POSITIVE DIFFERENTIAL OPERATORS IN KREIN SPACE $L^2(\mathbb{R})$

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Consider the weighted eigenvalue problem

$$Lu = \lambda\,(\operatorname{sgn} x)u, \tag{1}$$

on the whole real line $\mathbb{R}$ where $L = p(D)$ is a positive symmetric differential operator with constant coefficients. This problem is a model problem for a more general problem $Lu = \lambda\, wu$ with L a differential operator and w a function taking both positive and negative values.

Our starting point is the observation that the operator $A = (\operatorname{sgn} x)L$ is symmetric and positive with respect to the indefinite inner product $[u,v] = \int u(x)\overline{v(x)}\operatorname{sgn} x dx$. The space $L^2(\mathbb{R})$ with this inner product is a Krein space. Once we prove that the resolvent set $\rho(A)$ is nonempty, H. Langer's spectral theory can be applied. This spectral theory shows that the spectrum of A is real and its properties on bounded open intervals not containing 0 are the same as the corresponding properties of a selfadjoint operator in a Hilbert space. In particular, A has a spectral function defined on open intervals in $\mathbb{R}$ with the endpoints different from 0 and ∞. The positive (negative, respectively) spectral points are of positive (negative, resp.) type. Therefore 0 and ∞ are the only possible critical points. A critical point λ is *regular* if the spectral function is bounded near λ. In that case the spectral function can be extended to intervals with an endpoint λ. A critical point is *singular* if it is not regular. If neither 0 nor ∞ is a singular critical point, then A is similar to a selfadjoint operator in $L^2(\mathbb{R})$. We used this fact in [5] to prove that A is similar to a selfadjoint operator in the case $p(t) = t^2$.

In this paper we generalize this result to more general polynomials p. The results of this paper are used in the forthcoming paper [6] to extend the results of [5] to a class of partial differential operators. For example, in [6] for $n > 1$ we prove the following.
The operator $(\operatorname{sgn} x_n)\Delta$ *defined on* $H^2(\mathbb{R}^n)$ *is similar to a selfadjoint operator in* $L^2(\mathbb{R}^n)$.

The question of nonsingularity of the critical point ∞ has been considered in [4]. This question leads to the investigation of the domain of A. In the present case the operator

A is positive (not uniformly positive as in [4]) and this is why the critical point at 0 may appear as a critical point of infinite type. If the spectrum of A accumulates at 0 from both sides, then 0 is a critical point of A. To determine whether it is singular or regular we are led to investigate the range of A. This question is harder than the investigation of the domain. In Section 1 we give a necessary and sufficient condition for $\mathcal{R}(B) = \mathcal{R}(C)$ for multiplication operators $B,\ C$ in $L^2(\mathbb{R})$. We also prove several stability theorems for the regularity of the critical points 0 and ∞ of positive definitizable operators in a Krein space. As a consequence we get a stability theorem for the similarity to a selfadjoint operator in a Hilbert space. For related results in this direction see [7]. In Section 2 we consider the differential operators with constant coefficients in $L^2(\mathbb{R})$. We give a precise description of the spectrum of the operator A. Under some additional restrictions on p, we prove that A is similar to a selfadjoint operator in $L^2(\mathbb{R})$. It follows from the general operator theory in Krein spaces that an operator which is positive in the Krein space $(L^2(\mathbb{R}), [\,\cdot\,|\,\cdot\,])$ and similar to a selfadjoint operator in the Hilbert space $L^2(\mathbb{R})$ has the half-range completeness property. We use this fact in Section 3 to show that our results in Section 2 give sufficient conditions for the half-range completeness property for the problem (1).

The Sturm-Liouville problem with indefinite weight has attracted considerable attention; we mention the references quoted in [3, 4] for a partial list. The problem of nonsingularity of the critical points of definitizable operators in Krein spaces has been investigated in [2, 7, 8, 10]. For differential operators with indefinite weights the study of this problem has been motivated by the investigation of the half-range completeness property, cf. [1, 3]. The regularity of the critical point 0 has been considered in [5].

For definitions and basic results of the theory of definitizable operators see [9].

1 Abstract Results

In this section we use the method of [2, Lemma 1.8, Corollary 3.3 and Theorem 3.9] to investigate the regularity of the critical points 0 and ∞ of a positive definitizable operator A in the Krein space $(\mathcal{K}, [\,\cdot\,|\,\cdot\,])$.

The following two lemmas are restatements of [2, Theorem 3.9 and Corollary 3.3] in terms of the critical point 0. We prove the first. The proof of the second one is analogous.

LEMMA 1.1 *Let $A = JP$ be a positive definitizable operator in the Krein space $(\mathcal{K}, [\,\cdot\,|\,\cdot\,])$ such that 0 is not an eigenvalue of P. Assume that $\nu > 0$ and the operator JP^ν is definitizable. Then the following statement are equivalent:*

(a) *The point 0 is not a singular critical point of the operator JP.*

(b) *The point 0 is not a singular critical point of the operator JP^ν.*

PROOF The point 0 is not a singular critical point of JP if and only if it is not a singular critical point of the operator PJ which is similar to JP. Further, 0 is not a singular critical point of PJ if and only if ∞ is not a singular critical point of the operator JP^{-1}. It follows from [2, Theorem 3.9] that ∞ is not a singular critical point of JP^{-1} if and only if ∞ is not a singular critical point of $JP^{-\nu}$. Clearly, ∞ is not a singular critical point of $JP^{-\nu}$ if and only if 0 is not a singular critical point of $P^{\nu}J$. Because of the similarity of the operators, 0 is not a singular critical point of $P^{\nu}J$ if and only if 0 is not a singular critical point of JP^{ν}. This sequence of equivalent statements proves the lemma. □

It follows from [2, Lemma 1.8] that the operator $JP^{-\nu}$ is definitizable for $\nu = 2^m$ with m being a positive integer.

LEMMA 1.2 *Let A and B be definitizable operators in the Krein space $\mathcal{K}$ such that 0 is neither an eigenvalue of A nor of B. Assume that $\mathcal{R}(A) = \mathcal{R}(B)$. Then the following statements are equivalent.*

(a) *The point 0 is not a singular critical point of A.*

(b) *The point 0 is not a singular critical point of B.*

Let μ be a measure on $\mathbb{R}$, g and h nonnegative μ-measurable functions on $\mathbb{R}$. Denote by M_g the operator of multiplication by g in $L^2(\mathbb{R}, \mu)$. We will repeatedly use the following result, which gives necessary and sufficient conditions for the equality of the domains and the ranges of M_g and M_h.

LEMMA 1.3 *Let g and h be nonnegative measurable functions on $\mathbb{R}$.*

(a) *The following statements are equivalent:*

  (i) $\mathcal{D}(M_g) = \mathcal{D}(M_h)$

  (ii) *The functions $\frac{h}{1+g}$ and $\frac{g}{1+h}$ are essentially bounded.*

(b) *The following statements are equivalent:*

  (i) $\mathcal{R}(M_g) = \mathcal{R}(M_h)$.

  (ii) *There exists a constant $C \geq 0$ such that*

$$g \leq Ch(1+g) \ \mu\text{-a.e.} \quad and \quad h \leq Cg(1+h) \quad \mu\text{-a.e.} \ . \tag{2}$$

PROOF The statement (a) is evident.
(b) For a μ-measurable function f denote the set $\{x \in \mathbb{R} | f(x) = 0\}$ by N_f. Note that each of the conditions (a) and (b) implies that $N_g = N_h = N$. Therefore $\mathcal{N}(M_g) = \mathcal{N}(M_h)$ consists of functions $f \in L^2(\mathbb{R}, \mu)$ with the support contained in N. Let

$$G(x) = H(x) = 0 \ (x \in N) \ , G(x) = \frac{1}{g(x)}, \ H(x) = \frac{1}{h(x)} \ (x \in \mathbb{R} \setminus N) \ .$$

It follows from (a) that the condition (ii) is equivalent to $\mathcal{D}(M_G) = \mathcal{D}(M_H)$. Since $\mathcal{D}(M_G) = \mathcal{R}(M_g) \oplus \mathcal{N}(M_g)$, we conclude that (i) and (ii) are equivalent. $\square$

A polynomial p is *nonnegative* if $p(x) \geq 0$ for all $x \in \mathbb{R}$.

EXAMPLE 1 Let h be a nonnegative polynomial of degree $2k$ in one variable. If $g(t) = t^{2k}$, then h and g satisfy the conditions of Lemma 1.3 (a).

EXAMPLE 2 Let h be a nonnegative polynomial. Then $h(t) = ag(t)\tilde{h}(t)$, where $a > 0$, $\tilde{h}$ is a positive polynomial without real roots and $g(t) = (t - r_1)^{2k_1} \cdots (t - r_m)^{2k_m}$. Then h and g satisfy the condition (ii) of Lemma 1.3 (b).

THEOREM 1.4 *Let S be a selfadjoint operator in the Hilbert space $(\mathcal{K}, (\cdot\,|\,\cdot))$ such that JS^2 is a definitizable operator in the Krein space $(\mathcal{K}, [\cdot\,|\,\cdot])$. Let $\nu > 0$ and let h be a nonnegative continuous function. Assume that the operators $J|S|^\nu$ and $Jh(S)$ are definitizable.*

(a) *Assume that the functions $g(t) = |t|^\nu$ and h satisfy the conditions of Lemma 1.3 (a). Then the following statements are equivalent.*

  (i) *The point ∞ is not a singular critical point of JS^2.*

  (ii) *The point ∞ is not a singular critical point of $Jh(S)$.*

(b) *Assume that 0 is not an eigenvalue of S and that the functions $g(t) = |t|^\nu$ and h satisfy the condition (2). Then the following statements are equivalent.*

  (i) *The point 0 is not a singular critical point of JS^2.*

  (ii) *The point 0 is not a singular critical point of $Jh(S)$.*

PROOF We prove (b). The proof of (a) is similar. Lemma 1.1 implies that 0 is not a singular critical point of JS^2 if and only if it is not a singular critical point of $J|S|^\nu$.

It follows from Lemma 1.3 (b) that for any Borel measure μ the multiplication operators M_g and M_h in $L^2(\mathbb{R}, \mu)$ have the same range. The Spectral Theorem, see [11, Theorem 7.18], implies $\mathcal{R}(|S|^\nu) = \mathcal{R}(h(S))$. Therefore, $\mathcal{R}(J|S|^\eta) = \mathcal{R}(Jh(S))$. The conclusion follows from Lemma 1.2. $\square$

COROLLARY 1.5 *Let S be a selfadjoint operator in the Hilbert space $(\mathcal{K}, (\cdot\,|\,\cdot))$ such that 0 is not an eigenvalue of S and such that JS^2 is a definitizable operator in the Krein space $(\mathcal{K}, [\cdot\,|\,\cdot])$. Let η and ν be positive numbers and let h be a nonnegative continuous function. Let $g_1(t) = |t|^\eta$ and $g_2(t) = |t|^\nu$. Assume that the functions g_1 and h satisfy the conditions of Lemma 1.3 (a) and that the functions g_2 and h satisfy the condition (2). Assume that the operators $J|S|^\eta, J|S|^\nu$ and $Jh(S)$ are definitizable. Then the following statements are equivalent.*

(i) *The operator JS^2 is similar to a selfadjoint operator in $(\mathcal{K}, (\cdot\,|\,\cdot))$.*

(ii) *The operator $Jh(S)$ is similar to a selfadjoint operator in $(\mathcal{K}, (\cdot\,|\,\cdot))$.*

2 Differential Operators with Constant Coefficients

In this section we apply the results from Section 1 to a class of positive ordinary differential operators with constant coefficients.

In the following, a root of multiplicity m of a polynomial is counted as m roots. Denote by $\mathbb{C}_+$ (respectively $\mathbb{C}_-$) the set of all complex numbers z such that $\operatorname{Im} z > 0$ (respectively $\operatorname{Im} z < 0$).

We consider an even order polynomial

$$p(z) = a_0 z^{2n} + a_1 z^{2n-1} + \cdots + a_{2n-1} z + a_{2n} \ . \tag{3}$$

with real coefficients a_j.

For the reader's convenience we give a proof of the following lemma.

LEMMA 2.1 *Let p be a polynomial of degree $2n$ with real coefficients. Let α be a complex number.*

(a) *If α is nonreal, then the polynomial equation*

$$p(z) - \alpha = 0 \tag{4}$$

has exactly n solutions in $\mathbb{C}_+$ and exactly n solutions in $\mathbb{C}_-$.

(b) *If α is real, then the equation (4) has at most n solutions in $\mathbb{C}_+$.*

PROOF (a) Let $n_+(\alpha)$ be the number of solutions of (4) in $\mathbb{C}_+$. Since (4) has no real solutions, it follows that $n_+(\alpha)$ is constant for $\alpha \in \mathbb{C}_+$. Note that the equation $a_0 z^{2n} = \alpha$ has exactly n solutions with positive imaginary parts, an application of Rouche's theorem shows that $n_+(\alpha) = n$ for $|\alpha|$ sufficiently large.

The claim (b) is evident. □

Denote $D = -i\frac{d}{dx}$. We consider the spectral problem

$$p(D)f(x) = \lambda(\operatorname{sgn} x)f(x), \quad x \in \mathbb{R}, \tag{5}$$

For a polynomial q of degree k, $q(D)$ denotes the constant coefficient differential operator in the Hilbert space $L^2(\mathbb{R})$ defined on the Sobolev space $H^k(\mathbb{R})$.

Let J be the multiplication operator defined by

$$(Jf)(x) = (\operatorname{sgn} x)f(x),\ x \in \mathbb{R} \ .$$

Then the problem (5) can be written in terms of operators as

$$p(D)f = \lambda Jf, \quad f \in H^{2n}(\mathbb{R}) \ , \tag{6}$$

or, equivalently,

$$Jp(D)f = \lambda f, \quad f \in H^{2n}(\mathbb{R}) \ . \tag{7}$$

It is natural to study the problem (7) in the Krein space $\mathcal{K} = L^2(\mathbb{R})$ with the scalar product $[f,g] = \int_{\mathbb{R}} f(x)\overline{g(x)}\operatorname{sgn} x\, dx$. The multiplication operator J is a fundamental symmetry on $\mathcal{K}$ and the corresponding positive definite scalar product is the standard scalar product in $L^2(\mathbb{R})$.

Since p has real coefficients the operator $p(D)$ is selfadjoint in the Hilbert space $L^2(\mathbb{R})$. Therefore, the operator $Jp(D)$ is selfadjoint in the Krein space $\mathcal{K}$. A selfadjoint operator in a Krein space may have empty resolvent set. In the next theorem we show that this is not the case for the operator $Jp(D)$.

THEOREM 2.2 *Let p be an even order polynomial with real coefficients. Let $A = Jp(D)$.*

(a) *The spectrum of the operator A is real.*

(b) *The operator A has no eigenvalues. Its residual spectrum is empty.*

(c) *The continuous spectrum of A is given by*

$$\sigma_{\mathrm{c}}(A) = (-\infty, -m_p] \cup [m_p, +\infty), \ \ \textit{where} \ \ m_p = \min\{p(x) : x \in \mathbb{R}\} \ . \tag{8}$$

PROOF (a) Let ζ be an arbitrary nonreal complex number. We have to prove that the operator $A - \zeta I$ has a bounded inverse. Since the operators J and $p(D)$ are closed, it is sufficient to prove that $p(D) - \zeta J$ is a bijection of $H^{2n}(\mathbb{R})$ onto $L^2(\mathbb{R})$. Let $g \in L^2(\mathbb{R})$. The special restriction of $p(D)$ defined in $L^2(\mathbb{R}_\mp)$ with the domain consisting of all functions f in $H^{2n}(\mathbb{R}_\mp)$ such that $f^{(j)}(0) = 0$, $j = 0, \ldots, n-1$, is selfadjoint in the Hilbert space $L^2(\mathbb{R}_\mp)$. Therefore, the boundary value problems

$$\begin{aligned} (p(D)y)(x) \pm \zeta\, y(x) = g(x), \quad & x \in \mathbb{R}_\mp, \quad y \in H^{2n}(\mathbb{R}_\mp) \\ & y^{(j)}(0) = 0, \quad j = 0, \ldots, n-1 \end{aligned}$$

have unique solutions $y_\mp$ in $H^{2n}(\mathbb{R}_\mp)$.

Now consider the homogeneous equation

$$p(D)y - \zeta\, y = 0, \quad y \in H^{2n}(\mathbb{R}_+). \tag{9}$$

In order to find the fundamental set of solutions of (9) we have to solve the polynomial equation $p(-iz) - \zeta = 0$. Since ζ is nonreal, we can apply Lemma 2.1 (a) and conclude that this equation has n roots z_j^+, $j = 1, \ldots, n$, with negative real parts. These roots in the standard way lead to n linearly independent solutions ψ_j^+, $j = 1, \ldots, n$ of (9) which are in $H^{2n}(\mathbb{R}_+)$.

To find the fundamental set of solutions of the homogeneous equation

$$p(D)y + \zeta\, y = 0, \ y \in H^{2n}(\mathbb{R}_-). \tag{10}$$

we have to find the roots of $p(-iz)+\zeta=0$ with positive real parts. By Lemma 2.1 (a) there are n such roots; denote them by $z_j^-, j=1,\dots,n$. These roots in the standard way lead to n linearly independent solutions ψ_j^-, $j=1,\dots,n$ of (10) which are in $H^{2n}(\mathbb{R}_-)$. Since the set $\{z_j^+, j=1,\dots,n\}$ is disjoint from the set $\{z_j^-, j=1,\dots,n\}$, the set $\{\psi_j^+,\psi_j^-, j=1,\dots,n\}$ is linearly independent and moreover it is a basis of solutions of the homogeneous equation $q(D)y=0$, where $q(t)=\prod_{j=1}^{n}(t+iz_j^+)(t+iz_j^-)$. Therefore the Wronskian of $\{\psi_j^+,\psi_j^-, j=1,\dots,n\}$ does not have zeros.

Every solution $f\in H^{2n}(\mathbb{R})$ of the equation

$$p(D)f-\zeta Jf=g \tag{11}$$

must satisfy

$$f(x)=\begin{cases} y_-(x)+\sum_{j=1}^{n} c_j^-\psi_j^-(x), & x\in\mathbb{R}_- \\ y_+(x)+\sum_{j=1}^{n} c_j^+\psi_j^+(x), & x\in\mathbb{R}_+ \end{cases}$$

for some complex numbers c_j^-, c_j^+, $j=1,\dots,n$. The continuity of $f^{(j)}$, $j=0,1,\dots,2n-1$ at 0 leads to a system of $2n$ linear equations in c_j^-, c_j^+, $j=1,\dots,n$. The determinant of this system is the Wronskian of the functions ψ_j^+, ψ_j^-, $j=1,\dots,n$ evaluated at 0. Since this determinant is not 0, the system has unique solution. Therefore, the equation (11) has a unique solution, i.e., $p(D)-\zeta J$ is bijection of $H^{2n}(\mathbb{R})$ onto $L^2(\mathbb{R})$. Consequently, ζ is in the resolvent set of A.

(b) Let $\zeta\in\mathbb{R}$ and let $y\in H^{2n}(\mathbb{R})$ be a solution of the equation

$$p(D)y-\zeta Jy=0\ .$$

The restriction y_+ (y_-, resp.) of y to $\mathbb{R}_+$($\mathbb{R}_-$, resp.) satisfies the equation (9) ((10), respectively). Applying Lemma 2.1 (b) and arguing as in the proof of (a), we conclude that the equation (9)((10), respectively), has $k_+\le n$ ($k_-\le n$, resp.) linearly independent solutions ψ_j^+, $j=1,\dots,k_+$ (ψ_j^-, $j=1,\dots,k_-$, resp.). Moreover, the Wronskian of

$$\{\psi_1^+,\dots\ \psi_{k_+}^+,\ \psi_1^-,\dots,\psi_{k_-}^-\}$$

is nowhere 0. Since y_+ (y_-, resp.) is a linear combination of ψ_j^+, $j=1,\dots,k_+$ (ψ_j^-, $j=1,\dots,k_-$, respectively) the continuity of $y^{(m)}$ for $m=0,1,\dots,k_++k_--1$ at 0 implies $y_+=0$ and $y_-=0$. Hence $y=0$. Since A is selfadjoint in $\mathcal{K}$ it cannot have real numbers in residual spectrum.

(c) We use I. M. Glazman's decomposition method. Define $A_\pm$ in $L^2(\mathbb{R}_\pm)$ by $\mathcal{D}(A_\pm)=H^{2n}(\mathbb{R}_\pm)\cap H_0^n(\mathbb{R}_\pm)$ and $A_\pm y=\pm p(D)y,\ y\in\mathcal{D}(A_\pm)$. The operator A_- (A_+, respectively) is a selfadjoint operator in $L^2(\mathbb{R}_-)$ ($L^2(\mathbb{R}_+)$, resp.). The continuous spectrum of A_- (A_+, respectively) is $(-\infty, -m_p]$ ($[m_p, +\infty)$, resp.). The operator $A_- \oplus A_+$ is selfadjoint in $L^2(\mathbb{R})$ and its continuous spectrum is the union of the continuous spectra of A_- and A_+. The operators A and $A_- \oplus A_+$ have the same continuous spectrum. Therefore, by (b), $\sigma(A) = \sigma_c(A) = \sigma_c(A_-) \cup \sigma_c(A_+)$. □

THEOREM 2.3 *Let p be a nonnegative polynomial. Let $A = Jp(D)$.*

(a) *The operator A is a positive definitizable operator.*

(b) *The point ∞ is a regular critical point of A.*

(c) *The point 0 is a critical point of A if and only if $0 \in \sigma(A)$, or equivalently, if and only if $m_p = \min\{p(x) | x \in \mathbb{R}\} = 0$.*

PROOF (a) The definitizability of the positive operator A follows from Theorem 2.2.

The positivity of A and the equality (8) imply the statement (c) and the fact that ∞ is a critical point of $Jp(D)$.

Since the operators $A = Jp(D)$ and JD^{2n} are definitizable the operator D satisfies all the assumptions for S in Theorem 1.4 (a). By [5], ∞ is not a singular critical point of JD^2. By Example 1 the functions $h = p$ and $g(t) = t^{2n}$ satisfy the conditions of Lemma 1.3 (a). Therefore we can apply Theorem 1.4 (a) to conclude that ∞ is not a singular critical point of A. □

It follows from Theorem 2.3 that A is similar to a selfadjoint operator in $L^2(\mathbb{R})$ if $m_p > 0$. The same is true if $m_p = 0$ and 0 is a regular critical point of A. In the next theorem we give a sufficient condition for p under which 0 is a regular critical point of A.

Let a be an arbitrary real number. Denote by $V(a)$ the multiplication operator on $L^2(\mathbb{R})$ defined by $(V(a)f)(x) = e^{iax} f(x)$, $x \in \mathbb{R}$. Simple calculations show that the following proposition holds.

PROPOSITION 2.4 *The operators JD^{2n} and $J(D + aI)^{2n}$ are similar:*

$$V(a)^{-1} JD^{2n} V(a) = J(D + aI)^{2n}.$$

THEOREM 2.5 *Let p be a nonnegative polynomial with exactly one real root. Then 0 is a regular critical point of $A = Jp(D)$. The operator A is similar to a selfadjoint operator in $L^2(\mathbb{R})$.*

PROOF Let a be the single real root of p. By Proposition 2.4 the operators JD^2 and $J(D - aI)^2$ are similar. Therefore the operator $J(D - aI)^2$ is similar to a selfadjoint operator in $L^2(\mathbb{R})$. Put $S = D - aI$ and $q(x) = p(x + a)$.

Then q satisfies all the assumptions for h in Corollary 1.5 and $Jq(S) = Jp(D)$. Since JS^2 is similar to a selfadjoint operator in $L^2(\mathbb{R})$, Corollary 1.5 implies that $Jq(S) = Jp(D)$ is similar to a selfadjoint operator in $L^2(\mathbb{R})$. □

3 Half-range Completeness

Let A be a positive operator in the Krein space $\mathcal{K} = (L^2(\mathbb{R}), [\,\cdot\,|\,\cdot\,])$. Assume that A has a nonempty resolvent set. Let $\mathcal{K}_\pm$ be the set of all functions f in $L^2(\mathbb{R})$ which vanish on the set $\mathbb{R}_\mp$. Then $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ is a fundamental decomposition of $\mathcal{K}$.

Assume that neither 0 nor ∞ are singular critical points of A. Let E be the spectral function of A. Then the operator A is a selfadjoint operator in the Hilbert space $(\mathcal{K}, [(E(\mathbb{R}_+) - E(\mathbb{R}_-))\,\cdot\,,\,\cdot\,])$; see [9, Theorem 5.7]. The corresponding fundamental decomposition is $\mathcal{K} = \mathcal{L}_+ \oplus \mathcal{L}_-$, where $\mathcal{L}_\pm = E(\mathbb{R}_\pm)\mathcal{K}$. This fundamental decomposition reduces A. Let $P_\pm$ be the orthogonal projection in $\mathcal{K}$ to $\mathcal{K}_\pm$. Then the restriction

$$T_\pm := P_\pm|_{\mathcal{L}_\pm} : \mathcal{L}_\pm \longrightarrow \mathcal{K}_\pm$$

is a bounded and boundedly invertible bijection of $\mathcal{L}_\pm$ onto $\mathcal{K}_\pm$. Let $f_\pm \in \mathcal{K}_\pm$. Then $T_\pm^{-1} f_\pm \in \mathcal{L}_\pm$. Therefore

$$T_\pm^{-1} f_\pm = \int_{\mathbb{R}_\pm} dE(t) T_\pm^{-1} f_\pm \ .$$

Since $P_\pm$ is continuous we get

$$f_\pm = \int_{\mathbb{R}_\pm} dP_\pm E(t) T_\pm^{-1} f_\pm = \int_{\mathbb{R}_\pm} dF_\pm(t) f_\pm \ ,$$

where $F_\pm(\imath) = P_\pm E(\imath) T_\pm^{-1}$, for $\imath$ an open interval in $\mathbb{R}_\pm$. Then $F_\pm$ is a projection valued measure on $\mathbb{R}_\pm$.

We have proved that the elements $f_\pm$ from $\mathcal{K}_\pm$ can be represented as integrals over $\mathbb{R}_\pm$ with respect to the measure $F_\pm(\cdot) f_\pm$ which is obtained by orthogonally projecting the spectral measure $E(\cdot) T_\pm^{-1} f_\pm$ onto $\mathcal{K}_\pm$. This is exactly the continuous analogue of the familiar concept of half-range completeness property in the discrete spectrum case; see [1].

This property holds in particular for the operators from Theorem 2.5.

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