17.5182 x] + Sin[17.5182 x]) -
0.0000192765 Cos[18.5172 t] (18.5172 Cos[18.5172 x] + Sin[18.5172 x]) +
0.0000176844 Cos[19.5163 t] (19.5163 Cos[19.5163 x] + Sin[19.5163 x])
```

Next we define our approximation of the solution as a function with a name:

```
In[313]:= Clear[uu1]; uu1[x_, t_] = (efns1[x] * ((Cos[# t] & /@ eves1)) . (
  (coefs1)
  (efns1)
```

```
Out[313]:= -0.25269 Cos[0.787637 t] (0.787637 Cos[0.787637 x] + Sin[0.787637 x]) +
  0.39419 Cos[1.67161 t] (1.67161 Cos[1.67161 x] + Sin[1.67161 x]) -
  0.0269886 Cos[2.61621 t] (2.61621 Cos[2.61621 x] + Sin[2.61621 x]) +
  0.0181549 Cos[3.58655 t] (3.58655 Cos[3.58655 x] + Sin[3.58655 x]) -
  0.00401456 Cos[4.56859 t] (4.56859 Cos[4.56859 x] + Sin[4.56859 x]) +
  0.00299455 Cos[5.55668 t] (5.55668 Cos[5.55668 x] + Sin[5.55668 x]) -
  0.00106509 Cos[6.54824 t] (6.54824 Cos[6.54824 x] + Sin[6.54824 x]) +
  0.000853103 Cos[7.54196 t] (7.54196 Cos[7.54196 x] + Sin[7.54196 x]) -
  0.000389857 Cos[8.53712 t] (8.53712 Cos[8.53712 x] + Sin[8.53712 x]) +
  0.000326671 Cos[9.53327 t] (9.53327 Cos[9.53327 x] + Sin[9.53327 x]) -
  0.000174079 Cos[10.5301 t] (10.5301 Cos[10.5301 x] + Sin[10.5301 x]) +
  0.000150363 Cos[11.5275 t] (11.5275 Cos[11.5275 x] + Sin[11.5275 x]) -
  0.000088868 Cos[12.5254 t] (12.5254 Cos[12.5254 x] + Sin[12.5254 x]) +
  0.0000784415 Cos[13.5235 t] (13.5235 Cos[13.5235 x] + Sin[13.5235 x]) -
  0.0000499425 Cos[14.5219 t] (14.5219 Cos[14.5219 x] + Sin[14.5219 x]) +
  0.0000448011 Cos[15.5205 t] (15.5205 Cos[15.5205 x] + Sin[15.5205 x]) -
  0.0000301681 Cos[16.5192 t] (16.5192 Cos[16.5192 x] + Sin[16.5192 x]) +
  0.0000274027 Cos[17.5182 t] (17.5182 Cos[17.5182 x] + Sin[17.5182 x]) -
  0.0000192765 Cos[18.5172 t] (18.5172 Cos[18.5172 x] + Sin[18.5172 x]) +
  0.0000176844 Cos[19.5163 t] (19.5163 Cos[19.5163 x] + Sin[19.5163 x])
```

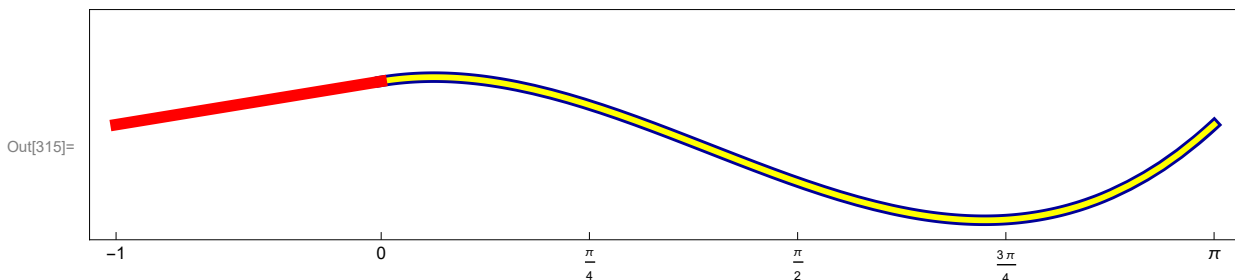
Does it evaluate properly?

```
In[314]:= uu1[1, 0.6]
```

```
Out[314]:= -0.0851306
```

Does it approximate the function ff1 well?

```
In[315]:= Plot[Evaluate[{{ff1[x]}, uu1[x, 0]}], {x, 0, Pi},
  PlotStyle -> {{Thickness[0.01], RGBColor[0, 0, 0.6]}, {Thickness[0.005], Yellow}},
  Epilog -> {{Red, Thickness[0.01], Line[{{-1, 0}, {0, uu1[0, 0]}}]}},
  PlotRange -> {{-1.1, Pi + 0.1}, {-1.2, 1.2}}, AspectRatio -> 1 / 5, Frame -> True,
  FrameTicks -> {{{}}, {}}, {Join[Range[0, Pi, Pi / 4], {-1}], {}},
  Axes -> False, ImageSize -> 600]
```

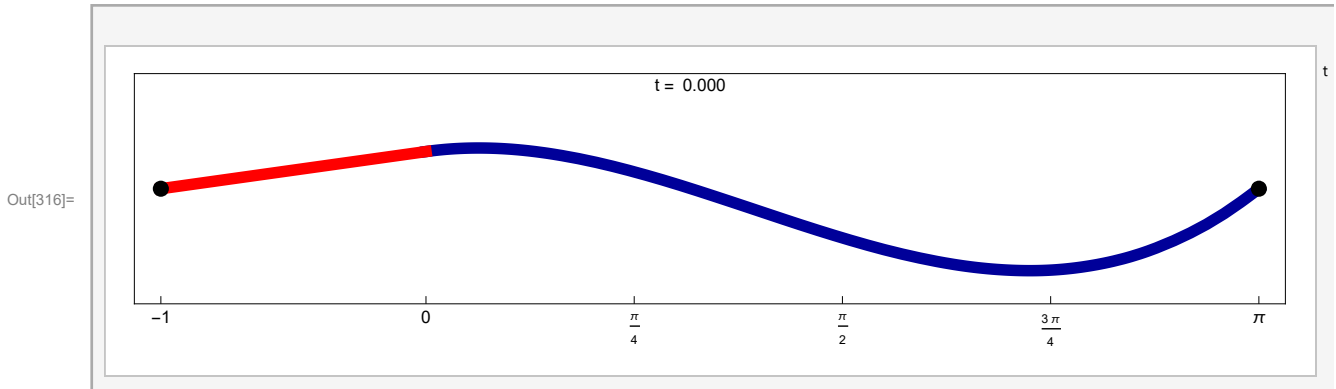


Ready to oscillate?

```

In[316]:= Manipulate[Plot[Evaluate[uu1[x, t]], {x, 0, Pi},
  PlotStyle -> {{Thickness[0.01], RGBColor[0, 0, 0.6]}},
  Epilog -> {
    {Red, Thickness[0.01], Line[{{-1, 0}, {0, Evaluate[uu1[0, t]}]}]},
    {PointSize[0.014], Point[#] & /@ {{-1, 0}, {Pi, 0}}},
    {Text["t =", {0.9, 1.25}], Text[NumberForm[N[t], {4, 3}], {1.05, 1.25}]}
  },
  PlotRange -> {{-1.1, Pi + 0.1}, {-1.4, 1.4}}, AspectRatio -> 1 / 5,
  Frame -> True, FrameTicks -> {{}, {}}, {Join[Range[0, Pi, Pi / 4], {-1}], {}},
  Axes -> False, ImageSize -> 600], {t, 0, 7.3, 0.05}]

```



Try different esthetics:

```

In[317]:= Module[{t}, t = 2 Pi;
  Plot[Evaluate[uu1[x, t]], {x, 0, Pi},
  PlotStyle -> {{Thickness[0.01], RGBColor[1, 0.5, 0]}},
  Epilog -> {
    {Red, Thickness[0.01], Line[{{-1, 0}, {0, Evaluate[uu1[0, t]}]}]},
    {PointSize[0.012], White, Point[#] & /@ {{-1, 0}, {Pi, 0}}},
    {Text["t =", {0.925 + 0.08, 1.25}], BaseStyle ->
      {FontWeight -> "Normal", FontColor -> RGBColor[1, 1, 1]}},
    Text[NumberForm[N[2 t], {3, 1}], {1.011 + 0.08, 1.25}],
    BaseStyle -> {FontWeight -> "Normal", FontColor -> RGBColor[1, 1, 1]}]}
  },
  PlotRange -> {{-1.1, Pi + 0.1}, {-1.4, 1.4}}, AspectRatio -> 1 / 5,
  Frame -> True, FrameTicks -> {{}, {}}, {Join[Range[0, Pi, Pi / 4], {-1}], {}},
  Axes -> False, ImageSize -> 800, Background -> Black]

```



In[318]= 2.14 / 2

Out[318]= 1.07

Another way to make an animation is to produce a lot of pictures and put them in a table. We do not want to see the result since it contains 147 pictures; therefore the semi-column is essential at the end of the next cell.

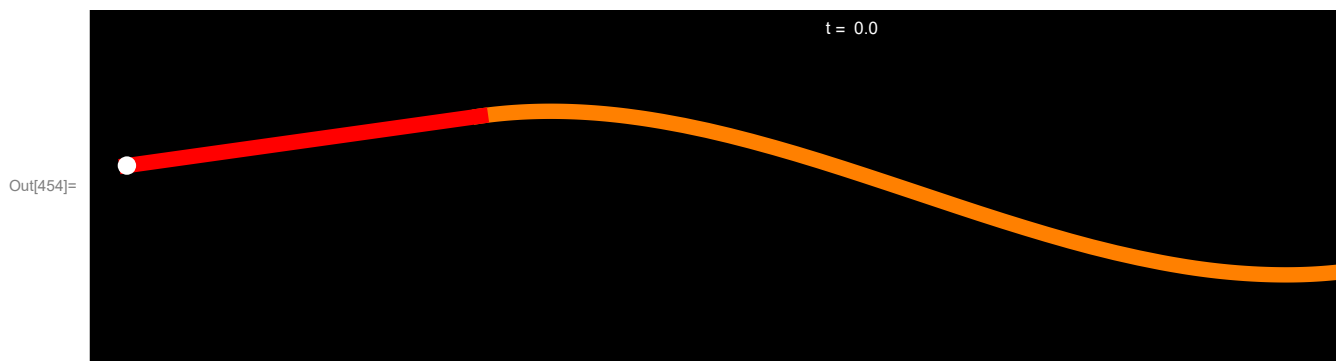
```
In[319]= aniuu1 = Table[Plot[Evaluate[uu1[x, t]],
  {x, 0, Pi}, PlotStyle -> {{Thickness[0.01], RGBColor[0, 0, 0.6]}},
  Epilog -> {
    {Red, Thickness[0.01], Line[{{-1, 0}, {0, Evaluate[uu1[0, t]}]}]},
    {PointSize[0.012], Point[#] & /@ {{-1, 0}, {Pi, 0}}},
    {Text["t =", {0.925 + 0.08, 1.25}], Text[NumberForm[N[t], {4, 3}],
      {1.011 + 0.08, 1.25}]}
  },
  PlotRange -> {{-1.1, Pi + 0.1}, {-1.4, 1.4}}, AspectRatio -> 1 / 5,
  Frame -> True, FrameTicks -> {{{}, {}}, {Join[Range[0, Pi, Pi / 4], {-1}], {}},
  Axes -> False, ImageSize -> 800], {t, 0, 7.3, 0.05}];
```

```
In[320]= aniuu1BB = Table[Plot[Evaluate[uu1[x, t]],
  {x, 0, Pi}, PlotStyle -> {{Thickness[0.01], RGBColor[1, 0.5, 0]}},
  Epilog -> {
    {Red, Thickness[0.01], Line[{{-1, 0}, {0, Evaluate[uu1[0, t]}]}]},
    {PointSize[0.012], White, Point[#] & /@ {{-1, 0}, {Pi, 0}}},
    {Text["t =", {0.925 + 0.08, 1.25}], BaseStyle -> {FontWeight -> "Normal", FontColor ->
      RGBColor[1, 1, 1]}], Text[NumberForm[N[t], {3, 1}], {1.011 + 0.08, 1.25},
      BaseStyle -> {FontWeight -> "Normal", FontColor -> RGBColor[1, 1, 1]}]}
  },
  PlotRange -> {{-1.1, Pi + 0.1}, {-1.4, 1.4}}, AspectRatio -> 1 / 5,
  Frame -> True, FrameTicks -> {{{}, {}}, {Join[Range[0, Pi, Pi / 4], {-1}], {}},
  Axes -> False, ImageSize -> 800, Background -> Black], {t, 0, 7.3, 0.05}];
```

In[321]= Length[aniuu1BB]

Out[321]= 147

In[454]= Show[aniuu1BB[[1]], ImageSize -> 800]



In[323]= ListAnimate[Show[#, ImageSize -> 600] & /@ aniuu1BB]

In[324]=  $2 \times 7.3 / 147$

Out[324]= 0.0993197

In[450]= `dds = 0.1 & /@ Range[Length[aniu1BB]]; (* duration of each frame that we want*)`

In[455]= `SetDirectory[NotebookDirectory[]]`

Out[455]= C:\Dropbox\Work\myweb\Courses\Math\_pages\Math\_430

```
(* Export["aniu1s1.gif", aniu1[[1]], "ImageSize"→800];
Export["aniu1BBAni.gif", aniu1BB, "AnimationRepetitions"→Infinity,
"ImageSize"→800, "DisplayDurations"→dds] *)
```

```
(* Export["aniu1BBs1.gif", aniu1BB[[1]], "ImageSize"→800];
```

```
Export["aniu1BBAni.gif", aniu1BB, "AnimationRepetitions"→Infinity,
"ImageSize"→800, "DisplayDurations"→dds] *)
```

Out[453]= aniu1BBs1.gif

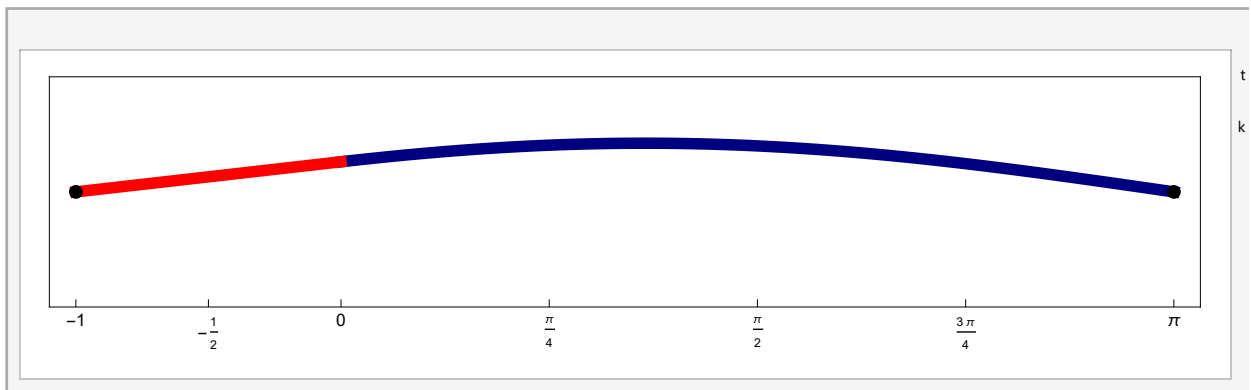
## Normal mode of vibrations

And finally here are the normal modes of vibrations. Those are the few solutions that we found before

In[329]= `Clear[nmv1]; nmv1[x_, t_] := (efs1[x] * ( (  $\frac{1}{\sqrt{\#^2 + 4}}$  Cos[# t] ) & /@ eves1 ))`

In[330]= `Manipulate[Plot[Evaluate[nmv1[x, t]][[k]],
{x, 0, Pi}, PlotStyle → {{Thickness[0.01], RGBColor[0, 0, 0.5]}},
Epilog → {
{Red, Thickness[0.01], Line[{{-1, 0}, {0, Evaluate[nmv1[0, t]][[k]]}}]},
{PointSize[0.012], Point[#] & /@ {{-1, 0}, {Pi, 0}}}
}, PlotRange → {{-1.1, Pi + .1}, {-1.4, 1.4}}, AspectRatio → 1 / 5, Frame → True,
FrameTicks → {{{}, {}}, {Join[Range[0, Pi, Pi / 4], {-1 / 2, -1}], {}}, Axes → False,
ImageSize → 600], {t, 0, 4 Pi, N[Pi / 64]}, {{k, 1}, Range[10], ControlType → Setter}]`

Out[330]=



## Solution 2

We want to solve

$$\partial_{tt}u[x, t] = \partial_{xx}u[x, t], \quad 0 \leq x \leq \pi, \quad t \geq 0, \quad u[0, t] + \partial_x u[0, t] = 0, \\ u[\pi, t] = 0, \quad u[x, 0] = \text{ff1}[x], \quad \partial_t u[x, 0] = 0.$$

So we need to solve the Sturm-Liouville eigenvalue problem

$$-y(x)'' = \lambda x y(x), \quad 0 \leq x \leq \pi, \\ y(0) + y'(0) = 0, \\ y(\pi) = 0.$$

### Look for negative eigenvalues

We look for negative eigenvalues  $\lambda = -\mu^2$

$$-y(x)'' = -\mu^2 y(x), \quad 0 \leq x \leq \pi, \\ y(0) + y'(0) = 0, \\ y(\pi) = 0.$$

```
In[331]:= DSolve[y''[x] == μ² y[x], y[x], x]
```

```
Out[331]= {{y[x] → e^{x μ} c₁ + e^{-x μ} c₂}}
```

I do not know how to force Mathematica into working with hyperbolic functions. So I force it by hand:

```
In[332]:= Simplify[(ExpToTrig[DSolve[y''[x] == μ² y[x], y[x], x]]) /. \\ {C[1] + C[2] → C[1], C[1] - C[2] → C[2]}
```

```
Out[332]= {{y[x] → c₁ Cosh[x μ] + c₂ Sinh[x μ]}}
```

Substitute the fundamental solution ( $\text{Cosh}[\mu x] C[1] + \text{Sinh}[\mu x] C[2]$ ) into the boundary conditions and collect the expression with  $C[1]$  and  $C[2]$ . This will give us two homogeneous equations.

```
In[333]:= Collect[(Cosh[μ x] C[1] + Sinh[μ x] C[2]) /. {x → 0} + \\ ((D[(Cosh[μ x] C[1] + Sinh[μ x] C[2]), x]) /. {x → 0}), {C[1], C[2]}
```

```
Out[333]= c₁ + μ c₂
```

```
In[334]:= Collect[(Cosh[μ x] C[1] + Sinh[μ x] C[2]) /. {x → Pi}, {C[1], C[2]}
```

```
Out[334]= c₁ Cosh[π μ] + c₂ Sinh[π μ]
```

We want nontrivial solutions for  $C[1]$  and  $C[2]$  of the system

$$C[1] + \mu C[2] = 0 \\ C[1] \text{Cosh}[\pi \mu] + C[2] \text{Sin}[\pi \mu] = 0$$

For a nontrivial solution to exist we need the determinant to be equal to 0

```
In[335]:= Det[{{1, μ}, {Cosh[Pi μ], Sinh[Pi μ]}}]
```

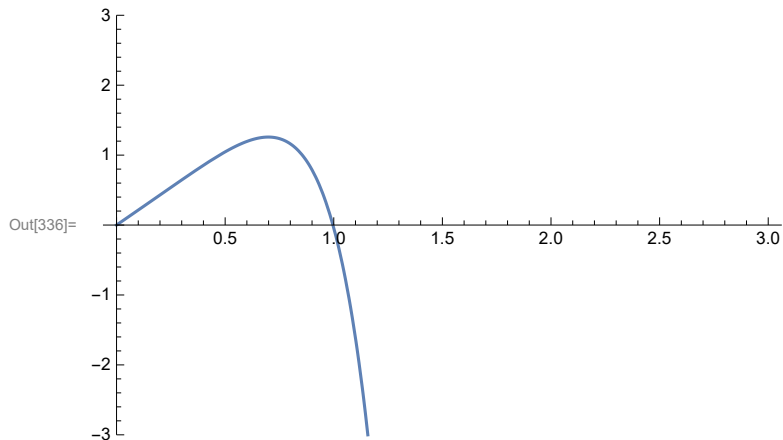
```
Out[335]= -μ Cosh[π μ] + Sinh[π μ]
```



The above function is the function whose zeros give as  $-\mu^2$  the negative eigenvalues

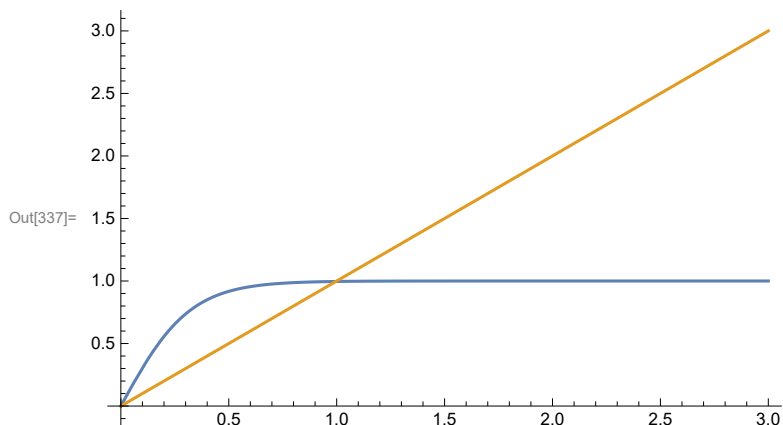
Since the above function has no positive zeros there are no negative evs. Below is a graphical verification of this claim.

```
In[336]:= Plot[-μ Cosh[π μ] + Sinh[π μ], {μ, 0, 3}, PlotRange → {-3, 3}]
```



The equation for the eigenvalue can be rewritten as  $\text{Tanh}[\pi \mu] = \mu$

```
In[337]:= Plot[{Tanh[π μ], μ}, {μ, 0, 3}]
```



```
In[338]:= evn2 = μ /. FindRoot[-μ Cosh[π μ] + Sinh[π μ] == 0, {μ, 1}]
```

Out[338]= 0.996182

The corresponding eigenfunction is

```
In[339]:= efn2[x_] = evn2 Cosh[evn2 x] - Sinh[evn2 x]
```

Out[339]= 0.996182 Cosh[0.996182 x] - Sinh[0.996182 x]

Verify

```
In[340]:= FullSimplify[D[efn2[x], {x, 2}] - evn22 efn2[x]]
```

Out[340]= 0.

## Zero is not an eigenvalues

Is zero an eigenvalue?

$$-y(x)'' = 0, \quad 0 \leq x \leq \pi,$$

$$y(0) + y'(0) = 0,$$

$$y(\pi) = 0.$$

In[341]:= **DSolve**[**y''[x] == 0, y[x], x]**

Out[341]= **{ {y[x] -> C<sub>1</sub> + x C<sub>2</sub> } }**

Substitute into the boundary conditions

In[342]:= **Collect**[**((C[1] + x C[2]) /. {x -> 0}) + ((D[C[1] + x C[2], x]) /. {x -> 0}), {C[1], C[2]}**]

Out[342]= **C<sub>1</sub> + C<sub>2</sub>**

In[343]:= **Collect**[**((C[1] + x C[2]) /. {x -> Pi}), {C[1], C[2]}**]

Out[343]= **C<sub>1</sub> + π C<sub>2</sub>**

We need a nontrivial solution for C[1] and C[2] of the system

$$C[1] + C[2] = 0$$

$$C[1] + \pi C[2] = 0$$

Since the determinant

In[344]:= **Det**[**{ {1, 1}, {1, Pi} }**]

Out[344]= **-1 + π**

is nonzero, the system has no nontrivial solutions, thus zero is not an eigenvalue.

## Look for positive eigenvalues

We look for positive eigenvalues  $\lambda = \mu^2$

$$-y(x)'' = \mu^2 y(x), \quad 0 \leq x \leq \pi,$$

$$y(0) + y'(0) = 0,$$

$$y(\pi) = 0.$$

In[345]:= **DSolve**[**-y''[x] == μ<sup>2</sup> y[x], y[x], x]**

Out[345]= **{ {y[x] -> C<sub>1</sub> Cos[x μ] + C<sub>2</sub> Sin[x μ] } }**

Substitute into the boundary conditions

In[346]:= **Collect**[**((C[1] Cos[x μ] + C[2] Sin[x μ]) /. {x -> 0}) + ((D[C[1] Cos[x μ] + C[2] Sin[x μ], x]) /. {x -> 0}), {C[1], C[2]}**]

Out[346]= **C<sub>1</sub> + μ C<sub>2</sub>**

```
In[347]= Collect[ ((C[1] Cos[x μ] + C[2] Sin[x μ]) /. {x → Pi}), {C[1], C[2]}]
```

```
Out[347]= c1 Cos[π μ] + c2 Sin[π μ]
```

We want nontrivial solutions for C[1] and C[2] of the system

$$C[1] + \mu C[2] = 0$$

$$C[1] \cos[\pi \mu] + C[2] \sin[\pi \mu] = 0$$

For a nontrivial solution to exist we need the determinant to be equal to 0

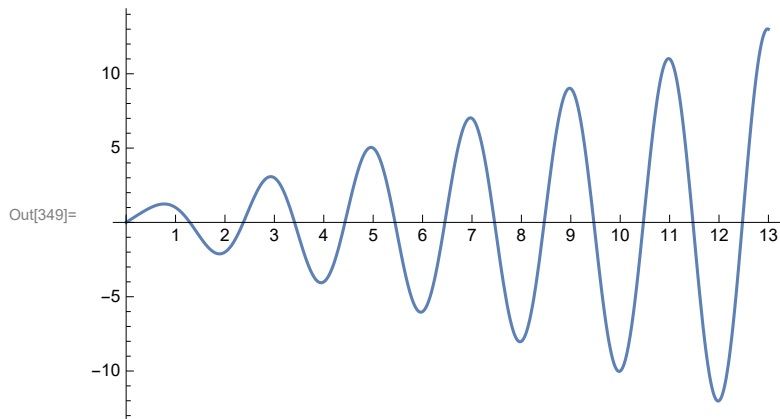
```
In[348]= Det[{{1, μ}, {Cos[π μ], Sin[π μ]}}]
```

```
Out[348]= -μ Cos[π μ] + Sin[π μ]
```

The above function is the function whose zeros give as  $\mu^2$  the negative eigenvalues

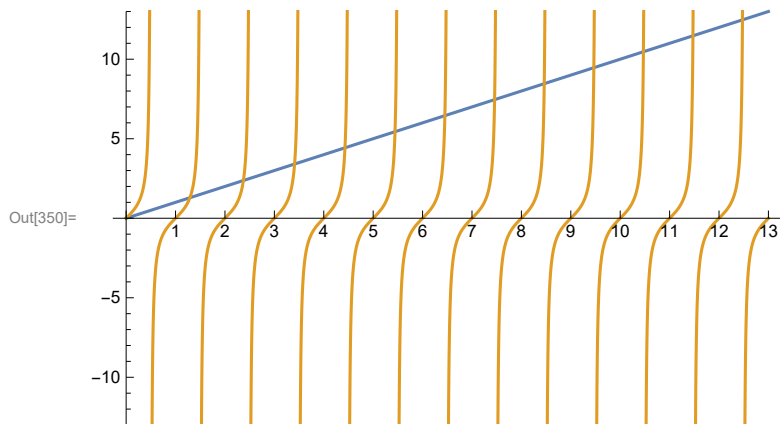
Getting solutions for  $\mu$  “visually”

```
In[349]= Plot[-μ Cos[π μ] + Sin[π μ], {μ, 0, 13}, Ticks → {Range[0, 20, 1], Automatic}]
```



Or, more informative graph

```
In[350]= Plot[{μ, Sin[π μ]/Cos[π μ]}, {μ, 0, 13}, Ticks → {Range[0, 20, 1], Automatic}, PlotRange → {-13, 13}]
```



*Mathematica* has a command to find approximate solutions:

```
In[351]= FindRoot[-μ Cos[π μ] + Sin[π μ] == 0, {μ, 1.2}]
```

```
Out[351]= {μ → 1.29011}
```

In the command above we need to give an initial guess. Then *Mathematica* finds a much better approximation which is nearby. *Mathematica*'s approximate value above is given as a substitution rule. To get just the numeric value do

```
In[352]= μ /. FindRoot[-μ Cos[π μ] + Sin[π μ] == 0, {μ, 1.2}]
```

```
Out[352]= 1.29011
```

However, we need many solutions. To do that I make the above command as a pure function

```
In[353]= (μ /. FindRoot[-μ Cos[π μ] + Sin[π μ] == 0, {μ, #}]) &[1.2]
```

```
Out[353]= 1.29011
```

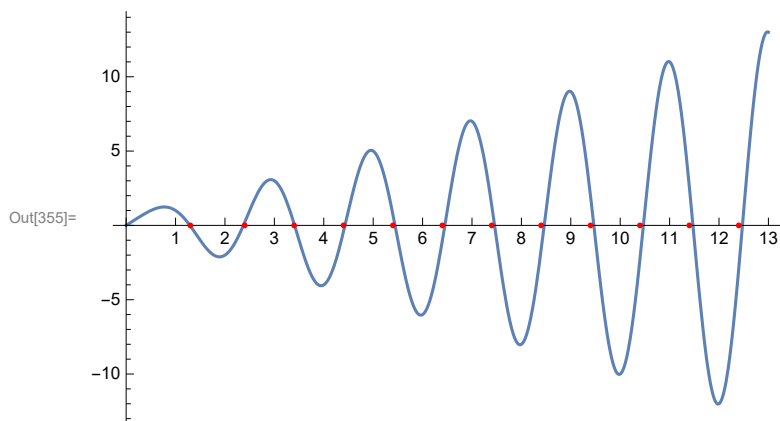
Next I make a list of initial guesses:

```
In[354]= Join[{1.3}, Range[2.4, 20, 1]]
```

```
Out[354]= {1.3, 2.4, 3.4, 4.4, 5.4, 6.4, 7.4, 8.4, 9.4,
  10.4, 11.4, 12.4, 13.4, 14.4, 15.4, 16.4, 17.4, 18.4, 19.4}
```

Next I verify that my initial guesses are close to the actual zeros of the function

```
In[355]= Plot[-μ Cos[π μ] + Sin[π μ], {μ, 0, 13},
  Epilog → {{Red, Point[{#, 0]} & /@ Join[{1.3}, Range[2.4, 20, 1]]}},
  Ticks → {Range[0, 20, 1], Automatic}]
```



Then I “map” my “zero finding function” to this list

```
In[356]= Map[(μ /. FindRoot[-μ Cos[π μ] + Sin[π μ] == 0, {μ, #}]) &, Join[{1.3}, Range[2.4, 20, 1]]]
```

```
Out[356]= {1.29011, 2.37305, 3.40918, 4.42932, 5.44216, 6.45105, 7.45757, 8.46256, 9.4665, 10.4697,
  11.4723, 12.4745, 13.4764, 14.478, 15.4795, 16.4807, 17.4818, 18.4828, 19.4837}
```

Next I name these approximate eigenvalues. The named list of many  $\mu$ -s will be essential in what follows.

```
In[357]:= Clear[eves2];
eves2 =
  (μ /. FindRoot[- μ Cos[π μ] + Sin[π μ] == 0, {μ, #}]) & /@ Join[{1.3}, Range[2.4, 20, 1]]
Out[358]= {1.29011, 2.37305, 3.40918, 4.42932, 5.44216, 6.45105, 7.45757, 8.46256, 9.4665, 10.4697,
  11.4723, 12.4745, 13.4764, 14.478, 15.4795, 16.4807, 17.4818, 18.4828, 19.4837}
```

```
In[359]:= Length[eves2]
```

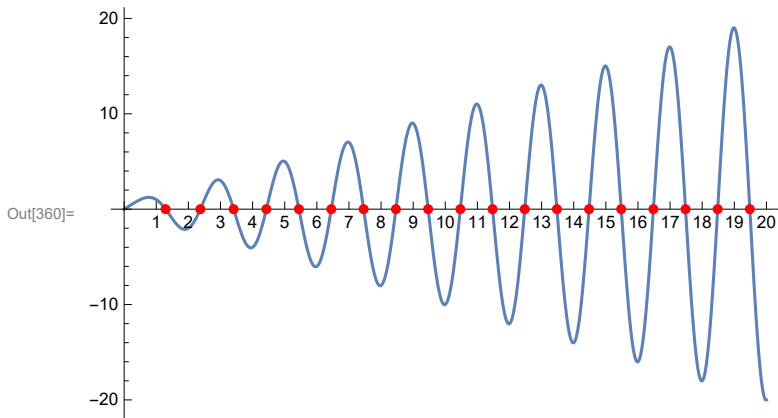
```
Out[359]= 19
```

Thus we found nineteen eigenvalues.

I visually inspect that there are no repetitions.

Now I can check graphically that I got the right solutions

```
In[360]:= Plot[- μ Cos[π μ] + Sin[π μ], {μ, 0, 20},
  Epilog -> {{Red, PointSize[0.015], Point[{#, 0]} & /@ eves2}},
  Ticks -> {Range[0, 20, 1], Automatic}]
```



Looks good.

Let us apply the original function to these approximations:

```
In[361]:= Chop[(- # Cos[π #] + Sin[π #]) & /@ eves2]
Out[361]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Quite good!

Based on the matrix that determined the eigenvalues we can now define the eigenfunctions: With  $\mu$  being the solution of  $\mu \cos[\pi \mu] + \sin[\pi \mu] = 0$ , a corresponding eigenfunction is found by solving for  $C_1$  and  $C_2$  the following system

```
In[362]:= MatrixForm[{{1, μ}, {Cos[π μ], Sin[π μ]}}]
Out[362]//MatrixForm=

$$\begin{pmatrix} 1 & \mu \\ \cos[\pi \mu] & \sin[\pi \mu] \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

```

Since the determinant of this system is zero, the system has many solutions. A simple solution is  $C_1 = \mu$ ,  $C_2 = -1$ . Below are twenty eigenfunctions corresponding to the eigenvalues that we found.

```
In[363]:= Clear[efs2];
efs2[x_] = (# Cos[# x] - Sin[# x]) & /@ eves2
Out[364]= {1.29011 Cos[1.29011 x] - Sin[1.29011 x],
2.37305 Cos[2.37305 x] - Sin[2.37305 x], 3.40918 Cos[3.40918 x] - Sin[3.40918 x],
4.42932 Cos[4.42932 x] - Sin[4.42932 x], 5.44216 Cos[5.44216 x] - Sin[5.44216 x],
6.45105 Cos[6.45105 x] - Sin[6.45105 x], 7.45757 Cos[7.45757 x] - Sin[7.45757 x],
8.46256 Cos[8.46256 x] - Sin[8.46256 x], 9.4665 Cos[9.4665 x] - Sin[9.4665 x],
10.4697 Cos[10.4697 x] - Sin[10.4697 x], 11.4723 Cos[11.4723 x] - Sin[11.4723 x],
12.4745 Cos[12.4745 x] - Sin[12.4745 x], 13.4764 Cos[13.4764 x] - Sin[13.4764 x],
14.478 Cos[14.478 x] - Sin[14.478 x], 15.4795 Cos[15.4795 x] - Sin[15.4795 x],
16.4807 Cos[16.4807 x] - Sin[16.4807 x], 17.4818 Cos[17.4818 x] - Sin[17.4818 x],
18.4828 Cos[18.4828 x] - Sin[18.4828 x], 19.4837 Cos[19.4837 x] - Sin[19.4837 x]}
```

Here is, for example, the third eigenfunction

```
In[365]:= efs2[x][[3]]
Out[365]= 3.40918 Cos[3.40918 x] - Sin[3.40918 x]
```

Verify the eigenvalue equation:

```
In[366]:= FullSimplify[Table[D[efs2[x][[k]], {x, 2}] + (eves2[[k]])^2 efs2[x][[k]], {k, 1, Length[eves2]}]]
Out[366]= {0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.}
```

Verify the boundary conditions

```
In[367]:= Table[Chop[(((efs2[x][[k]]) /. {x -> 0}) + (D[efs2[x][[k]], {x, 1}]) /. {x -> 0}),
((efs2[x][[k]]) /. {x -> Pi})}], {k, 1, Length[eves2]}]
Out[367]= {{0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0},
{0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}, {0, 0}}
```

Let us verify if the eigenfunctions are mutually orthogonal:

```
In[368]= Chop[Table[NIntegrate[efs2[x][[j]] × efs2[x][[k]], {x, 0, Pi}, MaxRecursion → 200,
  AccuracyGoal → 10, PrecisionGoal → 10], {j, 1, Length[eves2]}, {k, 1, Length[eves2]}]]
```

```
Out[368]= {{3.6852, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 9.91655, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 19.3274, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 31.8881, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 47.5932, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 66.4411, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 88.4312, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 113.563, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 141.837, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 173.253, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 207.81, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 245.509, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 286.349, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 330.332, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 377.455, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 427.721, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 481.128, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 537.676, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 597.367}}
```

We will need the integrals of the squares for later

```
In[369]= efsn2 = Chop[Table[NIntegrate[efs2[x][[j]] × efs2[x][[j]], {x, 0, Pi},
  MaxRecursion → 200, AccuracyGoal → 10, PrecisionGoal → 10], {j, 1, Length[eves2]}]]
```

```
Out[369]= {3.6852, 9.91655, 19.3274, 31.8881, 47.5932, 66.4411, 88.4312, 113.563, 141.837, 173.253,
  207.81, 245.509, 286.349, 330.332, 377.455, 427.721, 481.128, 537.676, 597.367}
```

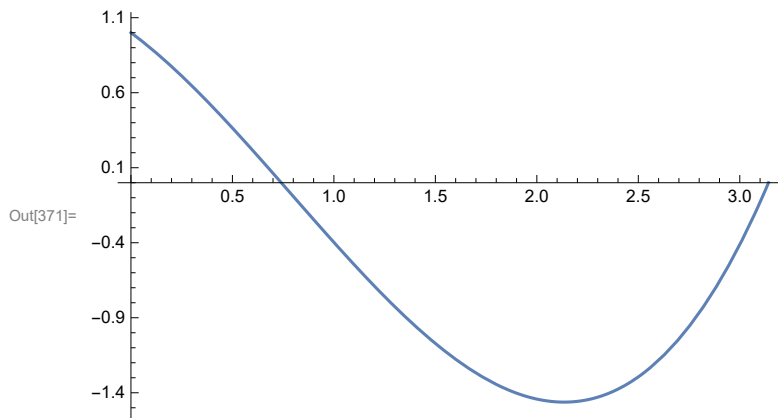
## Initial condition

```
In[370]= ff2[x_] = 
$$\left( .2869 \#^3 + \frac{-1 + \pi - .2869 \pi^3}{\pi^2} \#^2 - 1 \# + 1 \right) \&[x]$$

```

```
Out[370]= 1 - x - 0.684334 x2 + 0.2869 x3
```

```
In[371]= Plot[ff2[x], {x, 0, Pi}]
```



```
In[372]= coefs2 = Chop[Table[NIntegrate[(ff2[x]) efs2[x][[j]], {x, 0, Pi},  
MaxRecursion → 200, AccuracyGoal → 10, PrecisionGoal → 10], {j, 1, Length[efns2]}]]
```

```
Out[372]= {3.23486, -0.751968, 0.371083, -0.20701, 0.140855, -0.0969058, 0.0741292,  
-0.056213, 0.0457404, -0.0367102, 0.0310402, -0.0258585, 0.0224462,  
-0.0191997, 0.0169877, -0.0148199, 0.0133045, -0.0117855, 0.0107022}
```

**coefs2**

```
In[373]= efns2
```

```
Out[373]= {0.877796, -0.0758297, 0.0191999, -0.00649176, 0.00295956, -0.00145852, 0.000838269,  
-0.000494993, 0.000322485, -0.000211888, 0.000149368, -0.000105326, 0.0000783875,  
-0.0000581226, 0.0000450057, -0.0000346486, 0.0000276528, -0.0000219193, 0.0000179156}
```

```
In[374]= (efns2[x] * ((Cos[# t]) & /@ efs2)) . (
$$\frac{\mathbf{coefs2}}{\mathbf{efns2}}$$
)
```

```
Out[374]= 0.877796 Cos[1.29011 t] (1.29011 Cos[1.29011 x] - Sin[1.29011 x]) -  
0.0758297 Cos[2.37305 t] (2.37305 Cos[2.37305 x] - Sin[2.37305 x]) +  
0.0191999 Cos[3.40918 t] (3.40918 Cos[3.40918 x] - Sin[3.40918 x]) -  
0.00649176 Cos[4.42932 t] (4.42932 Cos[4.42932 x] - Sin[4.42932 x]) +  
0.00295956 Cos[5.44216 t] (5.44216 Cos[5.44216 x] - Sin[5.44216 x]) -  
0.00145852 Cos[6.45105 t] (6.45105 Cos[6.45105 x] - Sin[6.45105 x]) +  
0.000838269 Cos[7.45757 t] (7.45757 Cos[7.45757 x] - Sin[7.45757 x]) -  
0.000494993 Cos[8.46256 t] (8.46256 Cos[8.46256 x] - Sin[8.46256 x]) +  
0.000322485 Cos[9.4665 t] (9.4665 Cos[9.4665 x] - Sin[9.4665 x]) -  
0.000211888 Cos[10.4697 t] (10.4697 Cos[10.4697 x] - Sin[10.4697 x]) +  
0.000149368 Cos[11.4723 t] (11.4723 Cos[11.4723 x] - Sin[11.4723 x]) -  
0.000105326 Cos[12.4745 t] (12.4745 Cos[12.4745 x] - Sin[12.4745 x]) +  
0.0000783875 Cos[13.4764 t] (13.4764 Cos[13.4764 x] - Sin[13.4764 x]) -  
0.0000581226 Cos[14.478 t] (14.478 Cos[14.478 x] - Sin[14.478 x]) +  
0.0000450057 Cos[15.4795 t] (15.4795 Cos[15.4795 x] - Sin[15.4795 x]) -  
0.0000346486 Cos[16.4807 t] (16.4807 Cos[16.4807 x] - Sin[16.4807 x]) +  
0.0000276528 Cos[17.4818 t] (17.4818 Cos[17.4818 x] - Sin[17.4818 x]) -  
0.0000219193 Cos[18.4828 t] (18.4828 Cos[18.4828 x] - Sin[18.4828 x]) +  
0.0000179156 Cos[19.4837 t] (19.4837 Cos[19.4837 x] - Sin[19.4837 x])
```



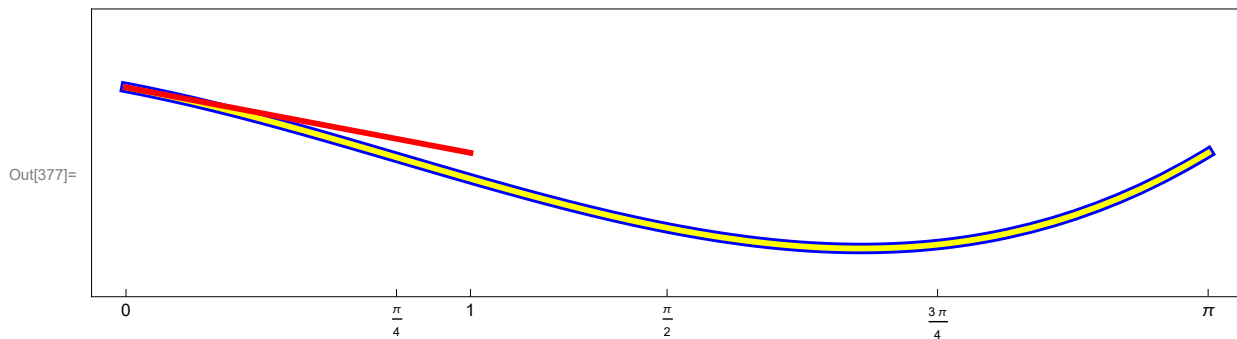
```
In[375]= Clear[uu2]; uu2[x_, t_] =
  (NIntegrate[(ff2[x]) efn2[x], {x, 0, Pi}, MaxRecursion -> 200, AccuracyGoal -> 10,
    PrecisionGoal -> 10] / NIntegrate[(efn2[x]) efn2[x], {x, 0, Pi},
    MaxRecursion -> 200, AccuracyGoal -> 10, PrecisionGoal -> 10])
  efn2[x] * (Cosh[evn2 t]) + (efs2[x] * ((Cos[# t]) & /@ eves2)) .  $\left(\frac{\text{coefs2}}{\text{efns2}}\right)$ 
```

```
Out[375]= 0.877796 Cos[1.29011 t] (1.29011 Cos[1.29011 x] - Sin[1.29011 x]) -
  0.0758297 Cos[2.37305 t] (2.37305 Cos[2.37305 x] - Sin[2.37305 x]) +
  0.0191999 Cos[3.40918 t] (3.40918 Cos[3.40918 x] - Sin[3.40918 x]) -
  0.00649176 Cos[4.42932 t] (4.42932 Cos[4.42932 x] - Sin[4.42932 x]) +
  0.00295956 Cos[5.44216 t] (5.44216 Cos[5.44216 x] - Sin[5.44216 x]) -
  0.00145852 Cos[6.45105 t] (6.45105 Cos[6.45105 x] - Sin[6.45105 x]) +
  0.000838269 Cos[7.45757 t] (7.45757 Cos[7.45757 x] - Sin[7.45757 x]) -
  0.000494993 Cos[8.46256 t] (8.46256 Cos[8.46256 x] - Sin[8.46256 x]) +
  0.000322485 Cos[9.4665 t] (9.4665 Cos[9.4665 x] - Sin[9.4665 x]) -
  0.000211888 Cos[10.4697 t] (10.4697 Cos[10.4697 x] - Sin[10.4697 x]) +
  0.000149368 Cos[11.4723 t] (11.4723 Cos[11.4723 x] - Sin[11.4723 x]) -
  0.000105326 Cos[12.4745 t] (12.4745 Cos[12.4745 x] - Sin[12.4745 x]) +
  0.0000783875 Cos[13.4764 t] (13.4764 Cos[13.4764 x] - Sin[13.4764 x]) -
  0.0000581226 Cos[14.478 t] (14.478 Cos[14.478 x] - Sin[14.478 x]) +
  0.0000450057 Cos[15.4795 t] (15.4795 Cos[15.4795 x] - Sin[15.4795 x]) -
  0.0000346486 Cos[16.4807 t] (16.4807 Cos[16.4807 x] - Sin[16.4807 x]) +
  0.0000276528 Cos[17.4818 t] (17.4818 Cos[17.4818 x] - Sin[17.4818 x]) -
  0.0000219193 Cos[18.4828 t] (18.4828 Cos[18.4828 x] - Sin[18.4828 x]) +
  0.0000179156 Cos[19.4837 t] (19.4837 Cos[19.4837 x] - Sin[19.4837 x]) +
  0.00018217 Cosh[0.996182 t] (0.996182 Cosh[0.996182 x] - Sinh[0.996182 x])
```

```
In[376]= uu2[1, 0.4]
```

```
Out[376]= -0.369246
```

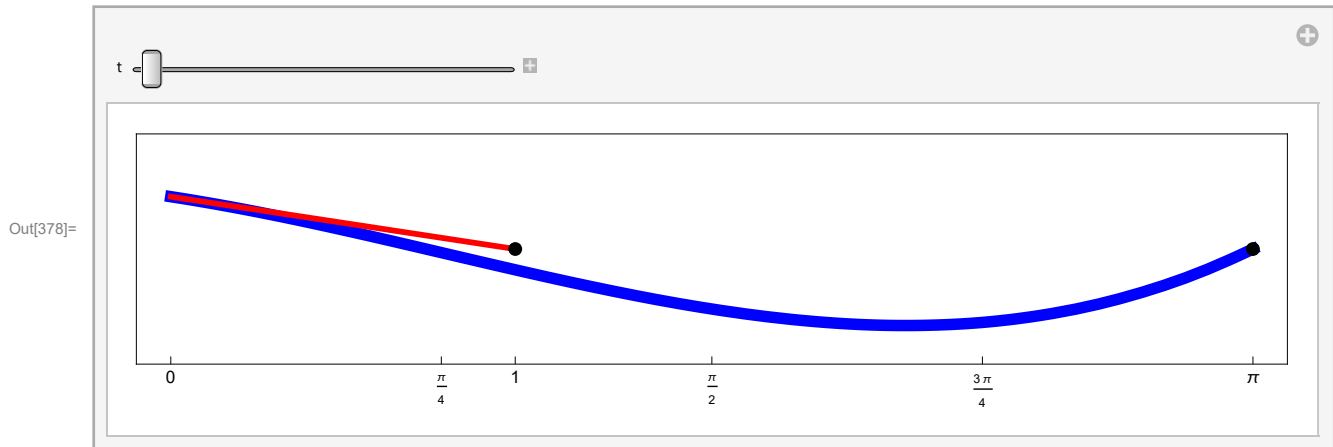
```
In[377]= Plot[Evaluate[{{ff2[x]}, uu2[x, 0]}], {x, 0, Pi},
  PlotStyle -> {{Thickness[0.01], Blue}, {Thickness[0.005], Yellow}},
  Epilog -> {{Red, Thickness[0.005], Line[{{1, 0}, {0, uu2[0, 0]}]}},
  PlotRange -> {{-0.1, Pi + .1}, {-2.2, 2.2}}, AspectRatio -> 1 / 4, Frame -> True,
  FrameTicks -> {{{}, {}}, {Join[{1}, Range[0, Pi, Pi / 4]}, {}]},
  Axes -> False, ImageSize -> 600]
```



```

In[378]:= Manipulate[Plot[Evaluate[uu2[x, t]], {x, 0, Pi},
  PlotStyle -> {{Thickness[0.01], Blue}},
  Epilog -> {{Red, Thickness[0.005], Line[{{1, 0}, {0, uu2[0, t]}}]},
    {PointSize[0.012], Point[#] & /@ {{1, 0}, {Pi, 0}}}
  }, PlotRange -> {{-0.1, Pi + .1}, {-2.2, 2.2}}, AspectRatio -> 1 / 5,
  Frame -> True, FrameTicks -> {{{}, {}}, {Join[{1}, Range[0, Pi, Pi / 4]}, {}]},
  Axes -> False, ImageSize -> 600], {t, 0, 11.5, 0.1}, ControlPlacement -> Top]

```



Because of the negative eigenvalue, this string brakes.

```

In[379]:= aniuu2BB = Table[Plot[Evaluate[uu2[x, t]],
  {x, 0, Pi}, PlotStyle -> {{Thickness[0.01], RGBColor[1, 0.5, 0]}},
  Epilog -> {
    {Red, Thickness[0.007], Line[{{1, 0}, {0, Evaluate[uu2[0, t]}}]},
    {PointSize[0.012], White, Point[#] & /@ {{1, 0}, {Pi, 0}}},
    {Text["t =", {1.45 + 0.08, 1.6}, BaseStyle -> {FontWeight -> "Normal",
      FontColor -> RGBColor[1, 1, 1]}}, Text[NumberForm[N[t], {3, 1}], {1.5 + 0.1, 1.6},
      BaseStyle -> {FontWeight -> "Normal", FontColor -> RGBColor[1, 1, 1]}]}
  },
  PlotRange -> {{-0.1, Pi + 0.1}, {-1.7, 1.7}}, AspectRatio -> 1 / 5,
  Frame -> True, FrameTicks -> {{{}, {}}, {Range[0, Pi, Pi / 4]}, {}},
  Axes -> False, ImageSize -> 800, Background -> Black], {t, 0, 11.5, 0.05}];

```

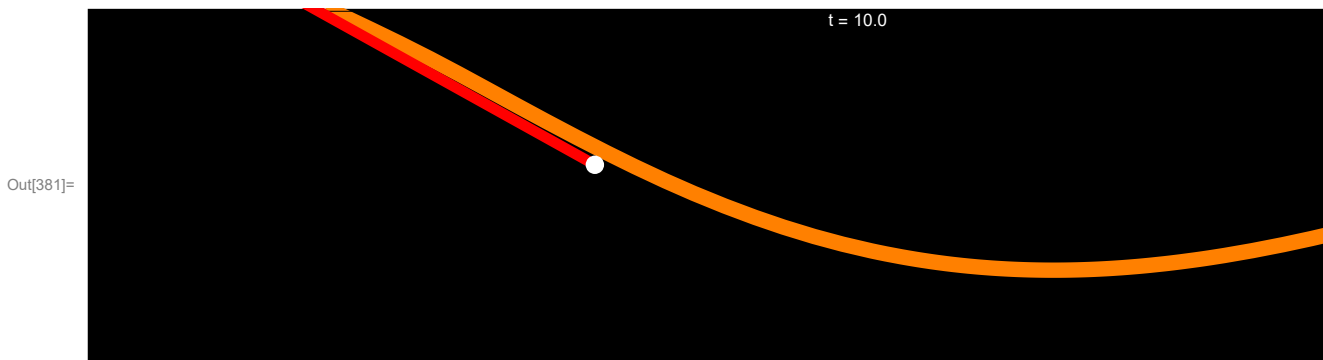
```

In[380]:= Length[aniu2BB]

```

Out[380]= 231

In[381]= `aniuu2BB[[200]]`



In[382]= `ListAnimate[Show[#, ImageSize -> 600] & /@ aniuu2BB]`

`(* dds=0.1&/@Range[Length[aniuu2BB]]; duration of each frame that we want*)`

`(* SetDirectory[NotebookDirectory[]] *)`

`(* Export["aniuu2BBs1.gif", aniuu2BB[[1], "ImageSize" -> 800];  
Export["aniuu2BBAni.gif", aniuu2BB, "AnimationRepetitions" -> Infinity,  
"ImageSize" -> 800, "DisplayDurations" -> dds] *)`

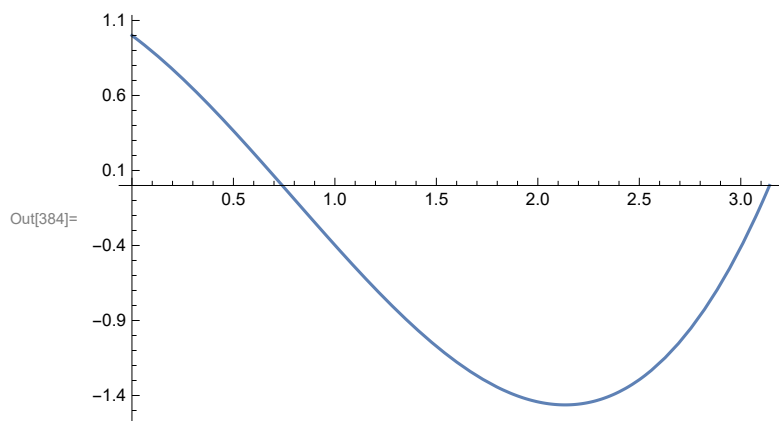
Out[458]= `aniuu2BBAni.gif`

## Solution 3

In[383]= `ff3[x_] = ((0.2869636751862268`)^3 +  $\frac{-1 + \pi - (0.2869636751862268`)^3 \pi^3}{\pi^2}$  #^2 - 1 # + 1) &[x]`

Out[383]= `1 - x - 0.684534 x^2 + 0.286964 x^3`

In[384]= `Plot[ff3[x], {x, 0, Pi}]`



In[385]= `Chop[NIntegrate[ff3[x] efn2[x], {x, 0, Pi},  
MaxRecursion -> 200, AccuracyGoal -> 10, PrecisionGoal -> 10]]`

Out[385]= `0`

```
In[386]= coefs3 = Chop[Table[NIntegrate[(ff3[x]) efs2[x][[j]], {x, 0, Pi},
      MaxRecursion → 200, AccuracyGoal → 10, PrecisionGoal → 10], {j, 1, Length[efns2]}]]
```

```
Out[386]= {3.23546, -0.752124, 0.371155, -0.207052, 0.140882, -0.0969253, 0.0741436,
  -0.0562243, 0.0457493, -0.0367175, 0.0310463, -0.0258637, 0.0224506,
  -0.0192036, 0.016991, -0.0148229, 0.0133072, -0.0117879, 0.0107043}
```

**coefs3**

```
In[387]= efns2
```

```
Out[387]= {0.877959, -0.0758453, 0.0192036, -0.00649308, 0.00296014, -0.00145882, 0.000838433,
  -0.000495092, 0.000322548, -0.00021193, 0.000149398, -0.000105347, 0.0000784029,
  -0.0000581342, 0.0000450146, -0.0000346555, 0.0000276583, -0.0000219237, 0.0000179191}
```

```
In[388]= (efs2[x] * ((Cos[# t]) & /@ eves2)) . (
$$\frac{\mathbf{coefs3}}{\mathbf{efns2}}$$
)
```

```
Out[388]= 0.877959 Cos[1.29011 t] (1.29011 Cos[1.29011 x] - Sin[1.29011 x]) -
  0.0758453 Cos[2.37305 t] (2.37305 Cos[2.37305 x] - Sin[2.37305 x]) +
  0.0192036 Cos[3.40918 t] (3.40918 Cos[3.40918 x] - Sin[3.40918 x]) -
  0.00649308 Cos[4.42932 t] (4.42932 Cos[4.42932 x] - Sin[4.42932 x]) +
  0.00296014 Cos[5.44216 t] (5.44216 Cos[5.44216 x] - Sin[5.44216 x]) -
  0.00145882 Cos[6.45105 t] (6.45105 Cos[6.45105 x] - Sin[6.45105 x]) +
  0.000838433 Cos[7.45757 t] (7.45757 Cos[7.45757 x] - Sin[7.45757 x]) -
  0.000495092 Cos[8.46256 t] (8.46256 Cos[8.46256 x] - Sin[8.46256 x]) +
  0.000322548 Cos[9.4665 t] (9.4665 Cos[9.4665 x] - Sin[9.4665 x]) -
  0.00021193 Cos[10.4697 t] (10.4697 Cos[10.4697 x] - Sin[10.4697 x]) +
  0.000149398 Cos[11.4723 t] (11.4723 Cos[11.4723 x] - Sin[11.4723 x]) -
  0.000105347 Cos[12.4745 t] (12.4745 Cos[12.4745 x] - Sin[12.4745 x]) +
  0.0000784029 Cos[13.4764 t] (13.4764 Cos[13.4764 x] - Sin[13.4764 x]) -
  0.0000581342 Cos[14.478 t] (14.478 Cos[14.478 x] - Sin[14.478 x]) +
  0.0000450146 Cos[15.4795 t] (15.4795 Cos[15.4795 x] - Sin[15.4795 x]) -
  0.0000346555 Cos[16.4807 t] (16.4807 Cos[16.4807 x] - Sin[16.4807 x]) +
  0.0000276583 Cos[17.4818 t] (17.4818 Cos[17.4818 x] - Sin[17.4818 x]) -
  0.0000219237 Cos[18.4828 t] (18.4828 Cos[18.4828 x] - Sin[18.4828 x]) +
  0.0000179191 Cos[19.4837 t] (19.4837 Cos[19.4837 x] - Sin[19.4837 x])
```

```
In[389]:= Clear[uu3]; uu3[x_, t_] = (efs2[x] * ((Cos[# t] & /@ eves2)) . (
  
$$\left. \begin{matrix} \text{coefs3} \\ \text{efns2} \end{matrix} \right)$$

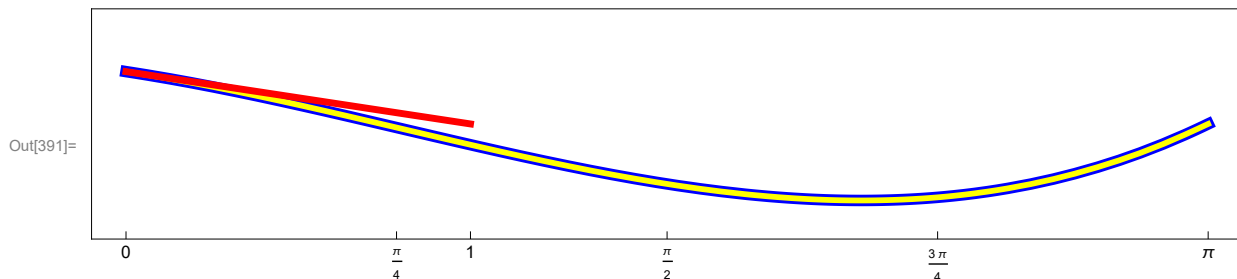
```

```
Out[389]= 0.877959 Cos[1.29011 t] (1.29011 Cos[1.29011 x] - Sin[1.29011 x]) -
  0.0758453 Cos[2.37305 t] (2.37305 Cos[2.37305 x] - Sin[2.37305 x]) +
  0.0192036 Cos[3.40918 t] (3.40918 Cos[3.40918 x] - Sin[3.40918 x]) -
  0.00649308 Cos[4.42932 t] (4.42932 Cos[4.42932 x] - Sin[4.42932 x]) +
  0.00296014 Cos[5.44216 t] (5.44216 Cos[5.44216 x] - Sin[5.44216 x]) -
  0.00145882 Cos[6.45105 t] (6.45105 Cos[6.45105 x] - Sin[6.45105 x]) +
  0.000838433 Cos[7.45757 t] (7.45757 Cos[7.45757 x] - Sin[7.45757 x]) -
  0.000495092 Cos[8.46256 t] (8.46256 Cos[8.46256 x] - Sin[8.46256 x]) +
  0.000322548 Cos[9.4665 t] (9.4665 Cos[9.4665 x] - Sin[9.4665 x]) -
  0.00021193 Cos[10.4697 t] (10.4697 Cos[10.4697 x] - Sin[10.4697 x]) +
  0.000149398 Cos[11.4723 t] (11.4723 Cos[11.4723 x] - Sin[11.4723 x]) -
  0.000105347 Cos[12.4745 t] (12.4745 Cos[12.4745 x] - Sin[12.4745 x]) +
  0.0000784029 Cos[13.4764 t] (13.4764 Cos[13.4764 x] - Sin[13.4764 x]) -
  0.0000581342 Cos[14.478 t] (14.478 Cos[14.478 x] - Sin[14.478 x]) +
  0.0000450146 Cos[15.4795 t] (15.4795 Cos[15.4795 x] - Sin[15.4795 x]) -
  0.0000346555 Cos[16.4807 t] (16.4807 Cos[16.4807 x] - Sin[16.4807 x]) +
  0.0000276583 Cos[17.4818 t] (17.4818 Cos[17.4818 x] - Sin[17.4818 x]) -
  0.0000219237 Cos[18.4828 t] (18.4828 Cos[18.4828 x] - Sin[18.4828 x]) +
  0.0000179191 Cos[19.4837 t] (19.4837 Cos[19.4837 x] - Sin[19.4837 x])
```

```
In[390]:= uu3[1, 0.]
```

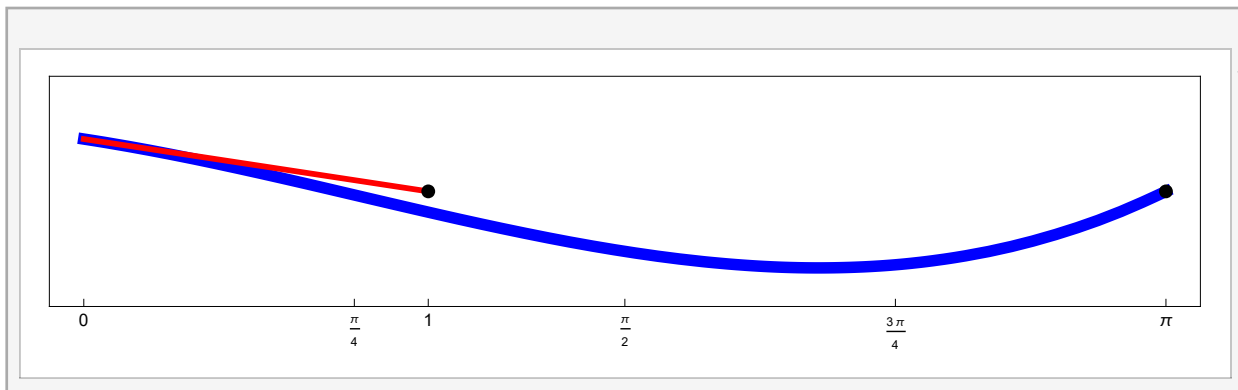
```
Out[390]= -0.397507
```

```
In[391]:= Plot[Evaluate[{{ff3[x]}, uu3[x, 0]}], {x, 0, Pi},
  PlotStyle -> {{Thickness[0.01], Blue}, {Thickness[0.005], Yellow}},
  Epilog -> {{Red, Thickness[0.006], Line[{{1, 0}, {0, uu3[0, 0]}}]}},
  PlotRange -> {{-0.1, Pi + .1}, {-2.2, 2.2}}, AspectRatio -> 1 / 5, Frame -> True,
  FrameTicks -> {{{}, {}}, {Join[{1}, Range[0, Pi, Pi / 4]], {}}, {}},
  Axes -> False, ImageSize -> 600]
```



```
In[392]= Manipulate[Plot[Evaluate[uu3[x, t]], {x, 0, Pi}, PlotStyle -> {{Thickness[0.01], Blue}},
  Epilog -> {{Red, Thickness[0.005], Line[{{1, 0}, {0, uu3[0, t]}}]},
    {PointSize[0.012], Point[#] & /@ {{1, 0}, {Pi, 0}}}},
  PlotRange -> {{-0.1, Pi + .1}, {-2.2, 2.2}}, AspectRatio -> 1 / 5, Frame -> True,
  FrameTicks -> {{{}, {}}, {Join[{1}, Range[0, Pi, Pi / 4]], {}}},
  Axes -> False, ImageSize -> 600], {t, 0, 19.5, 0.1}]
```

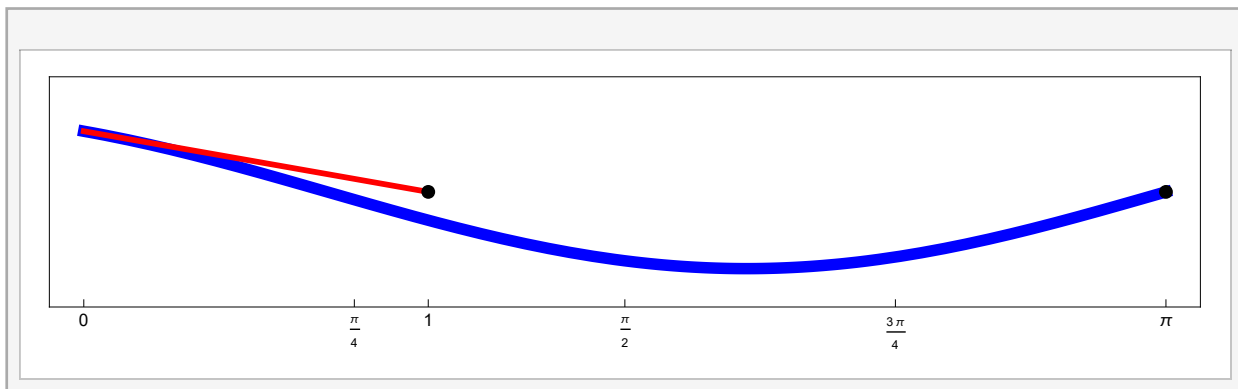
Out[392]=



```
In[393]= Clear[nmv2]; nmv2[x_, t_] := (efs2[x] * ( ( ( 1 / Sqrt[#^2 + 1] ) Cos[# t] ) & /@ eves2 ) )
```

```
In[394]= Manipulate[Plot[Evaluate[nmv2[x, t][[k]], {x, 0, Pi}, PlotStyle -> {{Thickness[0.01], Blue}},
  Epilog -> {{Red, Thickness[0.005], Line[{{1, 0}, {0, Evaluate[nmv2[0, t][[k]]}}]},
    {PointSize[0.012], Point[#] & /@ {{1, 0}, {Pi, 0}}}},
  PlotRange -> {{-0.1, Pi + 0.1}, {-1.5, 1.5}}, AspectRatio -> 1 / 5, Frame -> True,
  FrameTicks -> {{{}, {}}, {Join[{1}, Range[0, Pi, Pi / 4]], {}}, Axes -> False,
  ImageSize -> 600], {t, 0, 4 Pi, N[Pi / 64]}, {{k, 1}, Range[10], ControlType -> Setter}]
```

Out[394]=



In this example I carefully selected the initial shape, so that the string does not break. I selected a shape that is orthogonal to the eigenfunction corresponding to the negative eigenvalue.

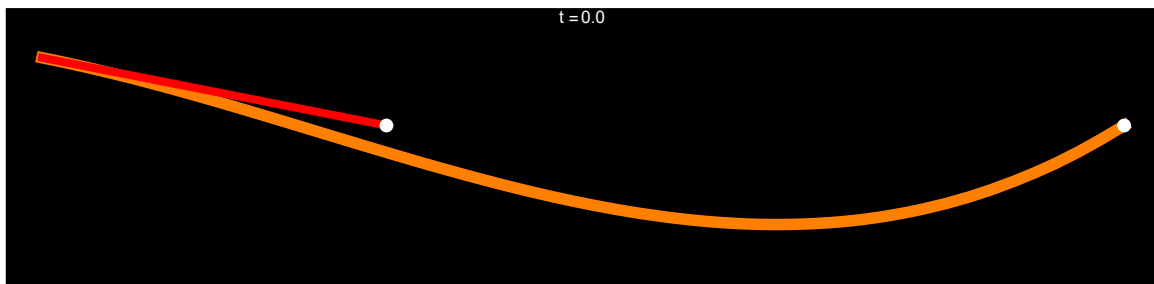
```
In[395]:= aniuu3BB = Table[Plot[Evaluate[uu3[x, t]],
  {x, 0, Pi}, PlotStyle -> {{Thickness[0.01], RGBColor[1, 0.5, 0]}},
  Epilog -> {
    {Red, Thickness[0.007], Line[{{1, 0}, {0, Evaluate[uu3[0, t]}]}]},
    {PointSize[0.012], White, Point[#] & /@ {{1, 0}, {Pi, 0}}},
    {Text["t =", {1.45 + 0.08, 1.6}, BaseStyle -> {FontWeight -> "Normal",
      FontColor -> RGBColor[1, 1, 1]}, Text[NumberForm[N[t], {3, 1}], {1.5 + 0.1, 1.6},
      BaseStyle -> {FontWeight -> "Normal", FontColor -> RGBColor[1, 1, 1]}]}
  },
  PlotRange -> {{-0.1, Pi + 0.1}, {-1.7, 1.7}}, AspectRatio -> 1 / 5,
  Frame -> True, FrameTicks -> {{{}, {}}, {Range[0, Pi, Pi / 4], {}}, {}},
  Axes -> False, ImageSize -> 800, Background -> Black], {t, 0, 10, 0.05}];
```

```
In[396]:= Length[aniu3BB]
```

```
Out[396]= 201
```

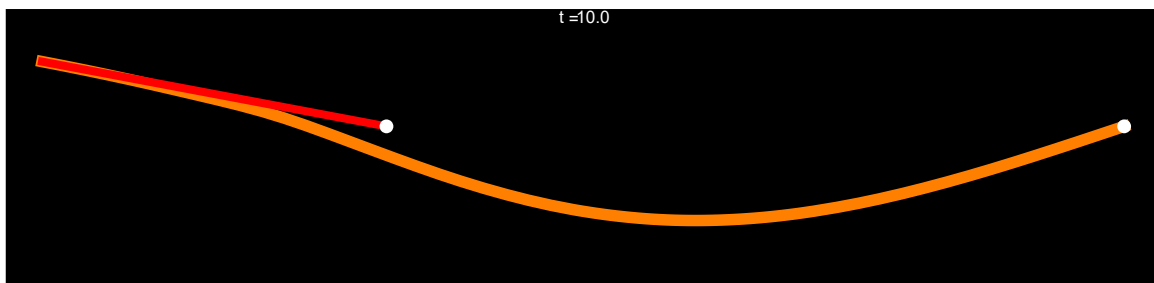
```
In[397]:= Show[aniu3BB[[1]], ImageSize -> 600]
```

```
Out[397]=
```



```
In[398]:= Show[aniu3BB[[201]], ImageSize -> 600]
```

```
Out[398]=
```



```
In[399]:= ListAnimate[Show[#, ImageSize -> 600] & /@ aniu3BB]
```

```
(* dds=0.1/@Range[Length[aniu3BB]]; duration of each frame that we want*)
```

```
(* Export["aniu3BBs1.gif", aniu3BB[[1]], "ImageSize" -> 800];
```

```
Export["aniu3BBAni.gif", aniu3BB, "AnimationRepetitions" -> Infinity,
  "ImageSize" -> 800, "DisplayDurations" -> dds] *)
```

```
Out[460]= aniu3BBAni.gif
```

## Solution 4

We want to solve

$$\begin{aligned} \partial_{tt}u[x, t] &= \partial_{xx}u[x, t], \quad 0 \leq x \leq \pi, \quad t \geq 0, \\ u[0, t] + 4 \partial_x u[0, t] &= 0, \\ u[\pi, t] &= 0, \\ u[x, 0] &= \text{ff1}[x], \\ \partial_t u[x, 0] &= 0. \end{aligned}$$

So we need to solve the Sturm-Liouville eigenvalue problem

$$\begin{aligned} -y(x)'' &= \lambda x y(x), \quad 0 \leq x \leq \pi, \\ y(0) + 4 y'(0) &= 0, \\ y(\pi) &= 0. \end{aligned}$$

## Look for negative eigenvalues

We look for negative eigenvalues  $\lambda = -\mu^2$

$$\begin{aligned} -y(x)'' &= -\mu^2 y(x), \quad 0 \leq x \leq \pi, \\ y(0) + 4 y'(0) &= 0, \\ y(\pi) &= 0. \end{aligned}$$

```
In[400]:= DSolve[y''[x] == μ² y[x], y[x], x]
```

```
Out[400]:= {{y[x] -> e^{x μ} c₁ + e^{-x μ} c₂}}
```

I do not know how to force Mathematica into working with hyperbolic functions. So I force it by hand:

```
In[401]:= Simplify[(ExpToTrig[DSolve[y''[x] == μ² y[x], y[x], x]]) /.
  {C[1] + C[2] -> C[1], C[1] - C[2] -> C[2]}]
```

```
Out[401]:= {{y[x] -> c₁ Cosh[x μ] + c₂ Sinh[x μ]}}
```

Substitute the fundamental solution ( $\text{Cosh}[\mu x] C[1] + \text{Sinh}[\mu x] C[2]$ ) into the boundary conditions and collect the expression with  $C[1]$  and  $C[2]$ . This will give us two homogeneous equations.

```
In[402]:= Collect[(Cosh[μ x] C[1] + Sinh[μ x] C[2]) /. {x -> 0} +
  (4 (D[(Cosh[μ x] C[1] + Sinh[μ x] C[2]), x]) /. {x -> 0}), {C[1], C[2]}]
```

```
Out[402]:= c₁ + 4 μ c₂
```

```
In[403]:= Collect[(Cosh[μ x] C[1] + Sinh[μ x] C[2]) /. {x -> Pi}, {C[1], C[2]}]
```

```
Out[403]:= c₁ Cosh[π μ] + c₂ Sinh[π μ]
```

We want nontrivial solutions for  $C[1]$  and  $C[2]$  of the system

$$\begin{aligned} C[1] + 4 \mu C[2] &= 0 \\ C[1] \text{Cosh}[\pi \mu] + C[2] \text{Sin}[\pi \mu] &= 0 \end{aligned}$$

For a nontrivial solution to exist we need the determinant to be equal to 0

```
In[404]:= Det[{{1, 4 μ}, {Cosh[Pi μ], Sinh[Pi μ]}}]
```

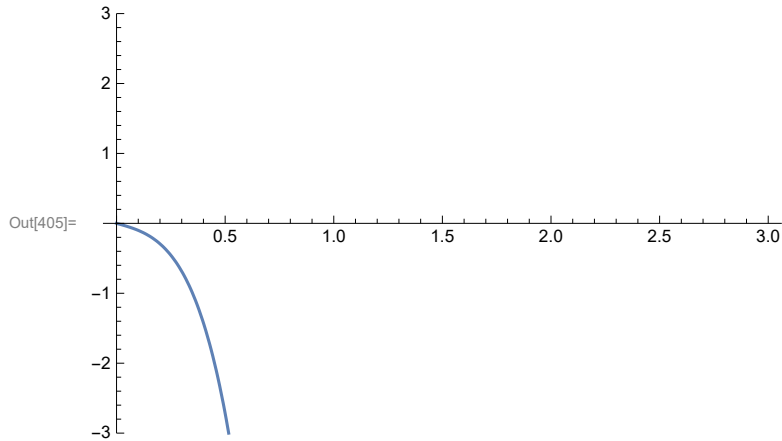
```
Out[404]:= -4 μ Cosh[π μ] + Sinh[π μ]
```

The above function is the function whose zeros give as  $-\mu^2$  the negative eigenvalues



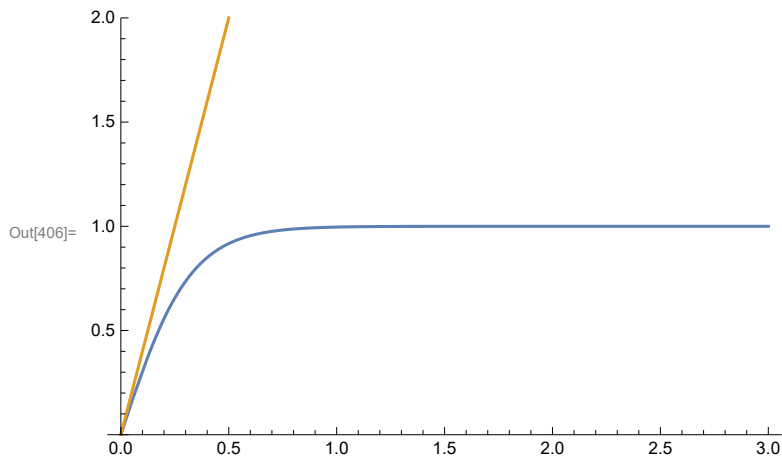
Since the above function has no positive zeros there are no negative evs. Below is a graphical verification of this claim.

```
In[405]:= Plot[-4 μ Cosh[π μ] + Sinh[π μ], {μ, 0, 3}, PlotRange → {-3, 3}]
```



The equation for the eigenvalue can be rewritten as  $\text{Tanh}[\pi \mu] = \mu$

```
In[406]:= Plot[{Tanh[π μ], 4 μ}, {μ, 0, 3}, PlotRange → {0, 2}]
```



Thus there are no negative eigenvalues in this case.

## Zero is not an eigenvalue

Is zero an eigenvalue?

$$-y(x)'' = 0, \quad 0 \leq x \leq \pi,$$

$$y(0) + 4y'(0) = 0,$$

$$y(\pi) = 0.$$

```
In[407]:= DSolve[y''[x] == 0, y[x], x]
```

```
Out[407]= {{y[x] → c1 + x c2}}
```

Substitute into the boundary conditions

```
In[408]:= Collect[(C[1] + x C[2]) /. {x -> 0} + 4 ((D[C[1] + x C[2], x]) /. {x -> 0}), {C[1], C[2]}]
```

```
Out[408]= C1 + 4 C2
```

```
In[409]:= Collect[(C[1] + x C[2]) /. {x -> Pi}, {C[1], C[2]}]
```

```
Out[409]= C1 + π C2
```

We need a nontrivial solution for C[1] and C[2] of the system

$$C[1] + 4 C[2] = 0$$

$$C[1] + \pi C[2] = 0$$

Since the determinant

```
In[410]:= Det[{{1, 4}, {1, Pi}}]
```

```
Out[410]= -4 + π
```

is nonzero, the system has no nontrivial solutions, thus zero is not an eigenvalue.

## Look for positive eigenvalues

We look for positive eigenvalues  $\lambda = \mu^2$

$$-y(x)'' = \mu^2 y(x), \quad 0 \leq x \leq \pi,$$

$$y(0) + 4 y'(0) = 0,$$

$$y(\pi) = 0.$$

```
In[411]:= DSolve[-y''[x] == μ2 y[x], y[x], x]
```

```
Out[411]= {{y[x] -> C1 Cos[x μ] + C2 Sin[x μ]}}
```

Substitute into the boundary conditions

```
In[412]:= Collect[(C[1] Cos[x μ] + C[2] Sin[x μ]) /. {x -> 0} +
  4 ((D[C[1] Cos[x μ] + C[2] Sin[x μ], x]) /. {x -> 0}), {C[1], C[2]}]
```

```
Out[412]= C1 + 4 μ C2
```

```
In[413]:= Collect[(C[1] Cos[x μ] + C[2] Sin[x μ]) /. {x -> Pi}, {C[1], C[2]}]
```

```
Out[413]= C1 Cos[π μ] + C2 Sin[π μ]
```

We want nontrivial solutions for C[1] and C[2] of the system

$$C[1] + 4 \mu C[2] = 0$$

$$C[1] \cos[\pi \mu] + C[2] \sin[\pi \mu] = 0$$

For a nontrivial solution to exist we need the determinant to be equal to 0

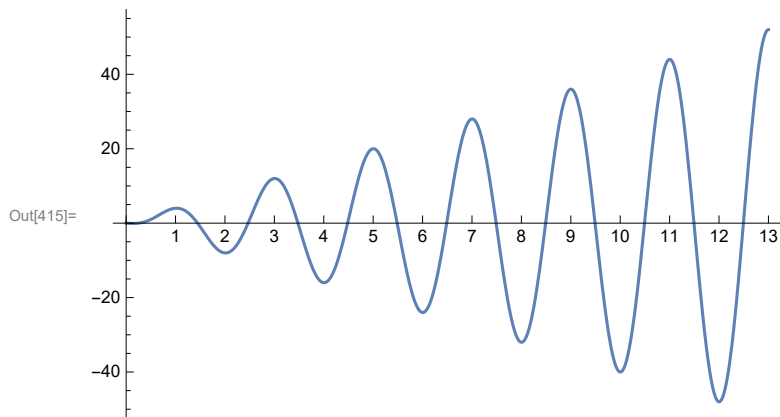
```
In[414]:= Det[{{1, 4 μ}, {Cos[π μ], Sin[π μ]}]}
```

```
Out[414]= -4 μ Cos[π μ] + Sin[π μ]
```

The above function is the function whose zeros give as  $\mu^2$  the negative eigenvalues

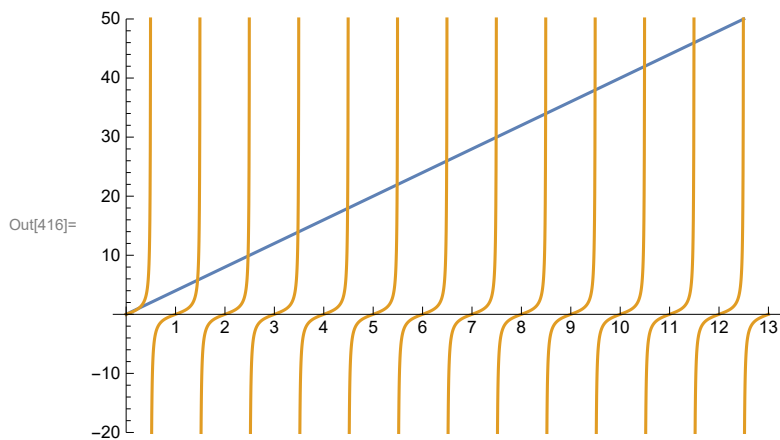
Getting solutions for  $\mu$  “visually”

```
In[415]:= Plot[-4 μ Cos[π μ] + Sin[π μ], {μ, 0, 13}, Ticks → {Range[0, 20, 1], Automatic}]
```



Or, more informative

```
In[416]:= Plot[{4 μ, Sin[π μ] / Cos[π μ]}, {μ, 0, 13},
  Ticks → {Range[0, 20, 1], Automatic}, PlotRange → {-20, 50}]
```



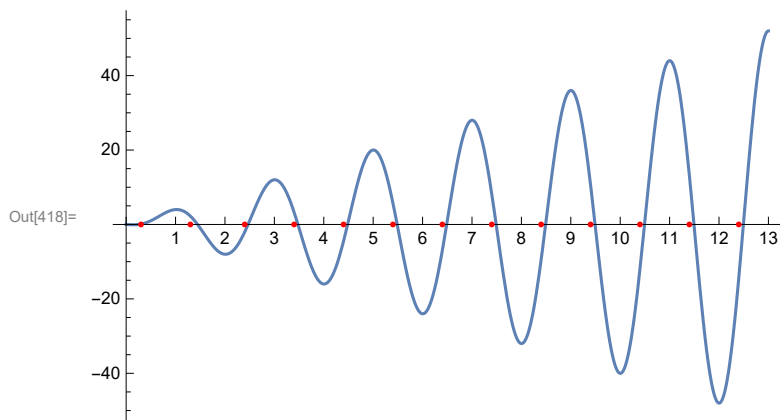
Next I make a list of initial guesses:

```
In[417]:= Join[ {.3, 1.3}, Range[2.4, 20, 1]]
```

```
Out[417]= {0.3, 1.3, 2.4, 3.4, 4.4, 5.4, 6.4, 7.4, 8.4, 9.4,
  10.4, 11.4, 12.4, 13.4, 14.4, 15.4, 16.4, 17.4, 18.4, 19.4}
```

Next I verify that my initial guesses are close to the actual zeros of the function

```
In[418]:= Plot[-4 μ Cos[π μ] + Sin[π μ], {μ, 0, 13},
  Epilog -> {{Red, Point[{#, 0]} & /@ Join[ {.3, 1.3}, Range[2.4, 20, 1] ]}},
  Ticks -> {Range[0, 20, 1], Automatic}]
```



Then I “map” my “zero finding function” to this list

```
In[419]:= Map[ (μ /. FindRoot[-4 μ Cos[π μ] + Sin[π μ] == 0, {μ, #}]) &,
  Join[ {.3, 1.3}, Range[2.4, 20, 1] ]]
```

```
Out[419]:= {0.25, 1.44549, 2.46786, 3.47715, 4.48226, 5.4855, 6.48774, 7.48938, 8.49063, 9.49162,
  10.4924, 11.4931, 12.4936, 13.4941, 14.4945, 15.4949, 16.4952, 17.4955, 18.4957, 19.4959}
```

Next I name these approximate eigenvalues. The named list of many  $\mu$ -s will be essential in what follows.

```
In[420]:= Clear[eves4];
  eves4 =
    (μ /. FindRoot[-4 μ Cos[π μ] + Sin[π μ] == 0, {μ, #}]) & /@ Join[ {.3, 1.3}, Range[2.4, 20, 1] ]
```

```
Out[421]:= {0.25, 1.44549, 2.46786, 3.47715, 4.48226, 5.4855, 6.48774, 7.48938, 8.49063, 9.49162,
  10.4924, 11.4931, 12.4936, 13.4941, 14.4945, 15.4949, 16.4952, 17.4955, 18.4957, 19.4959}
```

```
In[422]:= Length[eves4]
```

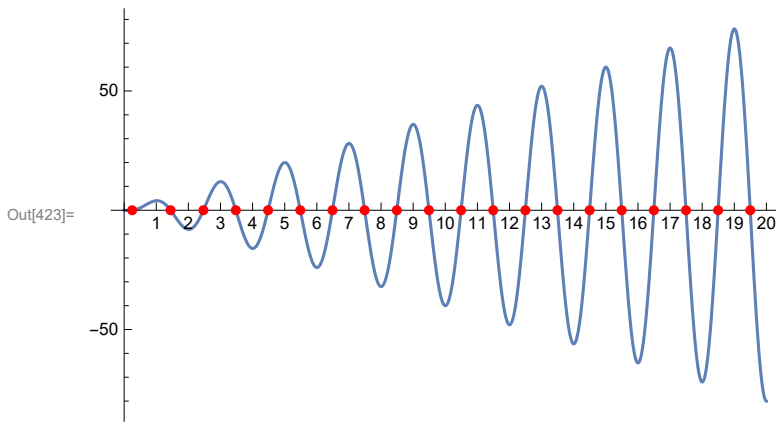
```
Out[422]:= 20
```

Thus we found twenty eigenvalues.

I visually inspect that there are no repetitions.

Now I can check graphically that I got the right solutions

```
In[423]:= Plot[-4 μ Cos[π μ] + Sin[π μ], {μ, 0, 20},
  Epilog -> {{Red, PointSize[0.015], Point[{#, 0]} & /@ eves4}},
  Ticks -> {Range[0, 20, 1], Automatic}]
```



Looks good.

Let us apply the original function to these approximations:

```
In[424]:= Chop[(-4 # Cos[π #] + Sin[π #]) & /@ eves4]
```

Out[424]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

Quite good!

Based on the matrix that determined the eigenvalues we can now define the eigenfunctions: With  $\mu$  being the solution of  $\mu \text{Cos}[\pi \mu] + \text{Sin}[\pi \mu] = 0$ , a corresponding eigenfunction is found by solving for  $C_1$  and  $C_2$  the following system

```
In[425]:= MatrixForm[{{1, 4 μ}, {Cos[π μ], Sin[π μ]}]}
```

Out[425]/MatrixForm=

$$\begin{pmatrix} 1 & 4\mu \\ \text{Cos}[\pi \mu] & \text{Sin}[\pi \mu] \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4\mu \\ \text{Cos}[\pi \mu] & \text{Sin}[\pi \mu] \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since the determinant of this system is zero, the system has many solutions. A simple solution is  $C_1 = 4\mu$ ,  $C_2 = -1$ . Below are twenty eigenfunctions corresponding to the eigenvalues that we found.

In[426]= **Clear[efs4];**

**efs4[x\_] = (4 # Cos[# x] - Sin[# x]) & /@ eves4**

Out[427]= {1. Cos[0.25 x] - Sin[0.25 x], 5.78195 Cos[1.44549 x] - Sin[1.44549 x],  
9.87146 Cos[2.46786 x] - Sin[2.46786 x], 13.9086 Cos[3.47715 x] - Sin[3.47715 x],  
17.9291 Cos[4.48226 x] - Sin[4.48226 x], 21.942 Cos[5.4855 x] - Sin[5.4855 x],  
25.951 Cos[6.48774 x] - Sin[6.48774 x], 29.9575 Cos[7.48938 x] - Sin[7.48938 x],  
33.9625 Cos[8.49063 x] - Sin[8.49063 x], 37.9665 Cos[9.49162 x] - Sin[9.49162 x],  
41.9697 Cos[10.4924 x] - Sin[10.4924 x], 45.9723 Cos[11.4931 x] - Sin[11.4931 x],  
49.9745 Cos[12.4936 x] - Sin[12.4936 x], 53.9764 Cos[13.4941 x] - Sin[13.4941 x],  
57.978 Cos[14.4945 x] - Sin[14.4945 x], 61.9795 Cos[15.4949 x] - Sin[15.4949 x],  
65.9807 Cos[16.4952 x] - Sin[16.4952 x], 69.9818 Cos[17.4955 x] - Sin[17.4955 x],  
73.9828 Cos[18.4957 x] - Sin[18.4957 x], 77.9837 Cos[19.4959 x] - Sin[19.4959 x]}

In[428]= **efs4[x] [[3]]**

Out[428]= 9.87146 Cos[2.46786 x] - Sin[2.46786 x]

In[429]= **Chop[Table[NIntegrate[efs4[x] [[j]] × efs4[x] [[k]], {x, 0, Pi}, MaxRecursion → 200,  
AccuracyGoal → 10, PrecisionGoal → 10], {j, 1, Length[eves4]}, {k, 1, Length[eves4]}]]]**

Out[429]= {{1.14159, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},  
{0, 52.084, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},  
{0, 0, 152.638, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},  
{0, 0, 0, 303.441, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},  
{0, 0, 0, 0, 504.505, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},  
{0, 0, 0, 0, 0, 755.834, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},  
{0, 0, 0, 0, 0, 0, 1057.43, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},  
{0, 0, 0, 0, 0, 0, 0, 1409.29, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},  
{0, 0, 0, 0, 0, 0, 0, 0, 1811.41, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},  
{0, 0, 0, 0, 0, 0, 0, 0, 0, 2263.8, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},  
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2766.45, 0, 0, 0, 0, 0, 0, 0, 0, 0},  
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3319.38, 0, 0, 0, 0, 0, 0, 0, 0},  
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3922.56, 0, 0, 0, 0, 0, 0, 0},  
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4576.01, 0, 0, 0, 0, 0, 0},  
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 5279.73, 0, 0, 0, 0, 0},  
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 6033.71, 0, 0, 0, 0},  
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 6837.96, 0, 0, 0},  
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 7692.47, 0, 0},  
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 8597.25, 0},  
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 9552.3}}

In[430]= **efns4 = Chop[Table[NIntegrate[efs4[x] [[j]] × efs4[x] [[j]], {x, 0, Pi},**

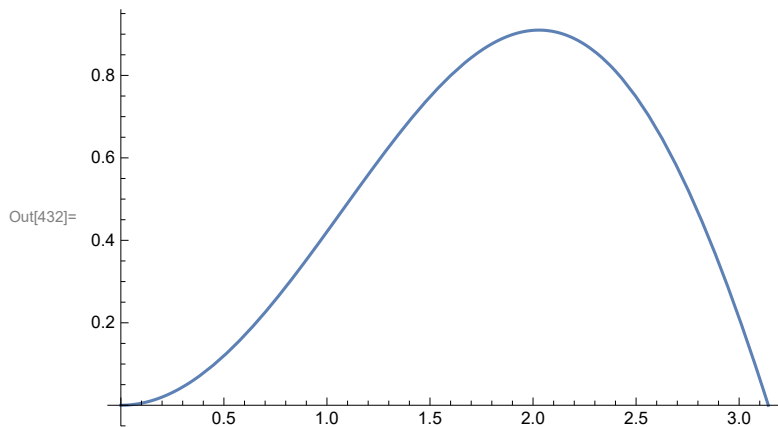
**MaxRecursion → 200, AccuracyGoal → 10, PrecisionGoal → 10], {j, 1, Length[eves4]}]]]**

Out[430]= {1.14159, 52.084, 152.638, 303.441, 504.505, 755.834, 1057.43, 1409.29, 1811.41, 2263.8,  
2766.45, 3319.38, 3922.56, 4576.01, 5279.73, 6033.71, 6837.96, 7692.47, 8597.25, 9552.3}

In[431]= **ff4[x\_] = (x Sin[x] / 2)**

Out[431]=  $\frac{1}{2} x \sin[x]$

In[432]= **Plot**[ff4[x], {x, 0, Pi}]



In[433]= **coefs4 = Chop**[Table[NIntegrate[(ff4[x]) efs4[x][[j]], {x, 0, Pi},  
MaxRecursion → 200, AccuracyGoal → 10, PrecisionGoal → 10], {j, 1, Length[efns4]}]]

Out[433]= {0.68671, -5.9285, 1.04022, -0.365932, 0.233143, -0.135893, 0.103631,  
-0.0714989, 0.0587623, -0.0442219, 0.037895, -0.0300844, 0.0264823, -0.0218027,  
0.0195563, -0.0165311, 0.0150358, -0.0129672, 0.0119215, -0.0104446}

**coefs4**

In[434]= efns4

Out[434]= {0.601536, -0.113826, 0.00681497, -0.00120594, 0.000462122,  
-0.000179792, 0.0000980029, -0.0000507341, 0.0000324401, -0.0000195344,  
0.000013698,  $-9.06328 \times 10^{-6}$ ,  $6.75127 \times 10^{-6}$ ,  $-4.76456 \times 10^{-6}$ ,  $3.70404 \times 10^{-6}$ ,  
 $-2.7398 \times 10^{-6}$ ,  $2.19887 \times 10^{-6}$ ,  $-1.6857 \times 10^{-6}$ ,  $1.38667 \times 10^{-6}$ ,  $-1.09341 \times 10^{-6}$ }

In[435]:= (efs4[x] \* ((Cos[# t]) & /@ eves4)) .  $\left( \frac{\text{coefs4}}{\text{efns4}} \right)$

Out[435]= 0.601536 Cos[0.25 t] (1. Cos[0.25 x] - Sin[0.25 x]) -  
 0.113826 Cos[1.44549 t] (5.78195 Cos[1.44549 x] - Sin[1.44549 x]) +  
 0.00681497 Cos[2.46786 t] (9.87146 Cos[2.46786 x] - Sin[2.46786 x]) -  
 0.00120594 Cos[3.47715 t] (13.9086 Cos[3.47715 x] - Sin[3.47715 x]) +  
 0.000462122 Cos[4.48226 t] (17.9291 Cos[4.48226 x] - Sin[4.48226 x]) -  
 0.000179792 Cos[5.4855 t] (21.942 Cos[5.4855 x] - Sin[5.4855 x]) +  
 0.0000980029 Cos[6.48774 t] (25.951 Cos[6.48774 x] - Sin[6.48774 x]) -  
 0.0000507341 Cos[7.48938 t] (29.9575 Cos[7.48938 x] - Sin[7.48938 x]) +  
 0.0000324401 Cos[8.49063 t] (33.9625 Cos[8.49063 x] - Sin[8.49063 x]) -  
 0.0000195344 Cos[9.49162 t] (37.9665 Cos[9.49162 x] - Sin[9.49162 x]) +  
 0.000013698 Cos[10.4924 t] (41.9697 Cos[10.4924 x] - Sin[10.4924 x]) -  
 9.06328 × 10<sup>-6</sup> Cos[11.4931 t] (45.9723 Cos[11.4931 x] - Sin[11.4931 x]) +  
 6.75127 × 10<sup>-6</sup> Cos[12.4936 t] (49.9745 Cos[12.4936 x] - Sin[12.4936 x]) -  
 4.76456 × 10<sup>-6</sup> Cos[13.4941 t] (53.9764 Cos[13.4941 x] - Sin[13.4941 x]) +  
 3.70404 × 10<sup>-6</sup> Cos[14.4945 t] (57.978 Cos[14.4945 x] - Sin[14.4945 x]) -  
 2.7398 × 10<sup>-6</sup> Cos[15.4949 t] (61.9795 Cos[15.4949 x] - Sin[15.4949 x]) +  
 2.19887 × 10<sup>-6</sup> Cos[16.4952 t] (65.9807 Cos[16.4952 x] - Sin[16.4952 x]) -  
 1.6857 × 10<sup>-6</sup> Cos[17.4955 t] (69.9818 Cos[17.4955 x] - Sin[17.4955 x]) +  
 1.38667 × 10<sup>-6</sup> Cos[18.4957 t] (73.9828 Cos[18.4957 x] - Sin[18.4957 x]) -  
 1.09341 × 10<sup>-6</sup> Cos[19.4959 t] (77.9837 Cos[19.4959 x] - Sin[19.4959 x])

In[436]:= Clear[uu4]; uu4[x\_, t\_] = (efs4[x] \* ((Cos[# t]) & /@ eves4)) .  $\left( \frac{\text{coefs4}}{\text{efns4}} \right)$

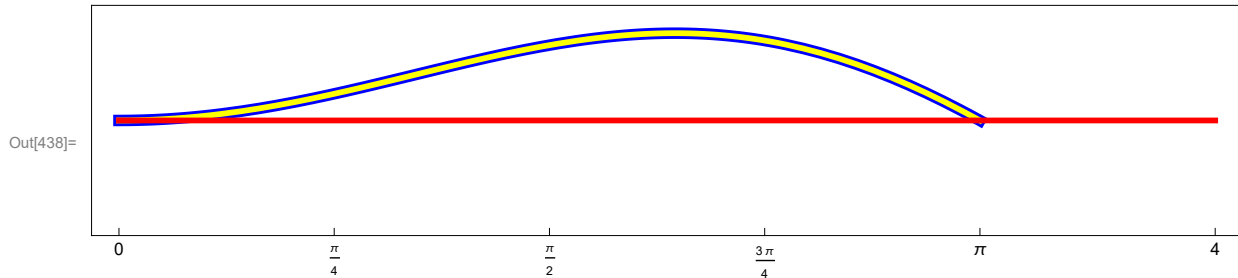
Out[436]= 0.601536 Cos[0.25 t] (1. Cos[0.25 x] - Sin[0.25 x]) -  
 0.113826 Cos[1.44549 t] (5.78195 Cos[1.44549 x] - Sin[1.44549 x]) +  
 0.00681497 Cos[2.46786 t] (9.87146 Cos[2.46786 x] - Sin[2.46786 x]) -  
 0.00120594 Cos[3.47715 t] (13.9086 Cos[3.47715 x] - Sin[3.47715 x]) +  
 0.000462122 Cos[4.48226 t] (17.9291 Cos[4.48226 x] - Sin[4.48226 x]) -  
 0.000179792 Cos[5.4855 t] (21.942 Cos[5.4855 x] - Sin[5.4855 x]) +  
 0.0000980029 Cos[6.48774 t] (25.951 Cos[6.48774 x] - Sin[6.48774 x]) -  
 0.0000507341 Cos[7.48938 t] (29.9575 Cos[7.48938 x] - Sin[7.48938 x]) +  
 0.0000324401 Cos[8.49063 t] (33.9625 Cos[8.49063 x] - Sin[8.49063 x]) -  
 0.0000195344 Cos[9.49162 t] (37.9665 Cos[9.49162 x] - Sin[9.49162 x]) +  
 0.000013698 Cos[10.4924 t] (41.9697 Cos[10.4924 x] - Sin[10.4924 x]) -  
 9.06328 × 10<sup>-6</sup> Cos[11.4931 t] (45.9723 Cos[11.4931 x] - Sin[11.4931 x]) +  
 6.75127 × 10<sup>-6</sup> Cos[12.4936 t] (49.9745 Cos[12.4936 x] - Sin[12.4936 x]) -  
 4.76456 × 10<sup>-6</sup> Cos[13.4941 t] (53.9764 Cos[13.4941 x] - Sin[13.4941 x]) +  
 3.70404 × 10<sup>-6</sup> Cos[14.4945 t] (57.978 Cos[14.4945 x] - Sin[14.4945 x]) -  
 2.7398 × 10<sup>-6</sup> Cos[15.4949 t] (61.9795 Cos[15.4949 x] - Sin[15.4949 x]) +  
 2.19887 × 10<sup>-6</sup> Cos[16.4952 t] (65.9807 Cos[16.4952 x] - Sin[16.4952 x]) -  
 1.6857 × 10<sup>-6</sup> Cos[17.4955 t] (69.9818 Cos[17.4955 x] - Sin[17.4955 x]) +  
 1.38667 × 10<sup>-6</sup> Cos[18.4957 t] (73.9828 Cos[18.4957 x] - Sin[18.4957 x]) -  
 1.09341 × 10<sup>-6</sup> Cos[19.4959 t] (77.9837 Cos[19.4959 x] - Sin[19.4959 x])



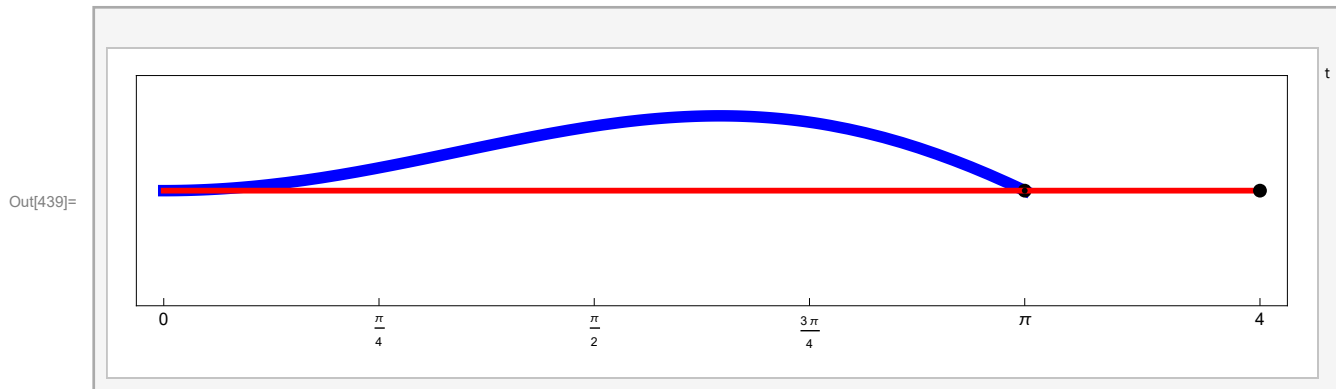
In[437]:= uu4[1, 0.]

Out[437]= 0.42072

In[438]:= Plot[Evaluate[{{ff4[x]}, uu4[x, 0]}], {x, 0, Pi},  
 PlotStyle → {{Thickness[0.01], Blue}, {Thickness[0.005], Yellow}},  
 Epilog → {{Red, Thickness[0.005], Line[{{4, 0}, {0, uu4[0, 0]}]}},  
 PlotRange → {{-0.1, 4 + .1}, {-1.2, 1.2}}, AspectRatio → 1 / 5, Frame → True,  
 FrameTicks → {{{}, {}}, {Join[{4}, Range[0, Pi, Pi / 4]], {}},  
 Axes → False, ImageSize → 600]



In[439]:= Manipulate[Plot[Evaluate[uu4[x, t]], {x, 0, Pi}, PlotStyle → {{Thickness[0.01], Blue}},  
 Epilog → {{PointSize[0.012], Point[#] & /@ {{Pi, 0}}},  
 {Red, Thickness[0.005], Line[{{4, 0}, {0, uu4[0, t]}]}}, {Point[{Pi, 0]}},  
 {PointSize[0.012], Point[#] & /@ {{4, 0}}}, PlotRange → {{-0.1, 4 + .1}, {-1.4, 1.4}},  
 AspectRatio → 1 / 5, Frame → True, Axes → False,  
 FrameTicks → {{{}, {}}, {Join[{4}, Range[0, Pi, Pi / 4]], {}},  
 ImageSize → 600], {t, 0, 26.11, 0.1}]



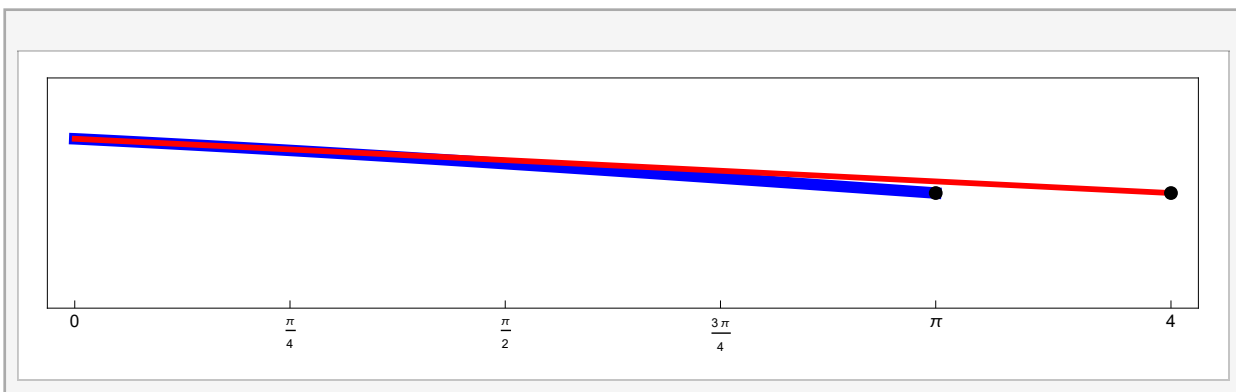
In[440]:= N[2 Pi / eves4[1]]

Out[440]= 25.1327

In[441]:= Clear[nmv4]; nmv4[x\_, t\_] := 
$$\left( \text{efs4}[x] * \left( \left( \frac{1}{\sqrt{16 \#^2 + 1}} \cos[\# t] \right) \& /@ \text{eves4} \right) \right)$$

```
In[442]= Manipulate[Plot[Evaluate[nmv4[x, t][[k]], {x, 0, Pi},
  PlotStyle -> {{Thickness[0.01], Blue}}, Epilog -> {{PointSize[0.012], Point[{Pi, 0]}},
    {Red, Thickness[0.005], Line[{{4, 0}, {0, Evaluate[nmv4[0, t][[k]]]}]},
    {PointSize[0.012], Point[{4, 0]}}
  }, PlotRange -> {{-0.1, 4 + 0.1}, {-1.5, 1.5}}, AspectRatio -> 1 / 5,
  Frame -> True, FrameTicks -> {{{}, {}}, {Join[{4}, Range[0, Pi, Pi / 4]], {}}, {}},
  Axes -> False, ImageSize -> 600], {t, 0, N[2 Pi / eves4[[1]]], 0.05},
  {{k, 1}, Range[10], ControlType -> Setter}]
```

Out[442]=



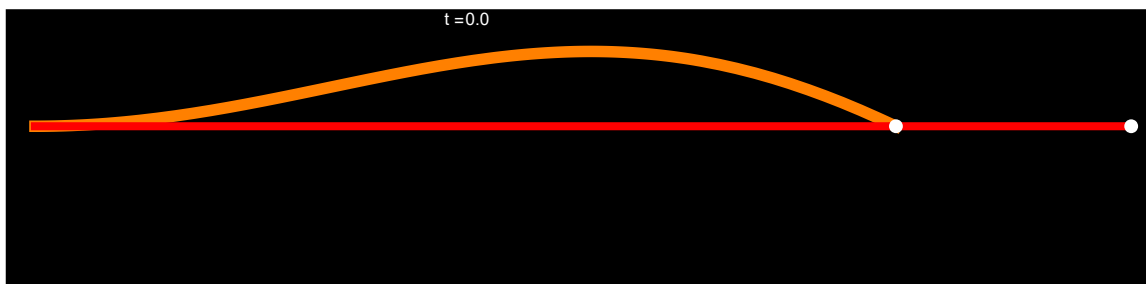
```
In[443]= aniuu4BB = Table[Plot[Evaluate[uu4[x, t]],
  {x, 0, Pi}, PlotStyle -> {{Thickness[0.01], RGBColor[1, 0.5, 0]}},
  Epilog -> {
    {Red, Thickness[0.007], Line[{{4, 0}, {0, Evaluate[uu4[0, t]]]}]},
    {PointSize[0.012], White, Point[#] & /@ {{4, 0}, {Pi, 0}}},
    {Text["t =", {1.45 + 0.08, 1.3}], BaseStyle -> {FontWeight -> "Normal", FontColor ->
      RGBColor[1, 1, 1]}},
    {Text[NumberForm[N[t], {3, 1}], {1.5 + 0.115, 1.3}],
      BaseStyle -> {FontWeight -> "Normal", FontColor -> RGBColor[1, 1, 1]}}
  },
  PlotRange -> {{-0.1, 4 + 0.1}, {-1.4, 1.4}}, AspectRatio -> 1 / 5,
  Frame -> True, FrameTicks -> {{{}, {}}, {Join[Range[0, Pi, Pi / 4], {4}], {}}, {}},
  Axes -> False, ImageSize -> 800, Background -> Black], {t, 0, 26, 0.1}];
```

```
In[444]= Length[aniu4BB]
```

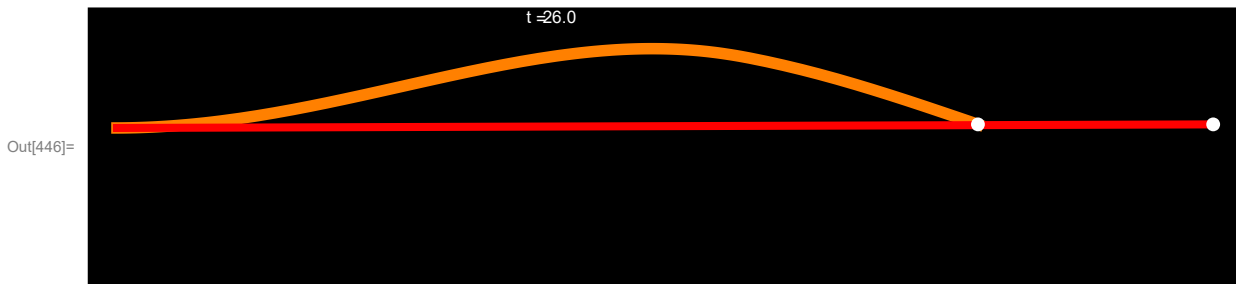
Out[444]= 261

```
In[445]= Show[aniu4BB[[1]], ImageSize -> 600]
```

Out[445]=



In[446]:= `Show[aniuu4BB[[261]], ImageSize -> 600]`



In[447]:= `ListAnimate[Show[#, ImageSize -> 600] & /@ aniuu4BB]`

`(* dds=0.1&@Range[Length[aniuu4BB]]; (* duration of each frame that we want*)`

`(* Export["aniuu4BBs1.gif", aniuu4BB[[1], "ImageSize" -> 800];  
Export["aniuu4BBAni.gif", aniuu4BB, "AnimationRepetitions" -> Infinity,  
"ImageSize" -> 800, "DisplayDurations" -> dds] *)`

Out[462]= `aniuu4BBAni.gif`

## Back to Solution 1, but with different sizes of a super-glued part

```

In[448]= Manipulate[Module[{meves1, mefs1, mff1, mcoefs1, mefns1, muu1},
  meves1 = Map[(μ /. FindRoot[h μ Cos[π μ] + Sin[π μ] == 0, {μ, #}]) &,
    Join[{.9, 1.7}, Range[2.6, 20, 1]]];
  mefs1[x_] = (h # Cos[# x] + Sin[# x]) & /@ meves1;
  mefns1 = Chop[Table[NIntegrate[mefs1[x][[j]] × mefs1[x][[j]], {x, 0, Pi}, MaxRecursion → 200,
    AccuracyGoal → 10, PrecisionGoal → 10], {j, 1, Length[meves1]}]];
  mff1[x_, h_] =  $\frac{h^{3/4}}{12} (-1 + x) (-\pi + x) \left(\frac{\pi}{1 + \pi} + x\right) \left(1 + \frac{x}{h}\right)$ ;
  mcoefs1 = Chop[Table[NIntegrate[(mff1[x, h]) mefs1[x][[j]], {x, 0, Pi}, MaxRecursion → 200,
    AccuracyGoal → 10, PrecisionGoal → 10], {j, 1, Length[meves1]}]];
  muu1[x_, t_] = (mefs1[x] * ((Cos[# t]) & /@ meves1)) .  $\left(\frac{mcoefs1}{mefns1}\right)$ ;
  Plot[Evaluate[muu1[x, t]], {x, 0, Pi}, PlotStyle → {{Thickness[0.01], Blue}},
  Epilog → {
    {Red, Thickness[0.01], Line[{{-h, 0}, {0, Evaluate[muu1[0, t]}]}]},
    {PointSize[0.012], Point[#] & /@ {{-h, 0}, {Pi, 0}}}
  },
  PlotRange → {{-2.1, Pi + 0.1}, {-1.2, 1.2}}, AspectRatio → 1 / 5, Frame → True,
  FrameTicks → {{}, {}}, {Join[Range[-2,  $-\frac{1}{4}$ ,  $\frac{1}{4}$ ], Range[0, Pi, Pi / 4]], {}},
  Axes → False, ImageSize → 500]
], {t, 0, 7.3, 0.05}, {{h, 1},  $\frac{1}{4}$ , 2,  $\frac{1}{4}$ , ControlType → Setter}]

```

Out[448]=

