A Krein Space Approach to Elliptic Eigenvalue Problems with Indefinite Weights

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1 Introduction

We consider the eigenvalue problem

$$Lu = \lambda r u, \tag{1}$$

where L is a symmetric elliptic operator and r is a locally integrable function on $\mathbf{R}^n$. If r is of constant sign then this problem leads to a selfadjoint problem in the Hilbert space $L^2(|r|)$. In this note we are interested in the case when r takes both positive and negative values on sets of positive measure. Then the spectrum of the problem (1) is not necessarily real any more. Moreover it is not apparent that the spectrum does not cover the whole complex plane. Even when it is discrete there can be nonsimple eigenvalues. For ordinary differential equations the related completeness problem of the eigenfunctions has been studied extensively in recent years see for example [3, 6] and the references quoted therein. The corresponding question in the case of continuous spectrum has been addressed in [6].

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When $n > 1$ some special properties of the spectrum of (1) have been considered in [1, 8, 9, 10, 11, 12]. In all of these papers the spectrum is discrete. To ensure the discreteness of the spectrum the problem (1) had to be considered on a bounded domain Ω with appropriate boundary conditions.

The important question of completeness of the eigenfunctions in the space $L^2(\Omega, |r|)$ has been recently considered in [19]. In that paper an operator-theoretic approach has been used to give sufficient conditions for the eigenvectors of the generalized selfadjoint eigenvalue problem $Su = \lambda Tu$ to form a Riesz basis. Applied to the problem (1) on the bounded domain Ω with the weight function r bounded and bounded away from zero and satisfying certain smoothness conditions, these results imply that eigenfunctions of (1) form a Riesz basis in $L^2(\Omega, |r|)$. In case that r is not bounded away from zero it is not simple to state explicit assumptions on r so that all the abstract conditions for the Riesz property are satisfied. In [19] some of these conditions on r are given explicitly and for the remaining ones it is shown that they correspond to standard problems in the theory of function spaces and interpolation theory.

In problem (1) the operator $\frac{1}{r}L$ is symmetric with respect to the form $[u|v] = \int u\bar{v}r$. Therefore, the space $L^2(|r|)$ endowed with this form is a natural underlying space for this problem. This space is a Krein space. See [2, 4, 17] for the definitions and properties of Krein spaces and linear operators in them; for the convenience of the reader we give a short review of basic notions at the end of the Introduction. The operator A associated with the formal operator $\frac{1}{r}L$ is selfadjoint in the Krein space $(L^2(|r|), [\,\cdot\,|\,\cdot\,])$. Under certain conditions A is definitizable. Therefore the Krein-Langer spectral theory of definitizable operators can be applied. This theory extends the classical spectral theory for selfadjoint operators in a Hilbert space. In particular it follows that the nonreal spectrum of A is symmetric with respect to the real line and it consists of at most finitely many eigenvalues of finite algebraic multiplicity. Furthermore, A has a spectral function. With the exception of finitely many critical points this spectral function has properties analogous to the properties of the spectral function of a selfadjoint operator in a Hilbert space. The appearance of critical points is one of the most interesting aspects of the theory and a lot of research has been devoted to the analysis of such points. In particular the character of ∞ as a critical point has been investigated in [5, 22]. In the case of an operator with discrete spectrum the only possible nonisolated critical point is ∞. Let ∞ be a critical point of A. We show below that there exists a Riesz basis consisting of eigenvectors and associated eigenvectors of A if and only if

∞ is a regular critical point of A. This observation enables us to obtain results about the Riesz property of eigenvectors from the results of [5]. In particular, the results from [19] about the equation (1) on a bounded domain Ω can also be deduced from operator-theoretic results in [5]. Since the Riesz basis property is a particular case of the more general concept of regularity of ∞ it turns out that the methods developed in [19] for the discrete case can be applied to the case of continuous spectrum. This remark is essential for our approach to the equation (1) in unbounded domains.

In Section 2 of this note we develop the operator-theoretic background for our study of problem (1). We give several necessary and sufficient conditions for ∞ not to be a singular critical point of a definitizable operator in a Krein space. These results are reformulations of results from [5, 6, 19] adapted to the problem under consideration.

In Section 3 we consider the equation (1) on an unbounded domain. In the case of the weight function r being bounded and bounded away from 0, we give explicit conditions for nonsingularity of the critical point ∞ for the operator A. Our results extend corresponding results from [19] with the Riesz property of the basis replaced by the nonsingularity of the critical point ∞. If r is not bounded away from zero then we show that the approach of [19] carries over to the case of unbounded domains. As in [19] some of the assumptions on r are stated implicitly. This method indicates that results from the theory of function spaces are relevant for a better understanding of this problem.

Let $\mathcal{K}$ be a vector space and let $[\,\cdot\,|\,\cdot\,]$ be an indefinite scalar product on $\mathcal{K}$. The pair $(\mathcal{K}, [\,\cdot\,|\,\cdot\,])$ is a *Krein space* if there exists a direct sum decomposition $\mathcal{K} = \mathcal{K}_+ \dot{+} \mathcal{K}_-$ such that $(\mathcal{K}_\pm, \pm[\,\cdot\,|\,\cdot\,])$ are Hilbert spaces. For such a decomposition the corresponding *fundamental symmetry* J is a linear operator defined by $J(x_+ + x_-) = x_+ - x_-$. Let $(u|v) = [Ju|v]$ for $u, v \in \mathcal{K}$. Then the space $(\mathcal{K}, (\,\cdot\,|\,\cdot\,))$ is a Hilbert space. Its topology is independent of the choice of $\mathcal{K}_+$ and $\mathcal{K}_-$. The definitions of symmetric, selfadjoint and positive operators in a Krein space parallel those in a Hilbert space. A selfadjoint operator A is *definitizable* if its resolvent set is nonempty and there exists a nonzero polynomial p such that $[p(A)u|u] \geq 0$ for all u in the domain of $p(A)$. A definitizable operator has a spectral function with finitely many critical points in $\mathbf{R} \cup \{\infty\}$; see [17]. A critical point α is *regular* if the spectral function is bounded in a neighborhood of α. A critical point is *singular* if it is not regular.

2 Regularity of the critical point ∞

Let $(\mathcal{H}_0, [\,\cdot\,|\,\cdot\,])$ be a Krein space and A an operator with nonempty resolvent set in $\mathcal{H}_0$. Let J be a fundamental symmetry on $\mathcal{H}_0$ and $(\,\cdot\,|\,\cdot\,)$ the corresponding Hilbert space scalar product and $\|\cdot\|$ the corresponding norm.

Let λ be a point in the resolvent set of A. Denote by $\mathcal{H}_2(A)$ the space $\mathcal{D}(A)$ with the norm $\|(A-\lambda)\cdot\|$. Define the space $\mathcal{H}_{-2}(A)$ as the completion of $\mathcal{H}_0$ with respect to the norm $\|(A-\lambda)^{-1}\cdot\|$. If B is another operator with nonempty resolvent set in $\mathcal{H}_0$ such that $\mathcal{D}(B) = \mathcal{D}(A)$, then $\mathcal{H}_2(A) = \mathcal{H}_2(B)$ and the corresponding norms on these spaces are equivalent. Analogously, if the adjoints of A and B in $[\,\cdot\,|\,\cdot\,]$ (or equivalently in $(\,\cdot\,|\,\cdot\,)$) have the same domain, then $\mathcal{H}_{-2}(A) = \mathcal{H}_{-2}(B)$ and the corresponding norms on these spaces are equivalent.

For $s \in (0,2)$, define $\mathcal{H}_s(A)$ by complex interpolation (see [18, 21]) between the Hilbert spaces $\mathcal{H}_2(A)$ and $\mathcal{H}_0(A)$:

$$\mathcal{H}_s(A) = [\mathcal{H}_2(A),\, \mathcal{H}_0(A)]_{1-s/2}\,.$$

For $s \in (-2,0)$, define

$$\mathcal{H}_s(A) = [\mathcal{H}_0(A),\, \mathcal{H}_{-2}(A)]_{-s/2}\,.$$

The spaces $\mathcal{H}_s$, $-2 \le s \le 2$, and their norms do not depend either on J or on λ. If there is no possibility of confusion we will write $\mathcal{H}_s$ instead of $\mathcal{H}_s(A)$.

Assume additionally that A is selfadjoint in the Krein space $(\mathcal{H}_0, [\,\cdot\,|\,\cdot\,])$. Then the operator JA is selfadjoint in the Hilbert space $(\mathcal{H}_0, (\,\cdot\,|\,\cdot\,))$. Put $B = J(|JA| + I)$. Then B is a positive selfadjoint operator with a bounded inverse in $(\mathcal{H}_0, [\,\cdot\,|\,\cdot\,])$. The operators A and B clearly have the same domain. Since they are also selfadjoint we have

$$\mathcal{H}_2(A) = \mathcal{H}_2(B) \qquad \text{and} \qquad \mathcal{H}_{-2}(A) = \mathcal{H}_{-2}(B)$$

and the corresponding norms are equivalent. Consequently

$$\mathcal{H}_s(A) = \mathcal{H}_s(B), \quad -2 \le s \le 2.$$

The operator JB is boundedly invertible and positive in $(\mathcal{H}_0, (\,\cdot\,|\,\cdot\,))$. By definition of the interpolation spaces it follows that

$$\mathcal{H}_s(A) = \mathcal{H}_s(B) = \mathcal{H}_s(JB) = \mathcal{D}\left((JB)^{s/2}\right), \;\; 0 \le s \le 2. \tag{2}$$

Similarly, since the adjoints of the operators A, B, BJ have the same domain, it follows that

$$\mathcal{H}_{-s}(A) = \mathcal{H}_{-s}(B) = \mathcal{H}_{-s}(BJ), \quad 0 \leq s \leq 2. \tag{3}$$

This space can be obtained as the completion of $\mathcal{H}_0$ with respect to the norm $\|(BJ)^{-s/2} \cdot \|$.

Next we characterize $\mathcal{H}_1(B)$ and $\mathcal{H}_{-1}(B)$ without referring to the fundamental symmetry J. Taking $s = 1$ in (2) we see that $\mathcal{H}_1(B)$ is the completion of $\mathcal{D}(B)$ with respect to the form $[B \cdot \,|\, \cdot]$. Analogously, with $s = 1$ in (3), we see that $\mathcal{H}_{-1}(B)$ is the completion of $\mathcal{H}_0$ with respect to the form $[B^{-1} \cdot \,|\, \cdot]$. These characterizations hold for any operator B which is positive in $(\mathcal{H}_0, [\,\cdot\,|\,\cdot\,])$ and has a bounded inverse.

The operators JB and BJ are positive selfadjoint operators in the Hilbert space $(\mathcal{H}_0, (\,\cdot\,|\,\cdot\,))$. They are similar: $J(JB)J = BJ$. Consequently $J(JB)^s = (BJ)^s J$. It follows from (2) and (3) that

$$\mathcal{H}_s(A) = \mathcal{H}_s(B) = \mathcal{H}_1(J(JB)^s) = \mathcal{H}_1((BJ)^s J), \;\; 0 \leq s \leq 2, \tag{4}$$

and

$$\mathcal{H}_{-s}(A) = \mathcal{H}_{-s}(B) = \mathcal{H}_{-1}((BJ)^s J) = \mathcal{H}_{-1}(J(JB)^s), \;\; 0 \leq s \leq 2. \tag{5}$$

Next we give criteria for the regularity of ∞, as a critical point of A in terms of the spaces $\mathcal{H}_s$.

Theorem 2.1 *Let A be a definitizable operator in the Krein space $(\mathcal{H}_0, [\,\cdot\,|\,\cdot\,])$. Then the following statements are equivalent.*

(i) ∞ *is not a singular critical point of A.*

(ii) *There exist $s \in (0, 2]$ and an operator W_s in $\mathcal{H}_0$ with the following three properties.*

  (A) *The operator W_s is positive and has a bounded and everywhere defined inverse.*

  (B) *The space $\mathcal{H}_s$ is invariant under W_s.*

  (C) *The restriction of W_s to $\mathcal{H}_s$ is bounded in $\mathcal{H}_s$.*

(iii) *There exist $s \in (0, 2]$ and an operator W_s in $\mathcal{H}_0$ with the properties* (A), (B) *and* (C′)*, where:*

(C′) *The operator* W_s *is bounded on* $\mathcal{H}_0$.

(iv) *For every* $s \in (0,2]$ *there exists an operator* W_s *in* $\mathcal{H}_0$ *which has the properties* (A), (B) *and* (C).

(v) *For every* $s \in (0,2]$ *there exists an operator* W_s *in* $\mathcal{H}_0$ *which has the properties* (A), (B) *and* (C′).

(vi) *There exists* $s \in (0,2]$ *such that*

$$[\mathcal{H}_s, \mathcal{H}_{-s}]_{1/2} = \mathcal{H}_0. \tag{6}$$

(vii) *For every* $s \in (0,2]$ *the interpolation equality* (6) *holds.*

Proof Let $B = J(|JA| + I)$. It follows from [5, Corollary 3.3] that (i) is equivalent to

(i-1) ∞ is not a singular critical point of B.

By [5, Theorem 3.9] the statement (i-1) is equivalent to

(i-2) ∞ is not a singular critical point of $J(JB)^{s/2}$ for all $s \in (0,2]$.

Next we fix s in $(0,2]$ and we apply [5, Theorem 2.5] to the operator $J(JB)^{s/2}$. In view of the equation (2) the equivalence of (vi), (ii) and (iii) in [5, Theorem 2.5] translates to the equivalence of (i-2), (ii) and (iii) above. Since s is arbitrary in $(0,2]$, it follows that (i) is also equivalent to (iv) and (v) above.

Now we prove that (i) is equivalent to (vi) for $s = 1$:

(vi-1) $$[\mathcal{H}_1(B), \mathcal{H}_{-1}(B)]_{1/2} = \mathcal{H}_0.$$

It is sufficient to prove the equivalence of (i-1) and (vi-1). Because of (3) the Hilbert space $\mathcal{H}_{-1}(B)$ is the completion of $\mathcal{H}_0$ with respect to the scalar product $[B^{-1} \cdot \,|\, \cdot] = (\,\cdot\,|\,\cdot\,)_{-1}$. Similarly the Hilbert space $\mathcal{H}_1(B)$ is the completion of $\mathcal{D}(B)$ with respect to the scalar product $[B \cdot \,|\, \cdot] = (\,\cdot\,|\,\cdot\,)_1$. The operator B, considered as an operator in $\mathcal{H}_{-1}(B)$, is a densely defined essentially selfadjoint operator (since $\|Bv\|_{-1} \geq C\|v\|_{-1}$ for all $v \in \mathcal{D}(B)$). Its closure B_e is a selfadjoint operator in $\mathcal{H}_{-1}(B)$. Similarly we define a selfadjoint operator B_r in $\mathcal{H}_1(B)$ as the closure in $\mathcal{H}_1(B)$ of the restriction of B to $\mathcal{D}(B^2)$. Then B_r is a restriction of B and B is a restriction of B_e. It follows that $|B_r|^{1/2}$ is a restriction of $|B_e|^{1/2}$. Therefore,

$$\begin{aligned} \||B_e|^{1/2}v\|_{-1}^2 &= \|B_e|B_e|^{-1/2}v\|_{-1}^2 = \||B_r|^{-1/2}v\|_1^2 \\ &= [B|B_r|^{-1/2}v||B_r|^{-1/2}v] = [B|B_r|^{-1}v|v] \end{aligned} \tag{7}$$

for all $v \in \mathcal{D}(B)$. Also $\|B_e v\|_{-1} = \|v\|_1$ for all $v \in \mathcal{D}(B)$. It follows that $\mathcal{D}(B_e) = \mathcal{H}_1(B)$ and consequently $[\mathcal{H}_1(B), \mathcal{H}_{-1}(B)]_{1/2} = \mathcal{D}(|B_e|^{1/2})$. Therefore, (vi-1) is equivalent to the following condition:

(vi-2) The topology on $\mathcal{D}(B)$ induced by the positive definite scalar product $[B|B_r|^{-1} \cdot | \cdot]$ coincides with the strong topology inherited from $\mathcal{H}_0$.

The equivalence of (i-1) and (vi-2) has been shown in [5, Theorem 2.5 (iii),(v)]. This proves the equivalence of (i-1) and (vi-1).

Again by [5, Theorem 3.9] the statement (i-1) is equivalent to ∞ not being a singular critical point of $J(JB)^s$ for arbitrary s in $(0, 2]$. By the equivalence of (i-1) and (vi-1) applied to the operator $J(JB)^s$ and by (4) and (5), it follows that (i-1) is equivalent to (6). Since $s \in (0, 2]$ was arbitrary, the equivalence of (i), (vi) and (vii) follows. □

The following fact is well known; see [5, Theorem 2.5].

Lemma 2.2 *Let A be a positive operator with a bounded inverse. Then the statements* (i) *through* (vii) *are equivalent to:*

(viii) *The operator A is similar to a selfadjoint operator in the Hilbert space $(\mathcal{H}_0, (\cdot|\cdot))$.*

Proposition 2.3 *Let A be a definitizable operator with discrete spectrum. Then the statements* (i) *through* (vii) *are equivalent to:*

(ix) *For any choice of bases in the algebraic eigenspaces of A the basis vectors can be renormalized so that their union forms a Riesz basis in the Krein space $(\mathcal{H}_0, [\cdot|\cdot])$.*

Proof Let D be an open disc containing all nonreal eigenvalues and all finite critical points of A. Then the spectral projection $E(D)$ is selfadjoint in the Krein space $\mathcal{H}_0$. Let $\mathcal{H}_0^\infty = (I - E(D))\mathcal{H}_0$. The orthogonal direct sum $\mathcal{H}_0 = E(D)\mathcal{H}_0 \oplus \mathcal{H}_0^\infty$ reduces A. Since the spectrum of A is discrete, the space $E(D)\mathcal{H}_0$ is finite-dimensional. Therefore, the eigenvectors and associated eigenvectors of A form a Riesz basis for $E(D)\mathcal{H}_0$. If the operator A has a definitizing polynomial of even degree, then the restriction A_∞ of A to $\mathcal{H}_0^\infty$ is a selfadjoint operator in the Hilbert space $(\mathcal{H}_0^\infty, |[\cdot|\cdot]|)$. Consequently, (i) and (ix) are both satisfied. If all definitizing polynomials of A are of odd degree, then the operator A_∞ is a positive or negative operator with discrete spectrum and bounded inverse. Also, ∞ is not a singular critical point of A_∞ if and only if ∞ is not a singular critical point of A. Consequently, it is sufficient to prove the lemma for a positive operator A with bounded inverse and discrete spectrum. By Lemma 2.2, (i) is equivalent to (viii). The equivalence of (viii) and (ix) is a consequence of the definition of a Riesz basis. □

Remark 2.4 The equivalence of (vi$_1$) and (ix), and a special case of the implication (ii) $\Rightarrow$ (vi-1) are proved in [19, Theorems 2.1 and 2.2].

Remark 2.5 The definition of A_∞ in the proof of Proposition 2.3 does not depend on the discreteness of the spectrum. Let A be a definitizable operator such that ∞ is not a singular critical point of A. By Lemma 2.2, the operator A_∞ is similar to a selfadjoint operator in the Hilbert space $\mathcal{H}_0^\infty$. Therefore, the operators $\mu(A_\infty - i\mu I)^{-1}$, for $\mu \in \mathbf{R}\setminus\{0\}$ are uniformly bounded. Since $A - A_\infty$ is bounded, there exist positive numbers C and μ_0 such that $i\mu$ belongs to $\rho(A)$ for $|\mu| > \mu_0$, and the estimate

$$\|(A - i\mu I)^{-1}\| \leq \frac{C}{|\mu|}\,,\ \ |\mu| > \mu_0 \tag{8}$$

holds.

Remark 2.6 For a definitizable operator A with discrete spectrum it is always true that for any choice of bases in the algebraic eigenspaces of A the basis vectors can be renormalized so that their union forms a Riesz basis in the Hilbert space $\mathcal{H}_1(A)$. In this case the form $[A\cdot|\cdot]$ has finite number of negative squares and the construction described in [17, Example (c), pp. 11-12] leads to a selfadjoint operator $\hat{A}$ in the Pontryagin space $(\mathcal{H}_1/\ker A, [A\cdot|\cdot])$ which has the same eigenstructure, except for the eigenvalue zero, as the operator A. It is well known that a selfadjoint operator in a Pontryagin space is a definitizable operator and ∞ is not its critical point.

In the next proposition we use the concept of relative compactness. For its definition and properties see [16]. The following proposition is a consequence of more general results of [14, Theorem 3.6]; see also [13, Theorem

3] or [7, Section 2]. For the convenience of the reader we present a simple proof of the result which will be used in this paper.

We say that a definitizable operator A is *quasi-uniformly positive* if there exists a subspace $\mathcal{M}$, $\mathcal{M} \subset \mathcal{D}(A)$, of finite codimension such that the form $[Ax|y]$, $x, y \in \mathcal{M}$, is uniformly positive.

Proposition 2.7 *Let A be a quasi-uniformly positive operator in the Krein space $(\mathcal{H}_0, [\,\cdot\,|\,\cdot\,])$. Assume that ∞ is not a singular critical point of A. Let V be a symmetric operator which is relatively compact with respect to A. Then the operator $A+V$ is also a quasi-uniformly positive operator and ∞ is not a singular critical point of $A+V$. Moreover, the essential spectrum of $A+V$ coincides with the essential spectrum of A.*

Proof Let J be a fundamental symmetry on $(\mathcal{H}_0, [\,\cdot\,|\,\cdot\,])$ and let $(\,\cdot\,|\,\cdot\,) = [J\cdot\,|\,\cdot\,]$ be the corresponding Hilbert space scalar product. The operator V is A-compact if and only if the operator JV is JA-compact. This equivalence follows from the identity

$$\begin{aligned} &JV(JA-\lambda I)^{-1} - JV(A-\lambda I)^{-1}J \\ &\quad = \lambda JV(A-\lambda I)^{-1}(J-I)(JA-\lambda I)^{-1} \\ &\quad = \lambda JV(JA-\lambda I)^{-1}(J-I)(A-\lambda I)^{-1}J \end{aligned}$$

which holds for every $\lambda \in \rho(JA) \cap \rho(A)$. The operator $JA + JV$ is a selfadjoint operator in the Hilbert space $(\mathcal{H}_0, (\,\cdot\,|\,\cdot\,))$ and consequently $A+V$ is selfadjoint in $(\mathcal{H}_0, [\,\cdot\,|\,\cdot\,])$.

Since an A-compact operator is A-bounded with the relative bound zero, for any $a > 0$ there exists $b > 0$ such that

$$\|Vx\| \le a\|Ax\| + b\|x\|\,, \;\; x \in \mathcal{D}(A). \tag{9}$$

It follows from the inequalities (8) and (9) that, for $|\mu| > \mu_0$ and $x \in \mathcal{H}_0$,

$$\begin{aligned} \|V(A-i\mu I)^{-1}x\| &\le a\|A(A-i\mu I)^{-1}x\| + b\|(A-i\mu I)^{-1}x\| \\ &= a\|I + i\mu(A-i\mu I)^{-1}x\| + b\|(A-i\mu I)^{-1}x\| \\ &\le \left[a(1+C) + \frac{b}{|\mu|}\right]\|x\|. \end{aligned}$$

Consequently, we have $\|V(A-i\mu I)^{-1}\| < 1$ if we choose a sufficiently small and $|\mu|$ sufficiently large. Since

$$A + V - i\mu I = \left[I + V(A-i\mu I)^{-1}\right](A-i\mu I),$$

it follows that $i\mu \in \rho(A+V)$ when $|\mu|$ is sufficiently large. In particular, $\rho(A+V)$ is nonempty.

Since A is quasi-uniformly positive, the essential spectrum of JA is strictly positive. As the operator JV is JA-compact, it follows that the essential spectrum of the operator $J(A+V)$ is also strictly positive. Therefore, the form $[(A+V)x|y],\ x,y \in \mathcal{D}(A)$, is uniformly positive on a subspace of finite codimension. By [17, (c) on page 11], the operator $A+V$ is definitizable. Hence $A+V$ is a quasi-uniformly positive operator.

As $\mathcal{D}(A+V) = \mathcal{D}(A)$, it follows from [5, Corollary 3.3] that ∞ is not a singular critical point of $A+V$.

The essential spectrum of a definitizable operator A in a Krein space consists of those points in the spectrum of A which are not isolated eigenvalues of finite algebraic multiplicity. The last statement follows from [16, Theorem IV.5.35]. □

3 Applications to differential operators

3.1 We apply the above results to the spectral problem

$$(-\Delta + 1)u = \lambda\, r\, u \quad \text{in} \quad \Omega = \mathbf{R}^n, \tag{10}$$

with a real weight function r which changes sign in Ω. This is the simplest elliptic eigenvalue problem with indefinite weight function in which continuous spectrum occurs. Such problems do not seem to be studied in the literature. We assume that the weight function r is measurable and there exist positive numbers m and M such that $m \le |r| \le M$ almost everywhere on $\mathbf{R}^n$. Furthermore, we assume that the sets

$$\Omega^+ = \{x \in \mathbf{R}^n : r(x) > 0\} \quad \text{and} \quad \Omega^- = \{x \in \mathbf{R}^n : r(x) < 0\}$$

are unions of finitely many domains with sufficiently smooth boundaries. Let $\mathcal{H}_0 = L^2(\mathbf{R}^n)$ and let $[u|v] = \int u\,\overline{v}\,r$. Then $(\mathcal{H}_0, [\,\cdot\,|\,\cdot\,])$ is a Krein space. Let χ_+ and χ_- be the operators of multiplication by the characteristic functions of Ω^+ and Ω^-, respectively. Then $J = \chi_+ - \chi_-$ is a fundamental symmetry on this Krein space. The corresponding Hilbert space is the weighted Hilbert space $L^2(\mathbf{R}^n, |r|)$. Its norm is equivalent to the usual L^2 norm. With the problem (10) we associate the operator

$$A = \frac{1}{r}\,(-\Delta + 1)$$

with domain $\mathcal{D}(A) = H^2(\mathbf{R}^n)$. The operator A is boundedly invertible in $L^2(\mathbf{R}^n)$ as a product of boundedly invertible operators. Clearly,

$$[Au|u] \geq \int |u|^2 \geq \frac{1}{M} \int |u|^2 |r| \quad \text{for all} \quad u \in \mathcal{D}(A). \tag{11}$$

Therefore, A is a uniformly positive selfadjoint operator in the Krein space $(\mathcal{H}_0, [\,\cdot\,|\,\cdot\,])$. Note that $\mathcal{H}_s = H^{2s}(\mathbf{R}^n)$. We apply Theorem 2.1 to A. We will show that the operator J satisfies the assumption (ii) from that theorem. It is known that under our assumptions on Ω^+ and Ω^- the operators χ_+ and χ_- map the space $H^s(\mathbf{R}^n)$ continuously into itself for $s < \frac{1}{2}$; see [18] or [21, Lemma and Remark 1 in Section 2.10.2], noting that $H_2^s(\mathbf{R}^n) = W_2^s(\mathbf{R}^n)$ in the notation of [21]. Therefore, J satisfies all the properties required for W_s in (ii) and (iii) for $s < \frac{1}{4}$. Consequently, ∞ is not a singular critical point of A. It follows from the equivalence of (i) and (viii) that A is similar to a selfadjoint operator in $L^2(\mathbf{R}^n)$. Therefore, the spectral function of A has all the pertinent properties of the spectral function of a selfadjoint operator in a Hilbert space. This fact is useful in the construction of the eigenfunction expansion associated with problem (10). Also, it follows that the initial value problem

$$\begin{aligned} i\, r\, \frac{\partial u}{\partial t} &= (-\Delta + 1)u \\ u(0) &= u_0 \end{aligned}$$

is well posed in $L^2(\mathbf{R}^n)$ for $-\infty < t < \infty$. This is a consequence of the fact that iA generates a bounded C_0 group of operators.

It follows from (11) that the spectrum of negative (respectively, positive) type of A is contained in $(-\infty, -\frac{1}{M}]$ (respectively, in $[\frac{1}{M}, +\infty)$); for the definitions see [17, page 36]. Also, the resolvent set of A contains all nonreal numbers and the interval $(-\frac{1}{M}, \frac{1}{M})$.

3.2 In the previous example we could have considered more general elliptic operators, weight functions and domains. Instead of $\mathbf{R}^n$, Ω can be a domain with sufficiently smooth boundary. In that case appropriate boundary conditions must be introduced. The restrictions on the sets Ω^+ and Ω^- were used only to ensure that the operators χ_+ and χ_- map $H^s(\Omega)$ into itself for sufficiently small s. The cited references show that this is true for more general Ω^+ and Ω^- allowing a more general weight function r. The operator $(-\Delta + 1)$ may be replaced by a symmetric elliptic operator L of order $2m$ satisfying the following conditions.

(α) The Dirichlet form of L is defined on a closed subspace of $H^m(\Omega)$ defined by boundary conditions in the usual way.

(β) The Dirichlet form of L has a finite number of negative squares.

(γ) The associated operator in $L^2(\Omega)$ has a bounded inverse.

These conditions represent an implicit restriction on the coefficients of the operator. The conditions (β) and (γ) ensure that the associated operator A is definitizable (see [17]). The condition (α) is used in order to prove that the condition (ii) of Theorem 2.1 is satisfied for sufficiently small s. For $\Omega \neq \mathbf{R}^n$ this follows from the results of Grisvard and Seeley (see [21, Theorem in Section 4.3.3]) in the same way as it was done in [19]. In this more general situation it is no longer true that A is similar to a selfadjoint operator in a Hilbert space. However, ∞ is not a singular critical point of A. Consequently the corresponding initial value problem for the operator iA is again well posed in $L^2(\mathbf{R}^n)$ for $-\infty < t < \infty$. The corresponding C_0-group is not necessarily bounded.

If Ω is bounded then the spectrum of A is discrete and the equivalence of (i) and (ix) from Proposition 2.3 implies that there is a Riesz basis for $L^2(\Omega)$ consisting of eigenvectors and associated eigenvectors of the corresponding eigenvalue problem. This result was proved in [19, 3.1].

Assume that L has k negative eigenvalues, counted according to their multiplicities. It follows from [6, Corollary 1.6] that $A+V$ has at least k eigenvalues λ, counted according to their algebraic multiplicities, for each of which there exists an eigenfunction $u \in L^2(\mathbf{R}^n)$ such that $\lambda \int |u|^2\, r \leq 0$. All the finite critical points, if there are any, are included among these points. Each such point has finite rank of indefiniteness; see [15, page 64]. Studying the regularity of such critical points is simpler than studying the regularity of ∞, which has infinite rank of indefiniteness.

For additional spectral properties of $A+V$ see [6, Section 1.3].

3.3 Here we show that the condition (γ) above can be relaxed to allow 0 to be an isolated eigenvalue of L. To this end we again consider a model example and omit various straightforward generalizations. The equation is

$$(-\Delta + 1 + q)u = \lambda\, r\, u \quad \text{in} \quad \Omega = \mathbf{R}^n, \tag{12}$$

where r is as above and where q is relatively compact with respect to $-\Delta$; see [20] for sufficient conditions. As a consequence the operator $L = -\Delta + 1 + q$

is defined on $\mathcal{D}(L) = H^2(\mathbf{R}^n)$. Its spectrum in $(-\infty, 1)$ consists of at most finitely many eigenvalues of finite multiplicity. Assume that 0 is an (isolated) eigenvalue of L. Thus L satisfies the assumptions (α) and (β) above. Let $A = \frac{1}{r}(-\Delta + 1)$ and $V = \frac{q}{r}$. As in the proof of Proposition 2.7, the fact that q is relatively compact with respect to $-\Delta$ implies that V is relatively compact with respect to A. It follows from Proposition 2.7 that the operator $A+V$ is a definitizable operator in $(\mathcal{H}_0, [\,\cdot\,|\,\cdot\,])$ and that ∞ is not its singular critical point. The essential spectrum of $A+V$ coincides with the essential spectrum of A. Therefore the spectrum of $A+V$ in $(-\frac{1}{M}, \frac{1}{M})$ has finite total multiplicity.

From the previous considerations it follows that the operator $A+V$ has the same properties as the operator described at the end of the subsection 3.2.

The operator $-\Delta + 1$ can again be replaced by a more general elliptic operator and $\mathbf{R}^n$ by a more general domain Ω.

3.4 In this subsection we indicate how the assumptions on r might be relaxed. We consider first the problem

$$-\Delta u = \lambda r u \text{ in } \; \Omega = \mathbf{R}^n. \tag{13}$$

Assume that the real weight function r is locally integrable, almost everywhere different from zero. Then $C_0^\infty(\mathbf{R}^n)$ is dense in $\mathcal{H}_0 := L^2(\mathbf{R}^n, |r|)$. Indeed, assume that $u \in \mathcal{H}_0$ is orthogonal to $C_0^\infty(\mathbf{R}^n)$ in $\mathcal{H}_0$. Let $v = |r|^{1/2}u$. Then $v \in L^2(\mathbf{R}^n)$ and $\int |r|^{1/2} v\phi = \int u\phi|r| = 0$ for all $\phi \in C_0^\infty(\mathbf{R}^n)$. Since $|r|^{1/2} \in L^2_{\mathrm{loc}}(\mathbf{R}^n)$, it follows that $z = v|r|^{1/2} \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$. Therefore, $\int z\phi = 0$ for all $\phi \in C_0^\infty(\mathbf{R}^n)$, implying that $z = 0$. Since $z = u|r|$ and $r \neq 0$ almost everywhere, it follows that $u = 0$ almost everywhere.

In order to define a definitizable operator in $\mathcal{H}_0$ associated with problem (13), we impose the following two additional conditions on r. We assume that there exists $C > 0$ such that

$$\int |u|^2|r| \leq C \int |\nabla u|^2 \quad \text{for all} \quad u \in C_0^\infty(\mathbf{R}^n). \tag{14}$$

We also assume that

$$g(u,v) := \int \nabla u \nabla v\,,\; u, v \in C_0^\infty(\mathbf{R}^n) \quad \text{is a closable form in} \quad \mathcal{H}_0\,. \tag{15}$$

Using the assumption (14) we close the form g and denote the the closure also by g. By the representation theorem (see [16, Theorem VI.2.1]) assumption

(15) implies the existence of a nonnegative selfadjoint operator G in the Hilbert space $\mathcal{H}_0$ such that $g(u,v) = \int Gu\,\overline{v}\,|r|$ for $u \in \mathcal{D}(G)$ and $v \in C_0^\infty(\mathbf{R}^n)$. It follows from assumption (14) that the operator G is bounded from below by $1/C$. Consequently, the operator G^{-1} is bounded on $\mathcal{H}_0$. Note that $Gu = -\frac{1}{|r|}\Delta u$ whenever $|r|^{-1/2}\Delta u \in L^2(\mathbf{R}^n)$.

The form $[u|v] = \int u\overline{v}r$ induces a Krein space structure on $\mathcal{H}_0$. The multiplication operator $Ju = (\operatorname{sgn} r)u$ is a fundamental symmetry on $\mathcal{H}_0$. Let $A = JG$. Then A is a positive selfadjoint operator in the Krein space $\mathcal{H}_0$. Since it has a bounded inverse $G^{-1}J$, it is a definitizable operator.

As in the preceding subsections we can replace $\mathbf{R}^n$ by a more general domain Ω and a more general Dirichlet form g. In that case the assumptions (14) and (15) must be changed accordingly.

Remark 3.1 Assumption (14) corresponds to the embedding assumption $H_1 \subset H_0$ from [19]. Note that the other embedding assumption $H_1 \subset E$ from [19] is sufficient to prove the closability of the form $(Lu|v)$ in H_0. Therefore, the above construction gives a definitizable operator in H_0 under the conditions of [19].

It remains to give sufficient conditions on r for ∞ not to be a singular critical point of A. To this end we may apply Theorem 2.1 with $W_s = J$, as in subsection 3.1. A way to verify the condition (iii) of Theorem 2.1 is given in [19, Lemma 3.1]. The proof there does not use the boundedness of Ω. Since our basic assumptions (14) and (15) of this subsection are implicit, we are not providing a detailed study of this problem at this point.

As noted in [19] the condition (iii) of Theorem 2.1 is a problem in interpolation theory of function spaces. We already used a result of this theory in subsection 3.1 for special r. To our knowledge the theory is not yet sufficiently developed to provide sufficient conditions in the case of general weight functions.

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