QUASI-UNIFORMLY POSITIVE OPERATORS IN KREIN SPACE

BRANKO ĆURGUS and BRANKO NAJMAN

Definitizable operators in Krein spaces have spectral properties similar to those of selfadjoint operators in Hilbert spaces. A sufficient condition for definitizability of a selfadjoint operator A with a nonempty resolvent set $\rho(A)$ in a Krein space $(\mathcal{H}, [\,\cdot\,|\,\cdot\,])$ is the finiteness of the number of negative squares of the form $[Ax|y]$ (see [10, p. 11]).

In this note we consider a more restrictive class of operators which we call *quasi-uniformly positive*. A closed symmetric form s is called *quasi-uniformly positive* if its isotropic part $\mathcal{N}_s$ is finite dimensional and the space $(\mathcal{D}(s), s(\,\cdot\,,\,\cdot\,))$ is a direct sum of a Pontryagin space with a finite number $\pi(s)$ of negative squares and $\mathcal{N}_s$. The number $\kappa(s) := \dim \mathcal{N}_s + \pi(s)$ is the number of nonpositive squares of s; it is called the *negativity index* of s. A selfadjoint operator A in a Krein space $(\mathcal{H}, [\,\cdot\,|\,\cdot\,])$ is *quasi-uniformly positive* if the form $a(x,y) = [Ax|y]$ defined on $\mathcal{D}(A)$ is closable and its closure $\overline{a}$ is quasi-uniformly positive. The number $\kappa(A) := \kappa(\overline{a})$ is the *negativity index* of A. Such operators often appear in applications, see [3, 4, 5] and Section 3 of this note.

It turns out that this class of operators is stable under relatively compact perturbations, see Corollaries 1.2 and 2.3. The perturbations as well as the operators are usually defined as forms, so the above definition is natural.

Most of the results in this note are known. In particular the perturbation results from Section 2 are consequences of the results of [7]. We have found it useful to state the results in the framework of quadratic forms and quasi-uniformly positive operators since the proofs and the statements are simpler but still sufficiently general for several important applications.

As an illustration of these results we consider the operator associated with the Klein-Gordon equation

$$\left[\left(\frac{\partial}{\partial t} - ieq\right)^2 - \sum_j \left(\frac{\partial}{\partial x_j} - ieA_j\right)^2 + m^2\right] u = 0.$$

Setting

$$u_1 = u, \quad u_2 = \left(-i\frac{\partial}{\partial t} - eq\right) u$$

we get a system of equations for (u_1, u_2). The associated operator is quasi-uniformly positive in a Krein space suggested by the physical interpretation of the equation. The obtained results are essentially known, see [8, 11].

In the first two sections of this note $(\mathcal{H}, [\cdot|\cdot])$ is a Krein space, $(\mathcal{H}, (\cdot|\cdot))$ is a Hilbert space and J is the corresponding fundamental symmetry.

1 Quasi-uniformly positive operators

In this section we prove that quasi-uniformly positive operators in a Krein space are definitizable.

PROPOSITION 1.1 *A quasi-uniformly positive operator A in the Krein space $(\mathcal{H}, [\cdot|\cdot])$ is definitizable.*

PROOF [1] Since $S = JA$ is quasi-uniformly positive in $(\mathcal{H}, (\cdot|\cdot))$, there exists a selfadjoint operator F_1 of finite rank such that $S + F_1$ is uniformly positive. Since $\mathcal{D}(S) = \mathcal{D}(A)$ is dense in $\mathcal{H}$, perturbing F_1 we see that there exists a selfadjoint operator F such that $\mathcal{R}(JF) \subset \mathcal{D}(S) = \mathcal{D}(A)$ and such that $H := S + F$ is uniformly positive. The operator JH is uniformly positive in the Krein space $\mathcal{H}$. Since 0 and all nonreal numbers are in the resolvent set of JH, the resolvent identity yields

$$(JH - z)^{-1} = zH^{-1/2}(H^{1/2}JH^{1/2} - z)^{-1}H^{-1/2}J + (JH)^{-1}, \quad z \neq \overline{z}.$$

Therefore

$$\sup_{\eta\in\mathbb{R}} \|(JH - i\eta)^{-1}\| < \infty. \tag{1}$$

From the resolvent identity and $\mathcal{R}(JF) \subset \mathcal{D}(JH)$, for arbitrary real numbers η, η_0 we get

$$\begin{gathered}(JH - i\eta)^{-1}JF = (JH - i\eta)^{-1}(JH - i\eta_0)^{-1}(JH - i\eta_0)JF = \\ -i(\eta - \eta_0)^{-1}\{(JH - i\eta)^{-1} - (JH - i\eta_0)^{-1}\}(JH - i\eta_0)JF.\end{gathered}$$

Now (1) implies

$$\lim_{\eta\to\pm\infty} \|(JH - i\eta)^{-1}JF\| = 0.$$

Therefore, for sufficiently large $|\eta|$ the operator $I + (JH - i\eta)^{-1}JF$ has bounded inverse. Since

$$A - i\eta = JH - i\eta - JF = (JH - i\eta)(I + (JH - i\eta)^{-1}JF), \tag{2}$$

it follows that $i\eta \in \rho(A)$ for sufficiently large $|\eta|$. Consequently [10, (c) p. 11] implies that A is definitizable. □

In the next proposition we use the concept of relative compactness for operators. For its definition and properties see [9].

PROPOSITION 1.2 *The class of quasi-uniformly positive operators in a Krein space is closed with respect to relatively compact additive perturbations.*

[1] The authors are grateful to Prof. Peter Jonas for providing this proof which is significantly shorter than the original one.

PROOF Let A be a quasi-uniformly positive operator in the Krein space $(\mathcal{H}, [\,\cdot\,|\,\cdot\,])$ and let V be a symmetric operator in $(\mathcal{H}, [\,\cdot\,|\,\cdot\,])$ which is relatively compact with respect to A. For every $\lambda \in \rho(JA) \cap \rho(A)$ the identity

$$\begin{aligned} JV(JA - \lambda I)^{-1} &- JV(A - \lambda I)^{-1} J \\ &= \lambda JV(A - \lambda I)^{-1}(J - I)(JA - \lambda I)^{-1} \\ &= \lambda JV(JA - \lambda I)^{-1}(J - I)(A - \lambda I)^{-1} J \end{aligned}$$

holds. Therefore the operator V is A-compact if and only if the operator JV is JA-compact. Since the operator JA is quasi-uniformly positive in the Hilbert space $(\mathcal{H}, (\,\cdot\,|\,\cdot\,))$, it follows from [9, Theorem IV.5.35] that the operator $JA + JV$ is quasi-uniformly positive in the Hilbert space. Consequently $A + V$ is quasi-uniformly positive in the Krein space. □

PROPOSITION 1.3 *Let A be a quasi-uniformly positive operator in the Krein space $(\mathcal{H}, [\,\cdot\,|\,\cdot\,])$ and let 0 be in the spectrum of A. Then 0 is an isolated eigenvalue of A of finite multiplicity. In particular, 0 is not a singular critical point of a quasi-uniformly positive operator.*

PROOF Let H be the operator introduced in the proof of Proposition 1.1. Then 0 is in the resolvent set of JH and $A - JH$ is an operator of finite rank. The proposition follows from the Weinstein-Aronszajn formulas, see [9, IV, §6]. □

PROPOSITION 1.4 *Let S be a quasi-uniformly positive operator in $(\mathcal{H}, (\,\cdot\,|\,\cdot\,))$ with discrete spectrum. Then the spectrum of JS is also discrete.*

PROOF Let H be the operator introduced in the proof of Proposition 1.1. From the Weinstein-Aronszajn formulas it follows that the spectrum of H is also discrete. Therefore $H^{-1} = (JH)^{-1}J$ is a compact operator. The resolvent identity implies that the resolvent of JH is compact. It follows from the equality (2) that the resolvent of JS is also compact. □

The converse of Proposition 1.4 is not true. As we show in the example below, there exists a uniformly positive operator S in $(\mathcal{H}, (\,\cdot\,|\,\cdot\,))$ with nonempty continuous spectrum such that the spectrum of JS is discrete.

EXAMPLE Consider the Hilbert space ℓ^2. Let e_n, $n = 1, 2, \ldots$ be the standard orthonormal basis and $(\,\cdot\,|\,\cdot\,)$ the standard scalar product in ℓ^2. Let $\mathcal{H}_k$, $k = 1, 2, \ldots$ be a subspace of ℓ^2 spanned by e_{2k-1}, e_{2k}. Then $\ell^2 = \bigoplus_{k=1}^{\infty} \mathcal{H}_k$. Let J be a fundamental symmetry on ℓ^2 such that the matrix representation of the restriction of J on $\mathcal{H}_k$ is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This and all the other matrix representations in $\mathcal{H}_k$ are with respect to the basis $\{e_{2k-1},\ e_{2k}\}$. Let S be a uniformly positive operator in the Hilbert space $(\ell^2, (\,\cdot\,|\,\cdot\,))$ such that the matrix representation of the restriction of S on $\mathcal{H}_k$, is $\begin{pmatrix} k & -(k-1) \\ -(k-1) & k \end{pmatrix}$. Clearly 1 is an eigenvalue of S of infinite multiplicity, i.e. the spectrum of S is not discrete. The operator JS is uniformly positive in the Krein space $(\ell^2, (J\cdot\,|\,\cdot\,))$. The eigenvalues of JS are $\pm\sqrt{2k-1}$ and the linear span of the corresponding eigenvectors is dense in ℓ^2. Therefore, the spectrum of JS is discrete.

However, if ∞ is not a singular critical point of JS, then the following proposition holds.

PROPOSITION 1.5 *Let A be a quasi-uniformly positive operator in the Krein space $(\mathcal{H}, [\cdot|\cdot])$. Assume that ∞ is not a singular critical point of A. Then A has discrete spectrum if and only if JA has discrete spectrum.*

PROOF We only have to prove that the discreteness of the spectrum of A implies the discreteness of the spectrum of JA. By [4, Proposition 2.3] there exists a Riesz basis consisting of eigenvectors and associated eigenvectors of A. This implies that A has compact resolvent. From the identity

$$(A - \lambda I)^{-1} - (JA - \lambda I)^{-1} = \\ (A - \lambda I)^{-1}(I - J)JA(JA - \lambda I)^{-1}$$

it follows that JA also has compact resolvent. □

The discreteness of the spectra of A and JA does not imply the nonsingularity of ∞. This can be seen from the following example.

EXAMPLE In the notation of the previous example, let A be an operator in ℓ^2 such that the matrix representation of the restriction of A on $\mathcal{H}_k$ is $\begin{pmatrix} k^2 & -k(k-1) \\ k(k-1) & -k^2 \end{pmatrix}$ in $\mathcal{H}_k$. The matrix representation of the restriction of JA in $\mathcal{H}_k$ is $\begin{pmatrix} k^2 & -k(k-1) \\ -k(k-1) & k^2 \end{pmatrix}$. The operator JA is uniformly positive in $(\ell^2, (\cdot|\cdot))$ and it has discrete spectrum. Therefore, the operator A is uniformly positive in the Krein space $(\ell^2, (J\cdot|\cdot))$ and its spectrum is also discrete. Since the cosine of the angle between the eigenvectors of A in $\mathcal{H}_k$ converges to 1, the point ∞ is a singular critical point of A.

Quasi-uniformly positive operators have important spectral properties. We list them for reader's convenience.

Let E be the spectral function of the quasi-uniformly positive operator A (see [10]). Let $\lambda \in \sigma(A) \cap \mathbb{R}$. Then λ is of *positive* type (*negative* type, respectively) if there exists an open interval Δ containing λ such that $(E(\Delta)\mathcal{H}, [\cdot|\cdot])$ $((E(\Delta)\mathcal{H}, -[\cdot|\cdot])$, respectively) is a Hilbert space. Further λ is a *critical point* if $[\cdot|\cdot]$ is indefinite on $E(\Delta)\mathcal{H}$ for every open interval Δ containing λ. The set of all spectral points of A of positive type (negative type, respectively) is denoted by $\sigma_+(A)$ ($\sigma_-(A)$, resp.). The set of all critical points of A is denoted by $c(A)$.

A critical point λ is said to be of *finite negative* (*positive*, respectively) *index* $\kappa_-(\lambda)$ $(\kappa_+(\lambda), \textit{respectively})$ if $(E(\Delta)\mathcal{H}, [\cdot|\cdot])$ is a Pontryagin space with a finite number $\kappa_-(\lambda)$ ($\kappa_+(\lambda)$, respectively) of negative (positive, respectively) squares for all sufficiently small open intervals Δ containing λ.

A critical point is of *finite index* if it is of finite positive or finite negative index. In the terminology of [1] such a point is said to be of finite type. Every critical point of finite index is an eigenvalue.

Recall that the negativity index $\kappa(A)$ of the quasi-uniformly positive operator A equals the total multiplicity of the nonpositive eigenvalues of the selfadjoint operator JA in the Hilbert space $(\mathcal{H}, (\cdot|\cdot))$.

If 0 is an eigenvalue of A then by Proposition 1.3 it is an isolated eigenvalue of finite algebraic multiplicity. From the canonical form of a Hermitian operator in a finite dimensional Krein space [6, Theorem 3.3] it follows that in the corresponding algebraic eigenspace there exists a basis consisting of mutually orthogonal Jordan chains $\{x_{i1},...,x_{in_i}\}$, $i=1,...,p$ with the property that $[x_{i1}|x_{in_i}]\neq 0$. We denote $\varepsilon_i = \mathrm{sgn}\,[x_{i1}|x_{in_i}]$.

Note that while the Jordan chains are not unique, the number p of the chains, their lengths $n_i, i=1,...,p$ and the signs $\varepsilon_i, i=1,...,p$ are invariants. We say that $\{p;n_1,...,n_p;\varepsilon_1,...,\varepsilon_p\}$ is *the Jordan chain data of A at* 0.

The following proposition follows from the results in [10] and [3, Section 1.3].

PROPOSITION 1.6 *Let A be a quasi-uniformly positive operator in the Krein space $(\mathcal{H},[\cdot|\cdot])$ with the negativity index $\kappa(A)$.*

(a) *The set of nonreal eigenvalues of A with positive imaginary parts consists of finitely many eigenvalues with finite total algebraic multiplicity κ_a.*

(b) *The sets $\sigma_+(A)\cap\mathbb{R}_-$ and $\sigma_-(A)\cap\mathbb{R}_+$ consist of finitely many isolated eigenvalues of finite total (geometric) multiplicities κ_b^- and κ_b^+.*

(c) *All finite critical points of A are of finite index; the set $c(A)\cap\mathbb{R}_-$ ($c(A)\cap\mathbb{R}_+$, respectively) consists of negative (positive, resp.) critical points of finite positive (negative, resp.) index. If 0 is a critical point than it is a critical point of finite both positive and negative index. Moreover, in that case $\kappa_-(0)+\kappa_+(0)$ equals the algebraic multiplicity of the eigenvalue 0.*

(d) *Let $\{p;n_1,...,n_p;\varepsilon_1,...,\varepsilon_p\}$ be the Jordan chain data of A at 0. Let $n^-(0)$ denote the number of indices i with the property $(-1)^{n_i}\,\varepsilon_i=-1$, and $n^+(0)$ the number of indices i with the property $\varepsilon_i=-1$. Then*

$$\kappa_a+\kappa_b^+ +\kappa_b^- +\sum_{\lambda\in c(A)\cap[0,\infty)}\kappa_-(\lambda)+\sum_{\lambda\in c(A)\cap(-\infty,0)}\kappa_+(\lambda)+n^-(0)=\kappa(A), \tag{3}$$

and

$$\kappa_a+\kappa_b^+ +\kappa_b^- +\sum_{\lambda\in c(A)\cap(0,\infty)}\kappa_-(\lambda)+\sum_{\lambda\in c(A)\cap(-\infty,0]}\kappa_+(\lambda)+n^+(0)=\kappa(A). \tag{4}$$

(e) *Every Jordan chain of A is of finite length. There are finitely many linearly independent Jordan chains of length ≥ 2; the sum of the lengths of these Jordan chains does not exceed $3\,\kappa(A)$.*

The proof of part (d) uses the canonical form of the Hermitian operators JP and JAP in the finite dimensional Krein space $(P\mathcal{H},[\cdot|\cdot])$, where P is the orthogonal projection onto the algebraic eigenspace of the eigenvalue 0. The formulas (3) and (4) explain how the nonpositive squares of the form a are "used". The estimate in (e) is very crude. Note that $\kappa(A)$ is the maximal codimension of a subspace of $\mathcal{D}(A)$ on which a is uniformly positive definite. Therefore, parts (d) and (e) can be used to estimate the respective spectral quantities.

2 Quasi-uniformly positive forms

In this section we consider sesquilinear forms a and v in the Hilbert space $(\mathcal{H}, (\,\cdot\,|\,\cdot\,))$ satisfying

(A) The form a is closed and uniformly positive.

(B) The form v is relatively a-bounded with the a-bound $\Gamma < 1$.

This means (see [9, page 319]) that $\mathcal{D}(v) \supseteq \mathcal{D}(a)$ and that for all $\gamma > \Gamma$ there exists $C \geq 0$ such that

$$|v(x,x)| \leq \gamma a(x,x) + C\|x\|^2, \quad x \in \mathcal{D}(a). \tag{5}$$

Let B be the positive operator associated with the form a in the Hilbert space $(\mathcal{H}, (\,\cdot\,|\,\cdot\,))$, see [9, Theorem VI.2.1]. Then $\mathcal{D}(B^{1/2}) = \mathcal{D}(a)$ by [9, Theorem VI.2.23]. It follows from [9, Lemma VI.3.1] that there exists a bounded selfadjoint operator D on $\mathcal{H}$ such that

$$v(x,y) = (DB^{1/2}x|B^{1/2}y), \quad x, y \in \mathcal{D}(a). \tag{6}$$

By [9, Theorem VI.3.9] the form $a_1 = a + v$ is closed, symmetric and bounded from below. Let B_1 be the selfadjoint operator associated with the form a_1 in the Hilbert space $(\mathcal{H}, (\,\cdot\,|\,\cdot\,))$. Then

$$\mathcal{D}(|B_1|^{1/2}) = \mathcal{D}(B^{1/2}). \tag{7}$$

Let $A = JB$ and $A_1 = JB_1$. The operator B is uniformly positive and $0 \in \rho(B)$. Consequently, $0 \in \rho(A)$ and A is definitizable in the Krein space $(\mathcal{H}, [\,\cdot\,|\,\cdot\,])$.

PROPOSITION 2.1 *Assume that the selfadjoint operator A_1 is definitizable in the Krein space $(\mathcal{H}, [\,\cdot\,|\,\cdot\,])$. Then ∞ is not a singular critical point of A_1 if and only if it is not a singular critical point of A.*

PROOF This follows from (7) and [2, Corollary 3.6]. □

It remains to find sufficient conditions to establish the definitizabilty of the operator A_1.

In the next proposition we need the notion of relative compactness of quadratic forms. We refer to [12, page 369]. It is equivalent to the compactness of the operator D in (6).

PROPOSITION 2.2 1. *If there exists $\gamma < 1$ such that the relation* (5) *holds with $C = 0$, then $a_1 = a + v$ is a uniformly positive form. Therefore the operator A_1 is uniformly positive in the Krein space $(\mathcal{H}, [\,\cdot\,|\,\cdot\,])$.*
2. *If the form v is a-compact, then the form $a_1 = a + v$ is quasi-uniformly positive in the Hilbert space $(\mathcal{H}, (\,\cdot\,|\,\cdot\,))$. Therefore A_1 is a definitizable operator in the Krein space $(\mathcal{H}, [\,\cdot\,|\,\cdot\,])$..*

PROOF 1. The form $a + v$ is uniformly positive. Hence B is uniformly positive.
2. Since v is a-bounded with the a-bound < 1, the form $a + v$ and therefore also the operator B, is bounded from below. By [12, page 369] the operators B and B_1 have the same essential spectrum. Therefore B is quasi-uniformly positive in $(\mathcal{H}, (\,\cdot\,|\,\cdot\,))$ and A_1 is quasi-uniformly positive in the Krein space $(\mathcal{H}, [\,\cdot\,|\,\cdot\,])$. By Proposition 1.1 the operator A_1 is definitizable. □

COROLLARY 2.3 *Let s be a quadratic form in a Hilbert space $(\mathcal{H}, (\cdot|\cdot))$. The following statements are equivalent:*

(i) *s is a quasi-uniformly positive form.*

(ii) *s is a relatively form-compact symmetric perturbation of a uniformly positive form in $(\mathcal{H}, (\cdot|\cdot))$.*

PROOF The implication (i) $\Rightarrow$ (ii) follows from the corresponding statement about operators. The converse implication is the statement 2 of Proposition 2.2. □

In the next corollary we summarize the results of this section.

COROLLARY 2.4 *If any of the two assumptions of the* Proposition 2.2 *is satisfied, then ∞ is not a singular critical point of A_1 if and only if it is not a singular critical point of A.*

PROPOSITION 2.5 *If the form v is a-compact, then the essential spectra of A and A_1 coincide. Additionally, A_1 has compact resolvent if and only if A has a compact resolvent.*

PROOF From (6) and the definition of B_1 it follows that for all $x \in \mathcal{D}(B_1)$ and for all $y \in \mathcal{D}(B^{1/2})$ we have

$$((B_1 - \lambda J)x|y) = ((I + D - \lambda B^{-1/2} J B^{-1/2}) B^{1/2} x | B^{1/2} y). \tag{8}$$

The operator $Q = B^{-1/2} J B^{-1/2}$ is a bounded selfadjoint operator in the Hilbert space $\mathcal{H}$. From (8) we have

$$(B_1 - \lambda J)x = B^{1/2}(I + D - \lambda Q) B^{1/2} x, \quad x \in \mathcal{D}(B_1). \tag{9}$$

For $\lambda \in \rho(A_1)\backslash\mathbb{R}$ its conjugate $\overline{\lambda}$ is also in $\rho(A_1)$. Therefore the range of the operator $I + D - \overline{\lambda} Q$ contains $\mathcal{D}(B^{1/2})$ and consequently its adjoint $I + D - \lambda Q$ is injective. Since the operator $I - \lambda Q$ is bounded and boundedly invertible it follows from the Fredholm alternative that the injective operator $I + D - \lambda Q$ has a bounded inverse. Inverting (9) we get

$$(B_1 - \lambda J)^{-1} = B^{-1/2}(I + D - \lambda Q)^{-1} B^{-1/2}. \tag{10}$$

We also note that $\lambda \in \rho(B)$ and

$$(B - \lambda J)^{-1} = B^{-1/2}(I - \lambda Q)^{-1} B^{-1/2}. \tag{11}$$

It follows from (10) and (11) that

$$(B_1 - \lambda J)^{-1} - (B - \lambda J)^{-1} = B^{-1/2}[(I + D - \lambda Q)^{-1} - (I - \lambda Q)^{-1}] B^{-1/2} =$$

$$= -B^{-1/2}(I - \lambda Q)^{-1} D (I + D - \lambda Q)^{-1} B^{-1/2}.$$

Thus the operator $(A_1 - \lambda I)^{-1} - (A - \lambda I)^{-1}$ is compact. By [9, Theorem IV.5.35] the operators $(A - \lambda I)^{-1}$ and $(A_1 - \lambda I)^{-1}$ have the same essential spectrum. As a consequence the operators A and A_1 have the same essential spectrum. □

3 Klein-Gordon equation

Let $\mathcal{G}$ be a Hilbert space with a scalar product $(\cdot|\cdot)$, H a positive selfadjoint operator in $\mathcal{G}$ such that $H \geq m^2 I > 0$. For $-1 \leq \alpha \leq 1$, let $\mathcal{G}_\alpha$ be the Hilbert space completion of $(\mathcal{D}(H^\alpha), (H^\alpha \cdot \,|H^\alpha \cdot))$. Denote by $\|\cdot\|_\alpha$ the norm of this Hilbert space.

If $\alpha \leq 0$ the space $\mathcal{G}_\alpha$ coincides with $\mathcal{D}(H^\alpha)$. The operator H can be extended to an isometry between $\mathcal{G}_\alpha$ and $\mathcal{G}_{\alpha-1}$.

Denote by $\mathcal{H}$ the Hilbert space $\mathcal{G}_{1/4} \oplus \mathcal{G}_{-1/4}$ and by $\langle\cdot|\cdot\rangle$ its natural scalar product. If $x \in \mathcal{G}_{1/4}$ then $|(x|y)| \leq \|x\|_{1/4}\|y\|_{-1/4}$ $(y \in \mathcal{G})$. Therefore the scalar product $(\cdot|\cdot)$ can be extended by continuity from $\mathcal{G}_{1/4} \times \mathcal{G}$ to $\mathcal{G}_{1/4} \times \mathcal{G}_{-1/4}$ and similarly from $\mathcal{G} \times \mathcal{G}_{1/4}$ to $\mathcal{G}_{-1/4} \times \mathcal{G}_{1/4}$. Define an indefinite scalar product on $\mathcal{H}$ by

$$[x|y] = (x_1|y_2) + (x_2|y_1), \quad x = (x_1, x_2),\ y = (y_1, y_2) \in \mathcal{H}.$$

The space $\mathcal{H}$ with the indefinite scalar product $[\cdot|\cdot]$ is a Krein space. The fundamental symmetry is

$$\mathbf{J} = \begin{bmatrix} 0 & H^{-1/2} \\ H^{1/2} & 0 \end{bmatrix}$$

Define the operator $\mathbf{A}$ in $\mathcal{H}$ on $\mathcal{D}(\mathbf{A}) = \mathcal{G}_{3/4} \oplus \mathcal{G}_{1/4}$ by

$$\mathbf{A} = \begin{bmatrix} 0 & I \\ H & 0 \end{bmatrix}.$$

The operator $\mathbf{A}$ is a selfadjoint operator in $(\mathcal{H}, [\cdot|\cdot])$. Since

$$[\mathbf{A}x|x] = (Hx_1|x_1) + (x_2|x_2), \quad x = (x_1, x_2) \in \mathcal{D}(\mathbf{A}), \tag{12}$$

the operator $\mathbf{A}$ is uniformly positive in $(\mathcal{H}, [\cdot|\cdot])$. The form $[\mathbf{A}x|y]$, $x, y \in \mathcal{D}(\mathbf{A})$ is closable. Let $\mathbf{a}$ be its closure. Let $\mathbf{B}$ be the uniformly positive operator associated with the form $\mathbf{a}$ in the Hilbert space $(\mathcal{H}, \langle\cdot|\cdot\rangle)$. It follows from (12) that the domain of $\mathbf{a}$ is $\mathcal{D}(\mathbf{a}) = \mathcal{H}_1(\mathbf{A}) = \mathcal{G}_{1/2} \oplus \mathcal{G}$ and that

$$\mathbf{a}(x, y) = \langle \mathbf{P}x|\mathbf{P}y\rangle, \quad x, y \in \mathcal{G}_{1/2} \oplus \mathcal{G}$$

with

$$\mathbf{P} = \mathbf{B}^{1/2} = \begin{bmatrix} H^{1/4} & 0 \\ 0 & H^{1/4} \end{bmatrix}.$$

The following lemma follows from the fact that the operators $\mathbf{A}$ and $\mathbf{J}$ commute.

LEMMA 3.1 *Infinity is not a singular critical point of* $\mathbf{A}$.

Let V be a $H^{1/2}$-bounded symmetric operator in $\mathcal{G}$. We define the form

$$\mathbf{v}(x, y) = (Vx_1|y_2) + (x_2|Vy_1), \quad x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{G}_{1/2} \oplus \mathcal{G}.$$

LEMMA 3.2 *Let V be a $H^{1/2}$-bounded symmetric operator with the relative bound β_0. Then the form $\mathbf{v}$ is $\mathbf{a}$-bounded in $\mathcal{H}$ with the relative $\mathbf{a}$-bound $\leq \sqrt{\beta_0}$.*

PROOF Let $\beta > \beta_0$. Then there exists $C > 0$ such that

$$\|Vx_1\|^2 \le \beta \|H^{1/2}x_1\|^2 + C\|x_1\|^2.$$

Noting that $\mathbf{v}(x,x) = 2\mathrm{Re}\,(Vx_1|x_2)$ it follows that

$$|\mathbf{v}(x,x)| \le 2\|Vx_1\|\,\|x_2\| \le \sqrt{\beta}\|x_2\|^2 + \frac{1}{\sqrt{\beta}}\|Vx_1\|^2.$$

Since H is uniformly positive, $\|x_1\|^2$ can be replaced by $\|H^{1/4}x_1\|^2$. Therefore

$$|\mathbf{v}(x,x)| \le \sqrt{\beta}\mathbf{a}(x,x) + \frac{C}{\sqrt{\beta}}\langle x|x\rangle.$$

□

COROLLARY 3.3 *If the $H^{1/2}$-bound of V is < 1 then the form $\mathbf{a}+\mathbf{v}$ defined on $\mathcal{G}_{1/2} \oplus \mathcal{G}$ is closed, symmetric and bounded from below.*

PROOF This follows from [9, Theorem VI.3.9]. □

In the rest of this section we assume that the operator V is $H^{1/2}$-bounded with the relative bound < 1.

Let $\mathbf{B_1}$ be the selfadjoint operator associated with $\mathbf{a}+\mathbf{v}$ in the Hilbert space $(\mathcal{H}, \langle\cdot|\cdot\rangle)$ and let $\mathbf{A_1} = \mathbf{J}\mathbf{B_1}$. The operator $\mathbf{A_1}$ is selfadjoint in the Krein space $(\mathcal{H}, [\cdot|\cdot])$. From Proposition 2.1 we conclude:

PROPOSITION 3.4 *If the selfadjoint operator $\mathbf{A_1}$ is definitizable then ∞ is not its singular critical point.*

It follows from the symmetry of V that it can be extended to a bounded operator from $\mathcal{G}_\alpha$ to $\mathcal{G}_{\alpha-1/2}$ for $0 \le \alpha \le 1/2$. A calculation shows that

$$\mathbf{v}(x,y) = \langle \mathbf{D}\mathbf{P}x|\mathbf{P}y\rangle, \quad x,y \in \mathcal{G}_{1/2} \oplus \mathcal{G}$$

with

$$\mathbf{D} = \begin{bmatrix} 0 & H^{-3/4}VH^{-1/4} \\ H^{1/4}VH^{-1/4} & 0 \end{bmatrix}.$$

The operator $\mathbf{D}$ is bounded in $\mathcal{H}$ and

$$\|\mathbf{D}\| = \|VH^{-1/2}\|. \tag{13}$$

If the operator V is $H^{1/2}$-compact than it is $H^{1/2}$-bounded with the relative bound 0. Moreover $H^{1/4}VH^{-1/4}$ is a compact operator from $\mathcal{G}_{1/4}$ into $\mathcal{G}_{-1/4}$ and $H^{-3/4}VH^{-1/4}$ is a compact operator from $\mathcal{G}_{-1/4}$ into $\mathcal{G}_{1/4}$. Consequently $\mathbf{D}$ is a compact operator in $\mathcal{H}$. From (13), Lemma 3.1, Corollary 2.4, Propositions 2.2 and 2.5 we conclude:

THEOREM 3.5 *Let V be a symmetric $H^{1/2}$-bounded operator with the relative bound < 1. Let $\mathbf{A_1}$ be the selfadjoint operator in the Krein space $(\mathcal{H}, [\cdot|\cdot])$ defined above.*
1. *Assume that $\|VH^{-1/2}\| < 1$. Then $\mathbf{A_1}$ is a uniformly positive operator which is similar to a selfadjoint operator in the Hilbert space $(\mathcal{H}, \langle\cdot|\cdot\rangle)$.*
2. *Assume that $VH^{-1/2}$ is compact. Then $\mathbf{A_1}$ is a definitizable operator and ∞ is not its singular critical point. The essential spectrum of $\mathbf{A_1}$ equals the essential spectrum of $\mathbf{A}$ and this is the set of all λ such that λ^2 is in the essential spectrum of H.*

References

[1] Binding. P., Najman, B.: Regularity of finite type critical points for self-adjoint operators in Krein space. Preprint.

[2] Ćurgus, B.: On the regularity of the critical point infinity of definitizable operators. Integral Equations Operator Theory 8 (1985), 462-488.

[3] Ćurgus, B., Langer, H.: A Krein space approach to symmetric ordinary differential operators with an indefinite weight function. J. Differential Equations 79 (1989), 31-61.

[4] Ćurgus, B., Najman, B.: A Krein space approach to elliptic eigenvalue problems with indefinite weights. Differential and Integral Equations 7 (1994), 1241-1252. .

[5] Ćurgus, B., Najman, B.: Quadratic eigenvalue problems. Mathematische Nachrichten, to appear.

[6] Gohberg, I., Lancaster, P., Rodman, L.: Matrices and Indefinite Scalar Products. Birkhäuser, Basel, 1983.

[7] Jonas, P.: On a problem of the perturbation theory of selfadjoint operators in Krein spaces. J. Operator Theory 25 (1991), 183-211.

[8] Jonas, P.: On the spectral theory of operators associated with perturbed Klein-Gordon and wave type equations. Preprint Karl-Weierstrass-Institut für Mathematik, Berlin, 1990.

[9] Kato, T.: Perturbation Theory for Linear Operators. 2nd ed. Springer-Verlag, Berlin, 1976.

[10] Langer, H.: Spectral function of definitizable operators in Krein spaces. Functional Analysis, Proceedings, Dubrovnik 1981. Lecture Notes in Mathematics 948, Springer-Verlag, Berlin, 1982, 1-46.

[11] Langer, H., Najman, B.: A Krein space approach to the Klein-Gordon equation. Unpublished manuscript.

[12] Reed, M., Simon, B.: Methods of Modern Mathematical Physics, vol. IV: Analysis of Operators. Academic Press, New York, 1978.

B. Ćurgus
Department of Mathematics,
Western Washington University,
Bellingham, WA 98225, USA
curgus@cc.wwu.edu

B. Najman
Department of Mathematics,
University of Zagreb,
Bijenička 30, 41000 Zagreb, Croatia
najman@cromath.math.hr

Math Review 1991 Mathematics Subject Classification 46B50 45C20